Orbits

Why don't satellites just fall down?

May people believe that spacecraft stay up because there is no gravity in space. <u>This is not true.</u> The space shuttle, for example, orbits less than 1000 km up. Gravity at this altitude is only slightly weaker than at the Earth's surface.
The force F of gravity between two objects of mass m and M, whose centres are a distance r apart, is given by Newton's famous equation:

$$F = \frac{GMm}{r^2},$$

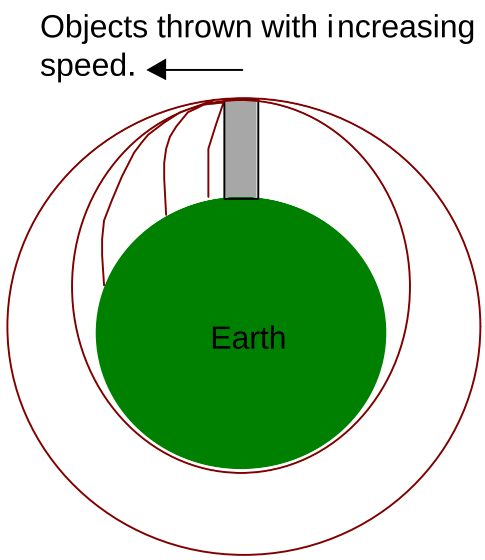
where $G = 6.67\times10^{-11}$ N m^2 kg^{-2}, and F is measured in Newtons. So, the force in orbit F_o is weaker than the force on the Earth's surface F_0 by the ratio:

$$\frac{F_0}{F_s} = \frac{\cancel{GMm}/r_o^{\,2}}{\cancel{GMm}/r_s^{\,2}} = \frac{r_s^{\,2}}{r_o^{\,2}} = \left(\frac{r_s}{r_o}\right)^2 = \left(\frac{6{,}400}{6{,}400+1000}\right)^2 = \left(\frac{64}{74}\right)^2 = 0.75$$

where the radius of the Earth is 6,400 km, and the orbit is 1000 km above the Earth's surface. So gravity is only 25% weaker up at the Space Shuttle's orbit. This cannot be why satellites stay up.
In reality, they stay up because they are moving sideways. If a satellite wasn't moving sideways, it would fall straight back down to Earth. There are two ways to see this:

Free Fall.

Consider a tower-block 1000 km high, protruding above the Earth's atmosphere. Imagine that you are dropping something off the top of this tower. If you just drop it (ignoring the rotation of the Earth), it will fall straight down, burning up in the atmosphere near the base of the tower.
But now, give it a sideways push as you drop it. As it falls, it will continue to move sideways, until it burns up. The harder you push it, the further away from the base of the tower it will land.
If you push it hard enough, it will miss the Earth altogether – by the time it's fallen 1000 km, it will have moved so far sideways that the Earth is no longer below it. If you're clever, you can get it moving in a circle around the Earth – perpetually falling but never hitting the bottom (see picture below).



Centrifugal Force

Another equivalent way of seeing the same thing is to use the concept of centrifugal force. When an object of mass *m* moves in a circular path of radius *r* at a velocity *v*, it will experience an outward centrifugal force given by the equation:

$$F = \frac{mv^2}{r}$$

For a satellite to stay in orbit, this centrifugal force must balance gravity. Thus,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}.$$

Simplifying, we find that

$$v^2 = \frac{GM}{r},$$

and hence that

$$v = \sqrt{\frac{GM}{r}}.$$

Thus for a satellite 1000 km above the Earth (and hence 7,400 km from the centre of the Earth) travelling in a circular orbit,

$$v = \sqrt{\frac{6.67\times10^{-11}\times5.976\times10^{24}}{7,400,000}} = 7.4kms^{-1}$$

(the mass of the Earth is 5.967×10^{24} kg).

Stability of Orbits

So, if we go at exactly the right speed, centrifugal force balances gravity and we can stay up. But are we safe up there, or is this a precarious balance? Consider the Earth in orbit around the Sun. Centrifugal force balances gravity at a speed of 30 kms^{-1}, and indeed, that is the speed of the Earth.

