ASTR4012/ASTR8002
Astrophysical Gas Dynamics
Assignment 4 – exam assignment

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due Wednesday, October 24, 2018

1 Rankine-Hugoniot shock jump consitions

(a) Find the compression ratio $r \equiv \rho_2/\rho_1$ at a plane shock front in terms of the upstream Mach number $\mathcal{M} = v_1/c_1$, where $c_1 = (\gamma P_1/\rho_1)^{1/2}$ is the upstream sound speed (subscripts 1 and 2 refer to the pre-shock and post-shock quantities, respectively). Assume that gas flows along the shock normal.

Hint: Use the hydrodynamic equations in conservation form to identify the flux of mass, momentum and energy entering and leaving the shock front.

(10 points)

(b) A supernova remnant shock moves at 5,000 km/s into a part of the interstellar medium with temperature of 10^4 K. Estimate the downstream (post-shock) temperature, assuming the medium consists of ionised hydrogen in full thermodynamic equilibrium.

(10 points)

(c) What would be the approximate temperature behind a shock front moving at 50,000 km/s, if full thermodynamic equilibrium were established? Why does this approximation fail?

(5 points)

2 Evolution of a supernova remnant

Assume first the supernova remnant in its adiabatic phase: all the mass of the remnant is concentrated in a thin shell located at the position of the shock at radius $r = r_s(t)$, where $M \approx (4\pi/3)r^3\rho_1$, with ρ_1 being the density of the ambient (interstellar) medium. Furthermore, the pressure $P(t)$ interior to the shock can be considered uniform and the equation of motion for the thin shell is given by

$$\frac{d}{dt}(M\dot{r}_s) = 4\pi r_s^2\, P, \quad (1)$$

where $\dot{r}_s = dr_s/dt$.

(a) Use the jump conditions for a strong shock of an ideal gas with an adiabatic index $\gamma = 5/3$ to estimate the thickness Δr of the shell in terms of r_s (assume the shell density equals the post-shock density).

(10 points)

(b) Given that in the Sedov phase, the total internal energy of the gas in the remnant equals 80% of the explosion energy $E_{\rm SN}$, show that the equations of motion have a solution of the form

$$r_s = At^{\alpha}. \tag{2}$$

Find the constants A and α.

(10 points)

(c) Assume now that the shell cools rapidly. Because the cooling rate of the gas is proportional to the density squared, there is a phase in the evolution when the thermal energy of the freshly shocked gas can no longer be shared evenly throughout the remnant, as it was the case in the energy-driven phase discussed above. Most of the energy is radiated away and a cool dense shell forms around the still hot interior. To a good approximation, the interior can be described as a hot adiabatic gas bubble of constant mass with an equation of state as before, $P \propto \rho^{\gamma}$ with $\gamma = 5/3$. The evolution of the blast wave is now driven by the adiabatic expansion of the bubble. Show that this pressure-driven 'snow plow phase' admits again a solution of the form

$$r_s \propto t^{\beta}, \tag{3}$$

and find the index β.

Hint: the equation of motion of the shell, Eq. (1), still applies in this phase.

(10 points)

3 Magnetic breaking of a rotating gas disc

Consider a uniform gaseous disc of density $\rho_{\rm cl}$ and half-thickness Z rotating rigidly with an initial angular velocity Ω_0. Furthermore assume the disc is threaded by a magnetic field $\mathbf{B}$ of strength B_0, initially uniform and parallel to the rotation axis of the disc. The magnetic field couples the disc ($|z| \leq Z$) with the external medium ($|z| > Z$) of density $\rho_{\rm ext}$, which is initially at rest. Use cylindrical coordinates (R, ϕ, z), assume symmetry around the rotation axis, i.e., $\partial\phi = 0$, and assume ideal MHD.

(a) For this system, derive the evolution equation for the angular velocity Ω, where $v_\phi = R\Omega$, and the evolution equation for the toroidal component of the magnetic field, B_ϕ. In your derivations, assume that $v_R \ll v_\phi$, $v_z \ll v_\phi$, and $B_R \ll B_0$, allowing you to drop 2nd-order terms in the Euler and induction equation. The final results should be

$$\frac{\partial \Omega}{\partial t} = \frac{B_0}{4\pi R\,\rho}\frac{\partial B_\phi}{\partial z}, \quad \text{and} \tag{4}$$

$$\frac{\partial B_\phi}{\partial t} = B_0 R\,\frac{\partial \Omega}{\partial z}. \tag{5}$$

(15 points)

(b) Show that the evolution of the external medium can be expressed by the wave equation

$$\frac{\partial^2 \Omega}{\partial t^2} = v_{\mathrm{A,ext}}^2 \frac{\partial^2 \Omega}{\partial z^2}, \tag{6}$$

with the Alfvén speed $v_{\mathrm{A,ext}} = B_0/(4\pi\rho_{\mathrm{ext}})^{1/2}$ in the external medium.

(5 points)

(c) Derive the evolution equation of the angular velocity at the surface of the disc ($|z| = Z$), using the torque per unit area $N = RB_0B_\phi/(4\pi)$ that the magnetic field exerts on the surface of the disc. The result should be

$$\frac{\partial^2 \Omega_{\mathrm{cl}}}{\partial t^2} = \frac{1}{Z}\frac{\rho_{\mathrm{ext}}}{\rho_{\mathrm{cl}}} v_{\mathrm{A,ext}}^2 \left.\frac{\partial \Omega}{\partial z}\right|_{|z|=Z} \tag{7}$$

Hint: The full torque is $I\partial\Omega/\partial t$ with the moment of inertia I for a cylinder.

(5 points)

(d) Combine Equations (6) and (7) to calculate the spin-down time of the disc.

Hint: Use the solution of Equation (6) at the disc surface and make the ansatz $\Omega \propto \exp(-\alpha t + \beta z)$ to determine the constants α and β.

(10 points)

Please submit your solutions via email to christoph.federrath@anu.edu.au or in person (either to me or Astrid Bardelang) by the assignment deadline. Note that this assignment serves as the exam assignment and counts double compared to the previous three assignments.