Observables

- So far we have talked about the Universe's dynamic evolution with two Observables
  - a –the scale factor (tracked by redshift)
  - t – time

So a cosmological test would be to compare age with redshift.

But cosmic clocks are in short supply.

Proper Distance

- Defined as what a ruler would measure at an instantaneous time. Use R-W Metric.

For a fixed t, angle, going from r=0 to r=r.

$$ds^2 = (cdt)^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right)$$

0

$$dt = 0, d\theta = 0, d\phi = 0$$

$$d_{proper} = \int ds = \int_0^r \left(\frac{a(t)dr}{\sqrt{1-kr^2}}\right) = a(t)\begin{Bmatrix} \arcsin(r) & k=+1 \\ r & k=0 \\ \text{arcsinh}(r) & k=-1 \end{Bmatrix} = a(t)S(r)$$

$$d_{proper}(t_0, r) = a_0 S(r) = \frac{a_0}{a} d_{proper}(t,r)$$

$$d_{comoving} \equiv \frac{a}{a_0} d_{proper} = aS(r)$$

First Go at Hubble Law

$$v = \frac{d}{dt}\left(d_{proper}(t,r)\right) = \frac{d}{dt}\left(aS(r)\right) = \dot{a}S(R) = \frac{\dot{a}}{a} d_{proper}$$

$$v = H(t)d_{proper} \approx H_0 d \text{ if d is small.}$$

Objects which have an apparent velocity proportional to their distance. The Hubble Law.

Somewhat tricky use of v here, which is not really A velocity, but rather a redshift.

Origin of Redshift

Massless particles (e.g. photons) travel on geodesics in GR – e.g.

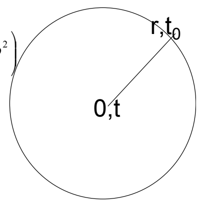
$$ds^2 = 0 \text{ for all massless particles}$$

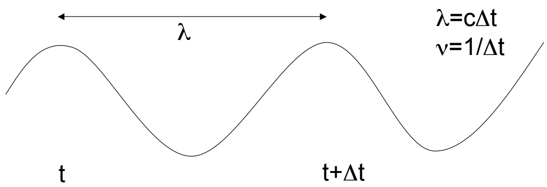
$$ds^2 = 0 = (cdt)^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right)$$

0 0

$$\int_t^{t_0} \frac{cdt}{a} = \int_0^r \left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)^{1/2}$$

$$\int_t^{t_0} \frac{cdt}{a} = \int_0^r \left(\frac{dr}{\sqrt{1-kr^2}}\right) = \begin{Bmatrix} \arcsin(r) & k=+1 \\ r & k=0 \\ \text{arcsinh}(r) & k=-1 \end{Bmatrix} \equiv S(r)$$





λ=cΔt

ν=1/Δt

When light is emitted from a object, its wavelength is represents the time interval between the wavelength peaks.

S(r) represents a comoving coordinate – it is invariant as the Universe expands

$$\int_t^{t+\Delta t} \frac{cdt}{a} = \int_{t_0}^{t_0+\Delta t_0} \frac{cdt}{a} = S(r)$$

In the limit that Δt (and hence Δt_0) are small, integral becomes

$$\frac{c\Delta t}{a} = \frac{c\Delta t_0}{a_0}$$

$$\frac{\Delta t_0}{\Delta t} = \frac{a_0}{a}$$

$$\frac{\Delta t_0}{\Delta t} = \frac{\lambda_0 / c}{\lambda_e / c} = \frac{\lambda_0}{\lambda_e} = \frac{a_0}{a}$$

$$\frac{\lambda_0}{\lambda_e} - 1 = \frac{a_0}{a} - 1$$

$$\frac{\lambda_0 - \lambda_e}{\lambda_e} = z = \frac{a_0}{a} - 1$$

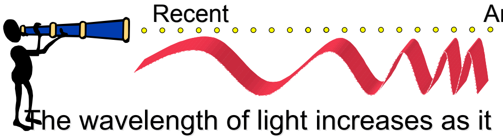
$z \equiv \frac{\lambda_0 - \lambda_e}{\lambda_e}$ Definition of the Redshift

λ_0 Observed wavelength of light (Observer's frame)

λ_e Emitted wavelength of light (rest frame)

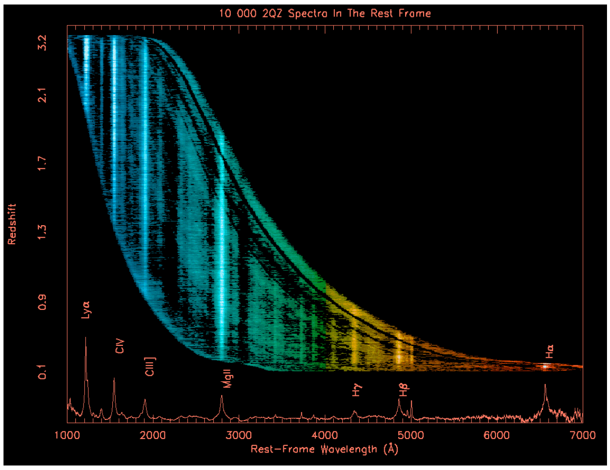
As the Universe Expands, Light is redshifted!