But what would happen if the Earth slowed down, even slightly, for some reason? Say an asteroid hit the Earth head on. We would be travelling ever so slightly slower, so the centrifugal force (which depends on the square of the velocity) will decrease. Gravity, however, remains just as strong. Surely we would fall inwards. And as we move inwards, gravity gets stronger still. So wouldn't we spiral inwards to our doom? Why not?

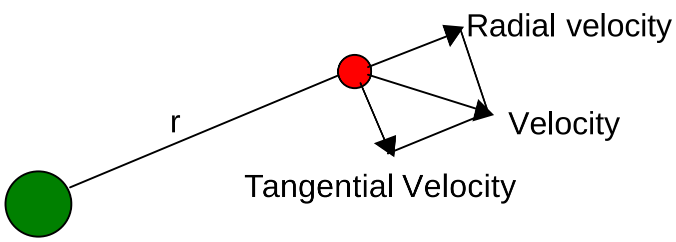
Luckily, the law of conservation of angular momentum saves us from frying. The angular momentum of an object moving in a circular orbit is $l=mvr$. This is conserved, so $v=l/mr$, ie. v is inversely proportional to r. Now centrifugal force is proportional to v^2/r. Substituting for v, we thus find that $F\mu\ 1/r^{-3}$. So the centrifugal force is proportional to the inverse *cube* of the radius. The gravitational force is only proportional to the inverse *square* of the radius.

So, as an object with a given angular momentum moves inwards, gravity increases, but the centrifugal force increases faster. Thus at some point, the centrifugal force will once again balance gravity.

So – orbits are stable – if you nudge them, they will come back into balance.

Energy Considerations

Another way to see all this is to consider the energy of an object in orbit. Break its velocity v into two perpendicular components – the radial velocity (towards or away from the thing it is orbiting – v_r, and the tangential velocity v_t.



The kinetic energy of this object is KE = $\frac{1}{2} mv^2 = \frac{1}{2} m(v_r^2+v_t^2)$ (by Pythagorus' theorem). Its angular momentum $l = mv_t r$, and is conserved. What is its potential energy? Take an object at distance r from the source and push it outwards against gravity until it reaches infinity (we choose to define the potential energy at infinity as zero). The energy used for each change in radius is just the force times the distance, and the force is given by Newton's law, so we just have to integrate this out.

$$PE = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} = -\frac{GMm}{r_0}$$

The total energy is thus:

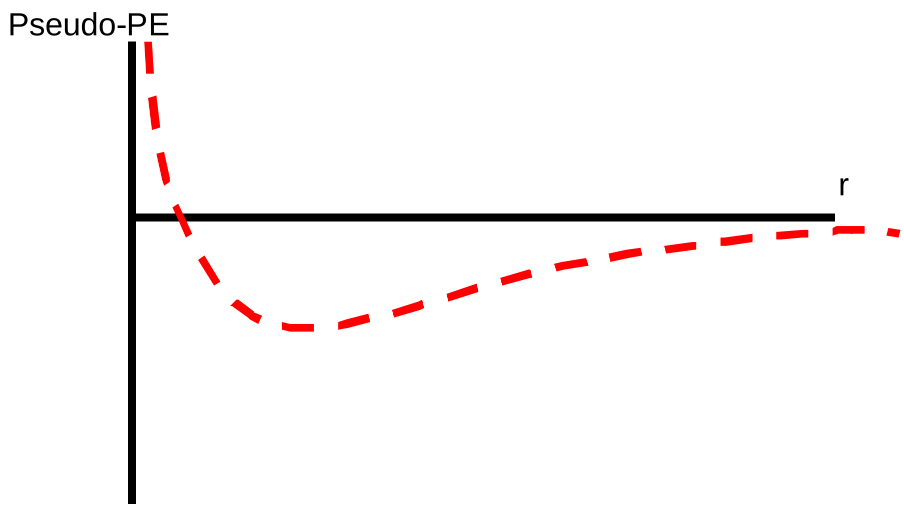
$$E = PE + KE = \tfrac{1}{2} m(v_r^{\ 2} + v_t^{\ 2}) - \tfrac{GMm}{r}$$

As angular momentum is conserved, we can substitute $v_t = l/mr$. Everything except the radial velocity component is thus a function of radius.

$$E = \tfrac{1}{2} m v_r^{\ 2} + \left(\frac{l^2}{2mr^2} - \frac{GMm}{r} \right)$$

