Dipanjan.

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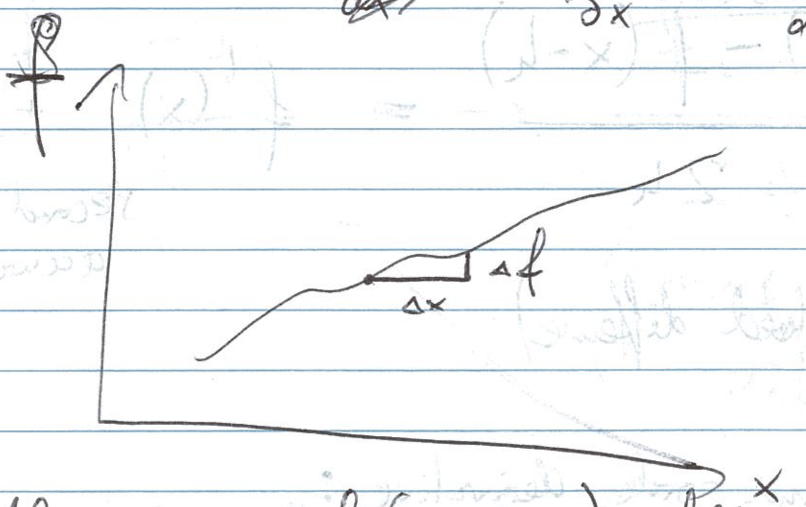
Poisson eq:

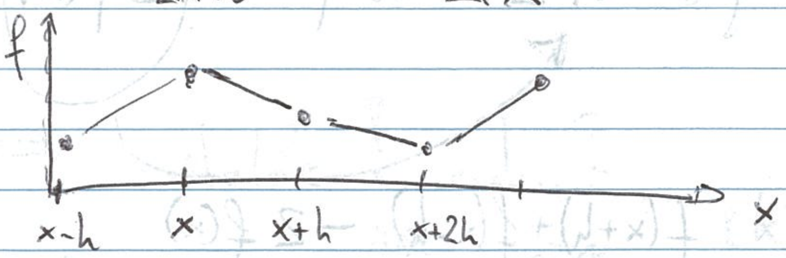
$$\Delta \Phi(x) = \rho(x)$$ (matter density ρ or charge density in Electro)

$$\Delta = \frac{\partial^2}{\partial x^2} = \nabla^2$$

or in 3D $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$



$$\frac{df}{dx} \approx \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



Numerical recipes (Chapter 19 PDEs)

Taylor expansion:

$$f(x+h) = f(x) + h f' + \frac{h^2}{2} f'' + \frac{h^3}{3!} f'''$$

$$f(x-h) = f(x) - h f' + \frac{h^2}{2} f'' - \frac{h^3}{3!} f'''$$

1st-order derivative:

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + O(h^2)$$

second order accurate

(centered difference)

Now 2nd-order derivative:

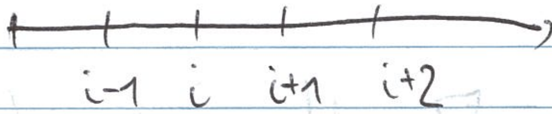
$$f(x+h) + f(x-h) = h^2 f'' + 2 f(x) + 2\frac{h^4}{4!}$$

$$\Rightarrow f'' = \frac{f(x+h) + f(x-h) - 2 f(x)}{h^2} + O(h^4)$$

advection

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial x} \qquad v = \text{const}$$

discretisation



$$\frac{f_i^{n+1} - f_i^{n}}{\Delta t} = -v \, \frac{f_{i+1}^{n} - f_{i-1}}{2\Delta x}$$

(using centred differences)

$$\Rightarrow \quad f_i^{n+1} = f_i^{n} - \frac{v\Delta t}{2\Delta x}\left(f_{i+1}^{n} - f_{i-1}\right)$$

$\Rightarrow$ unstable! why?

use waves $f^n = \xi^n e^{ik \overset{x}{j\Delta x}}$ (we switched to j for spatial index)

Lax scheme:

$$f_j^n \rightarrow \frac{1}{2}\left(f_{j+1}^n + f_{j-1}^n\right)$$

⇒ add a diffusion term

⇒ makes it stable

diffusion coefficient is $\frac{\Delta x^2}{2\Delta t}$ -ish.

↳ so numerical diffusion is reduced with higher resolution (smaller Δx)

↳ diffuse pro (which shows advection)

[Centred vs. LAX

Poisson eq.

$$\nabla^2 f = \rho \quad \text{in 2D} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{f_{j+1,l} - f_{j-1,l} - 2_{jl}}{\Delta x^2} +$$

$$\frac{f_{j,l+1} - f_{j,l-1} - 2_{jl}}{\Delta x}$$

$$\rightarrow f_{j+1,l} + f_{j-1,l} + f_{j,l+1} + f_{j,l-1}$$

$$-4 f_{jl} = \Delta x^2 \rho_{jl}$$

Fourier method

$$f_{jl} = \frac{1}{JL} \sum_{m}^{J-1} \sum_{n}^{L-1} \hat{u}_{m,n} e^{-\frac{2\pi i j m}{J}} \times e^{-\frac{2\pi i l n}{L}}$$

$$FT\left[\nabla^2 f\right] = FT\left[\rho\right]$$

$$\hookrightarrow \hat{u}_{m,n} \left(e^{-\frac{2\pi i m}{J}} + e^{\frac{2\pi i m}{J}} + \dots \right)$$

Overrelaxation: f^* is intermediate gues

means $\omega > 1$

$$f^{new} = \omega f^* + (1-\omega) f^{old}$$