A3002 – Cosmology – Problem Set

Due September 25th. If you need an extension, you should ask well in advance and justify why you need an extension. Problem sets turned in late will be penalised 15%. No problem sets will be accepted after October 14th.

Problem 1: (3 pts)
In lecture 2, we derived a the equation of motion for the Universe in a quasi-newtonian approach, and found the following differential equation

$$\left(\frac{d\xi}{d\tau}\right)^2 = \frac{1}{\xi} \pm 1,$$

$$\xi = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0} \qquad \tau = \frac{\left(|1-\Omega_0|\right)^{3/2}}{\Omega_0} H_0 t$$

Numerically, find the solution to this equation for the unbound case, and compare it to the analytic solution over the range of $0.01<\tau<5$. Comment on this comparison.

$$\xi = \frac{1}{2}\left(\cosh\eta - 1\right) = \frac{|1-\Omega_0|}{\Omega_0}\frac{r}{r_0}$$

$$\tau = \left(\frac{1}{2}\sinh\eta - \frac{1}{2}\eta\right) = \frac{|1-\Omega_0|^{3/2}}{\Omega_0} H_0 t$$

You will notice at $\xi,\tau=0$, the differential equation becomes undefined. So you need to be clever to figure out what values of τ and ξ to start on.

Problem 2: (3 pts)
Show that, for a matter dominated Universe (e.g. no Cosmological constant, no photons, and nothing else of interest, the k=-1 solution of the Friedman equations is given by:

$$\frac{a}{a_0} = \frac{\Omega_M}{2(1-\Omega_M)}(\cosh\eta - 1)$$

$$t = \frac{1}{H_0}\frac{\Omega_M}{2(1-\Omega_M)^{3/2}}(\sinh\eta - \eta)$$

Problem 3: (3pts)
Assuming a matter-dominated Universe (as above) and a value of the Hubble constant of 72 km/s/Mpc, calculate the age of the Universe in billions of years for 3 values of Ω_M=1,0.27,2.

Problem 4: Assuming that Ω_M=0.27, and H_0=72 km/s/Mpc

4.1: (1pt) The current temperature of the Universe is 2.73 degrees K as seen in the Cosmic Microwave Background. Given that Blackbody radiation has an energy density given by the expression $\rho_{rad} = \frac{4}{c}\sigma T^4$, where σ is the Stefan

Boltzmann Constant, compute the current energy density in radiation and compare this to the critical density.

4.2: (1pt) Compute the redshift where Ω_M and Ω_γ (density in photons) are equal.

4.3 (3pts) Assuming radiation is blackbody (spectrum described by Planck's law), show that the cosmic microwave background radiation continues to follow Planck's law, with $T(z)=T_0(1+z)$.

4.4 (1pt) Using your results from above, what was the temperature when radiation and matter made equal contributions to the overall density of the Universe.

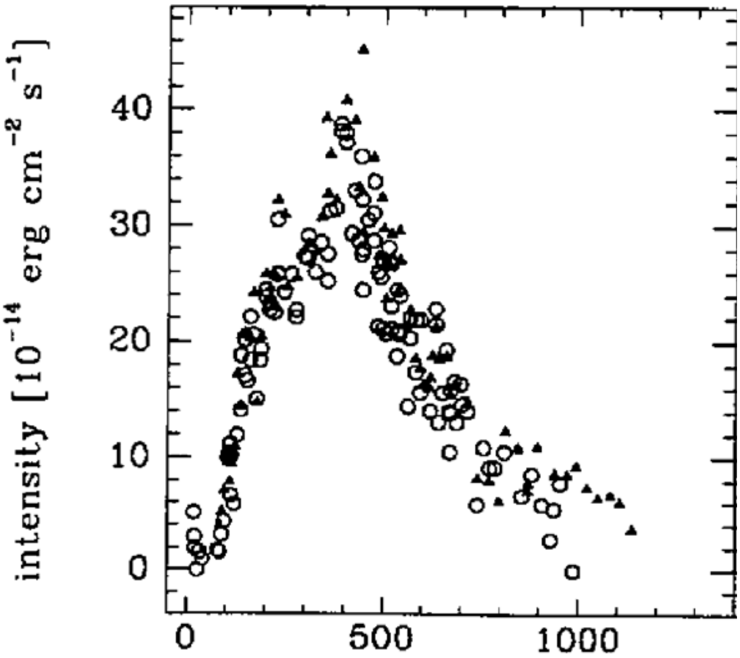

Figure 0: Strength of ionized gas as function of days after explosion.