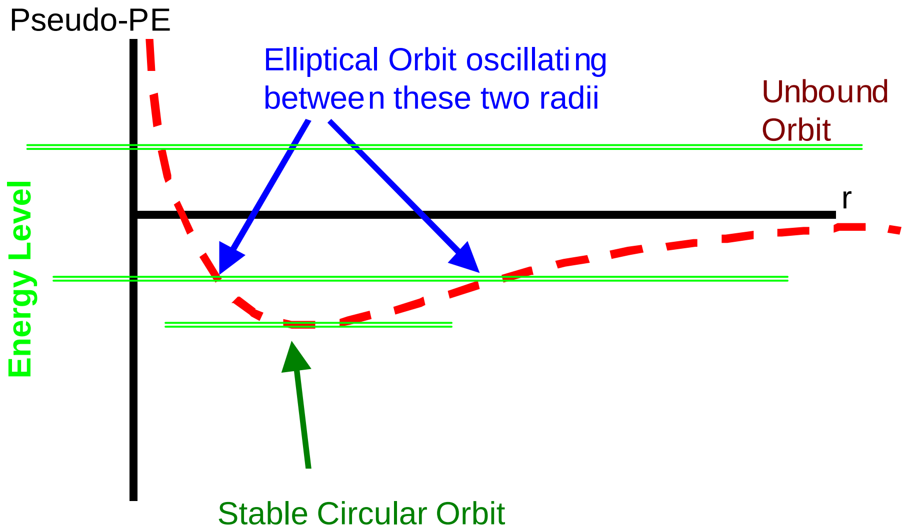
This equation is actually motion in one dimension only – all we have is the radial position and velocity. It actually looks just like the normal 1D energy equation, with a KE term that depends on the square of the velocity, and a PE term that depends only on position. So we can see our orbiting object as moving in one dimension, with a rather strange pseudo potential energy given by the term in brackets above.

What does this pseudo-PE term look like? When *r* is very small, the $1/r^2$ term will dominate, so the pseudo PE will be large and positive – indeed the pseudo-PE becomes infinite as *r* goes to zero. When *r* is large, the 1/r term dominates, so the pseudo-PE will be small and negative. As *r* goes to infinity, the pseudo-PE goes to zero. So, a graph of pseudo-PE against radius goes something like this:



The graph has a minimum (exactly where depends on the angular momentum *l*), which is why orbits are stable. If an object has just enough energy to sit at this minimum, $v_r=0$ (there's no spare energy for radial motion), so it will be in a perfect circular orbit.

If the energy of an object is greater than this minimum, but still negative, the object will oscillate in radius between r_{min} and r_{max}. This is an elliptical orbit! If the energy is positive, the object will fly out into space (it has achieved escape velocity) and will never come back (see the picture below).



An object with negative energy, but more energy than that needed for a stable orbit, will undergo approximately simple harmonic motion in this potential well. If and only if gravity follows the inverse square law, the period of this oscillation in radius will exactly equal the orbital period – this is why orbits are ellipses. If gravity obeyed any other law, the periods will not match, and the orbits will not repeat themselves, but will look more like flower petals.

This pseudo-PE curve is very informative – it tells us how hard it is to send spacecraft to various destinations. For example, notice that it takes more energy to go to $r=0$ than it takes to go to *r=*¥. Thus it is easier to send a spacecraft to Alpha Centauri than it is to drop it into the Sun!

This same curve applies to orbits about black holes (almost – see below). This gives the lie to most science fiction novels – it is actually very hard to fall into a black hole. Unless you have zero angular momentum, falling into a black hole requires vast amounts of effort on your behalf.

This has long been a puzzle in astrophysics. Objects like Cataclysmic Variables and Quasars are believed to be powered by matter falling into black holes. But how can it get rid of enough angular momentum to fall in? We'll come back to this later.

One complication – this calculation has used only Newtonian physics. Close to the event horizon of a black hole, we have to use relativity, and this changes things quite a bit. It turns out that if you are really close to the black hole (the exact distance depends on the angular momentum of the black hole, but is at most a few times the Schwartzschild radius), there is no such thing as a stable orbit, and without the use of rockets you will inevitably fall in. This basically occurs because as you get close, you would need to go very fast to fight off the severe gravity. But as your speed approaches the speed of light, your mass increases, so you get sucked in harder still.