

ASTR4012/ASTR8002

Astrophysical Gas Dynamics

Assignment 2

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1 Pressure of an ideal gas

The Maxwell-Boltzmann distribution of a thermal gas is given by

$$f_{\text{MB}}(v) d^3v = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left(-\frac{mv^2}{2k_B T} \right) d^3v, \quad (1)$$

where m is the mass of each particle, n the number density, v the velocity, T the temperature, and k_B is the Boltzmann constant. Show that the pressure is given by

$$P = nk_B T, \quad (2)$$

by using the following integral that defines the pressure from the distribution function,

$$P = \frac{4\pi}{3m} \int_0^\infty p^4 f(p) dp, \quad (3)$$

where $p = mv$ is the momentum of each particle. *Hint: get started by transforming the velocity distribution to the momentum distribution, by using the principle of probability conservation.*

(5 points)

2 Parker Instability

Consider an isothermal gas in the galactic disc, which is threaded by a horizontal magnetic field. Assume a constant gravitational field perpendicular to the disc plane in the z direction, i.e., $\mathbf{g} = -g\hat{z}$, and a magnetic field parallel to the disc plane along x , which only varies with z , i.e., $\mathbf{B} = B(z)\hat{x}$. For simplicity, study the system in two dimensions (x and z) using cartesian coordinates.

- (a) Assume that the system is in hydrostatic equilibrium with a constant ratio of the magnetic to thermal pressure, i.e.,

$$\alpha \equiv \frac{B^2}{8\pi P} = \text{const.} \quad (4)$$

What is the pressure distribution as a function of z ? Use the isothermal gas relation $P = c_s^2 \rho$, where c_s is the constant sound speed and the scale height $H = (1 + \alpha)c_s^2/g$ to express the result.

Hint: The Euler equation with a magnetic field reads

$$\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla \left(P + \frac{B^2}{8\pi} \right) + \frac{1}{4\pi\rho} (\mathbf{B} \cdot \nabla) \mathbf{B}. \quad (5)$$

(Have a thought about the additional terms compared to without B fields.)

(5 points)

- (b) Now consider this system slightly perturbed out of its equilibrium. Then, from the linear perturbation analysis, one gets the following dispersion relation in the xz -plane,

$$n^4 + c_s^2 \left[(1 + 2\alpha) \left(k^2 + \frac{k_0^2}{4} \right) \right] n^2 + k_x^2 c_s^4 \left[2\alpha k^2 + k_0^2 \left[\left(1 + \frac{3\alpha}{2} \right) - (1 + \alpha)^2 \right] \right] = 0, \quad (6)$$

where $n \equiv i\omega$ and $k_0 \equiv H^{-1}$, and the Fourier modes in the x and z direction for the perturbed variables are proportional to

$$e^{i(\omega t - k_x x)}, \quad \text{and} \quad e^{i(\omega t - k_z z)}, \quad (7)$$

with $k^2 = k_x^2 + k_z^2$. (Feel free to derive the dispersion relation, Eq. 6, yourself from the magnetohydrodynamical equations in the relevant geometry.)

Show that in the absence of a magnetic field all roots (in terms of n^2) of this dispersion relation are negative, i.e., $n^2 < 0$. What is the physical implication of this result regarding instability?

(5 points)

- (c) In the case of a non-vanishing magnetic field, derive the instability criterion for the Parker instability (magnetic Rayleigh-Taylor instability),

$$\left(\frac{2k}{k_0} \right)^2 < 2\alpha + 1. \quad (8)$$

Hint: use the roots of n^2 to find at least one unstable mode, i.e., $n^2 > 0$.

(5 points)

- (d) Show that the instability criterion, Equation (8), is equivalent to

$$\lambda_x > \Lambda_x \equiv 4\pi H \left(\frac{1}{2\alpha + 1} \right)^{1/2} \quad (9)$$

and

$$\lambda_z > \Lambda_z \equiv \frac{\Lambda_x}{[1 - (\Lambda_x/\lambda_x)^2]^{1/2}}, \quad (10)$$

with the wavelengths $\lambda_x = 2\pi/k_x$ and $\lambda_z = 2\pi/k_z$.

(5 points)

Please submit your solutions via email to christoph.federrath@anu.edu.au or in person (either to me or Astrid Bardelang) by the assignment deadline.