Problem 5. On 22nd February 1987, a supernova appeared in the Large Magellanic Cloud. When this object was observed with the IUE Ultraviolet Satellite, it initially contained spectral features associated with the supernova, but about 100 days after the explosion, suddenly, very narrow emission lines of Nitrogen appeared and gained in brightness for 300 days, before fading away. See figure 1 for the light curve of these narrow Nitrogen lines. When the Hubble Space Telescope was launched, one of the first pictures it took was of SN 1987A (figure 2), and the situation was revealed. There was a narrow ring of material (+ two additional fainter rings we will ignore here) surrounding the SN 1.66"x1.11" in diameter along the major and minor axis, respectively (Figure 2).

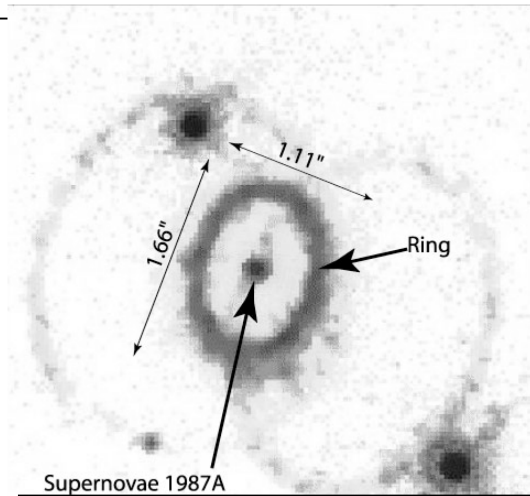

Figure 0: Image of SN 1987A from HST

Let's assume that the UV and X-rays from the Supernova, when it first exploded, ionized this ring of gas, and the light curve that we see is then a result of light travel time effects. We can then measure the distance to SN 1987A in a purely geometric way!

5.1 (1 pt) Draw a diagram of the situation where the Earth is a distance *D* from the SN down the y-axis, the ring is spherical and is located at radius *r* from the SN, and is inclined an angle *i* with respect to the x-y plane. The position of the ring is measured by θ, an angle in the plane of the ring, which is at 0 degrees down the x-axis, 90 degrees when projected onto the y-axis. Using measurements from figure 2, what is the inclination angle?

5.2 (3pts) Show that the difference in light travel time from the SN to the Earth versus the SN to the ring to the Earth is given by the expression

$$\Delta t = (1 - \sin\theta \cos i)\frac{r}{c},$$

if r << D.

5.3 (5pts) Each parcel of gas in the ring, when it gets hit by the initial burst of SN light, gets ionized, and then recombines when it encounters a friendly electron. This recombination produces the narrow emission lines of Nitrogen shown in figure 1. If we assume that the chance of a recombination occurring is constant per unit time, what form (equation) will the number of recombinations per unit time take for a group of atoms all ionized at the same time. Use this, in combination with the time delay from 5.2, to model the light curve (an approximate fit to the data is fine) shown in figure 1. Comment on the whether or not you think your modelled light curve is appropriate for the situation. Hint: You need to numerically solve this problem, by breaking the ring into small pieces as a function of θ. Figure out the behaviour of each parcel of gas, and then add them all up to find the light curve.

5.4 (3pts) Using expression from problem 5.2, derive the expression for the time after SN explosion date, when the first light from the ionised ring reaches Earth. Using the expression from problem 5.2, derive the expression for the time after SN explosion date when the maximum brightness of the ring occurs. Estimate, from figure 1, the turn-on date and maximum brightness of the ring, and then estimate the physical distance, *r* from the SN to the ring, from these two expressions. Assuming your model is perfect description of reality, estimate the uncertainty in the measurement, due to the uncertainties in your estimate of the turn on and maximum brightness of the ring.

5.5 (2pts) Using a weighted average (weighted by the square of the relative uncertainty of the two estimates) of the two values you have derived for *r*, and the angular size of the ring, calculate the distance to SN 1987A. In measuring their Hubble Constant of 72 km/s/Mpc, the Key Project assumed that the LMC was located 50kpc from Earth. What Hubble constant would they have gotten if they had used your distance to the LMC?