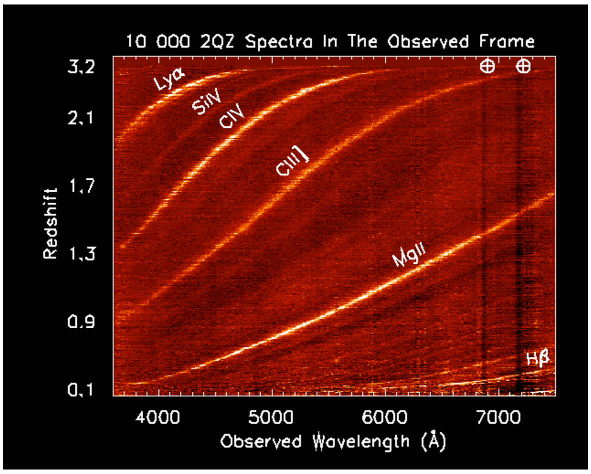
The Expansion Stretches Light Waves



The wavelength of light increases as it travels through the expanding Universe - “Redshift”

The longer the light has been travelling, the more it’s Redshifted – Not a Doppler Shift!





Luminosity Distance

- We observe the Universe via luminous objects. How bright they appear as a function of redshift is a fundamental observable.

$$f = \frac{L}{4\pi d^2} \text{ inverse square law for light in Euclidean Universe}$$

$$d_L \equiv \left(\frac{L}{4\pi f}\right)^{1/2}$$

Imagine we have a monochromatic bit of light
Emitted in all direction. How bright will it appear
As a function of distance

- Observed flux is Luminosity spread out over the surface area of a sphere with radius of the proper distance.

$$f = \frac{L}{4\pi d_p^2}\left(\frac{a}{a_0}\right)^2$$

But! There is time dilation – photons arrive more slowly (same idea of redshift) as scale factor changes

But! There is energy dimunition.Wavelength increases…So E per photon diminished.

Add two factors of a/a_0

$$f = \frac{L}{4\pi d_p^2}\left(\frac{a}{a_0}\right)^2 \quad \& \quad d_L \equiv \left(\frac{L}{4\pi f}\right)^{1/2}$$

$$d_L = d_p\left(\frac{a}{a_0}\right) = d_p(1+z)$$

Angular Size Distance…

How big does a rod (of length l) look as a function of distance?

$$\Delta\theta = \frac{l}{d} \quad \text{for Euclidean}$$

$$d_A \equiv \frac{l}{\Delta\theta}$$

dt,dr,dϕ = 0

$$ds^2 = (cdt)^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right)$$

$$l = \int ds = \int a(t) r d\theta = ar\Delta\theta$$

$$d_A = ar = d_{comoving}(k=0) = \frac{a}{a_0} d_{proper} = \frac{d_p}{(1+z)} = \frac{d_L}{(1+z)^2}$$

True as derived here for flat Universe
But always true!

Move to observable space…

$$a(t) = a_0\left(1 + \frac{da}{dt}(t_0)\frac{(t-t_0)}{a_0} - \frac{1}{2}\frac{d^2a}{dt^2}(t_0)\frac{(t-t_0)^2}{a_0^2} + \ldots\right)$$

Taylor Expansion

$$q_0 \equiv -\frac{\frac{d^2a}{dt^2}(t_0)a_0}{\frac{da}{dt}(t_0)^2} \quad \text{Deceleration Parameter.}$$

$$H_0 \equiv \frac{\frac{da}{dt}(t_0)}{a_0} \quad \text{Hubble Constant}$$

$$a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^{\,2}(t-t_0)^2 + \ldots\right]$$

Substitute in deceleration and Hubble Constant

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right)H_0^{\,2}(t_0-t)^2 + \ldots$$

$$(t_0 - t) = \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right]$$

Non-trivial algebra…

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right)H_0^{\,2}(t_0-t)^2 + \ldots$$

$$(t_0 - t) = \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right]$$

for a light ray, using R - W metric and $ds^2 = 0$

$$\int_t^{t_0}\frac{cdt}{a} = \int_0^r \frac{dr}{\sqrt{1-kr^2}} = S(r) = r \text{ for the flat case}$$

Substitute in For z

$$\frac{c}{a_0}\int_t^{t_0}(1+z)dt = \frac{c}{a_0}\int_t^{t_0}\left[1 + H_0(t-t_0) + \left(1 + \frac{q_0}{2}\right)H_0^{\,2}(t-t_0)^2 + \ldots\right]dt = r$$

$$r = \frac{c}{a_0}\left[(t_0 - t) + \frac{H_0^{\,2}(t_0-t)^2}{2} + \ldots\right]$$

Integrate

$$r = \frac{c}{a_0 H_0}\left[z - \frac{1}{2}(1+q_0)z^2 + \ldots\right]$$

Substitute in for t_0-t

$$d_{proper} = \int ds = \int_0^r\left(\frac{a(t)dr}{\sqrt{1-kr^2}}\right) = a(t)\begin{Bmatrix} \arcsin(r) & k=+1 \\ r & k=0 \\ \text{arcsinh}(r) & k=-1 \end{Bmatrix} = a(t)S(r)$$

$$d_{proper} = ar$$

$$a(t) = a_0\left[1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^{\,2}(t-t_0)^2 + \ldots\right]$$

$$(t_0 - t) = \frac{1}{H_0}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right]$$

Substitute in for t_0-t in a(t)

$$r = \frac{c}{a_0 H_0}\left[z - \frac{1}{2}(1+q_0)z^2 + \ldots\right]$$

Multiply a(t) and r expansions to get

$$d_{proper} = \frac{c}{H_0(1+z)}\left[z + \frac{1}{2}(1-q_0)z^2 + \ldots\right]$$

$$d_L = (1+z)d_{proper} = (1+z)^2 d_A$$