“Back-of-the-Envelope” Calculations”

Or: The Seven Habits of Highly Effective Astronomers

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Background

One of the most important skills that any professional astronomer needs is the ability to very quickly get a highly approximate answer to a problem, without getting bogged down in details. This has many purposes:

- Most research projects start off with a bright/crazy idea, along the lines of `I wonder if a black hole could be producing these flashes' or `perhaps that galaxy is full of red supergiant stars, which is why it is so red'. Most of these bright ideas turn out to be wrong, irrelevant or unimportant. For most astronomers, 1 in every 10 ideas turns out to be correct, productive or exciting: really brilliant astronomers may have a ratio as large as 1/5. Any astronomer who has all his/her ideas work is obviously doing boring, unadventurous science. Given that most ideas do not work, it is vital to be able to tell quickly whether it is worth pursuing a particular idea: if you work out every idea in enormous detail, you will never get to the few interesting ones. This is the most common mistake that young researchers make: they get bogged down in some horribly mathematical detailed calculation, full of nasty integrals and messy algebra, and in the end discover that the effect they are computing is 10^{27} too small to be detected, something they could have worked out on the back of an envelope right at the beginning if they'd tried, thus saving them a month of work. So, whenever starting a research project, try and work out the answer very very approximately. If your idea is crazy, you will quickly find out. Only if this initial quick guesstimate comes out with an interesting answer should you bother doing the calculation in detail.

- No really complex calculation or computer simulation ever gives the right answer first time (unless you are an incredibly persnickety, careful, anally precise individual, in which case you are probably enrolled either in accountancy or maths, and not in astronomy). Worse still, for real research, there is no answer in the back of the textbook, and no higher authority to give you a high mark for getting it right (God may know, but she isn't telling). What this means is that you should never believe the result of any detailed calculation unless it roughly agrees with some simple approximate estimate that you can do on the back of an envelope, and understand fully. This is a second common mistake of young researchers: they get a result, and go out and publish it, do vast amounts of work relying on it, while all the time it was clearly wrong. I see the same problem all the time in undergraduate assignments: someone comes up with an answer that is clearly, wildly, hopelessly wrong (e.g. the mass of some star is 0.7 kg, or an asteroid is travelling at 10^{10} m s^{-1} (faster than light), or the Earth took 10^{11} years to form (much longer than the age of the universe). Always check your answers for plausibility: even if you don't have time to re-do them properly, you will get credit for saying “this is clearly garbage, but I don't have time to fix it”.

- In most real situations, many different effects are operating. For example, consider the elementary physics problem of a falling object. You normally calculate how long it takes to hit the ground considering only gravity, height and maybe air resistance. In reality, however, its course will be influenced by radiation pressure, the Earth's magnetic field, the gravitational pull of Jupiter, the curvature of space-time, coriolis force, cosmic ray bombardment, and the perfume being worn by the person who dropped it, to name but a few. Most of these effects are tiny: far too small to have a measurable effect. Nonetheless, many people waste vast amounts of time computing, in tedious detail, these tiny and unimportant effects. A `back-of-the-envelope' calculation can often tell you, right away, that one of these effects is far too small to be worth calculating properly.

Hints for doing `Back-of-the Envelope' estimates

Doing guesstimate type calculations is actually far harder than doing things properly, in enormous detail. You need a really strong grasp of physics, intuition about which parts of a problem are important, and imagination to dream up short-cuts. The only real way to learn is through practice, and you will get plenty of that in this course. What I can give you is a few hints, based on the personal experience of many professional astronomers. Everybody has their own style for doing these simple calculations: you should develop your own, but hopefully these hints will give you somewhere to start.

Hint 1: Don't worry about factors of 2, π, etc.

The aim of most approximate calculations is to get an answer that is correct to about an order of magnitude (a factor of ~10). So don't worry about poxy little things like factors of two, or π, or 4/7. Throw away most constants! The area of a circle is r^2, the gravitational field of the Earth is 10 m s^{-2}, etc.
Another useful approximation is:

$$\sin\theta \sim \theta \sim \tan\theta$$

as long as the angles are smallish ($\theta < 15^\circ$) and measured in radians.

Hint 2: Guess numbers

Every professional astronomer should have memorised a bunch of basic numbers, like the typical density of rock, mass of a star, radius of a galaxy, and so on. When you find that some new number is needed, you can often guess its value by comparison with numbers you already know.

For example, if you are working out the pressure at the base of Olympus Mons, as a first guess, assume that Mars rocks have the same density as Earth rocks. Go further - assume that all solid bodies in the universe have the same density. With the exception of neutron stars, you will be correct to about a factor of five.

Exercise 1:
Roughly how many piano tuners are there in New York?

Exercise 2:
The car-park outside a shopping mall is completely full. You are cruising around in your car waiting for a space to become free. Roughly how long will you typically have to wait?

Exercise 3:
If you connect a car engine up to a generator, how many light bulbs could it keep illuminated?

Hint 3: Tinker with the geometry

Feel free to be very cavalier with the geometry of the problem you are working on. For example, the Milky Way galaxy has a complex flattened shape, but for many purposes it can be approximated as a point source. Assume that mountains are square blocks, that asteroid are cubes or spheres: whatever makes the calculation easier.

One specific hint: replace smoothly varying functions (which have to be integrated over) with discrete functions. For example, consider the issue of whether a star passing near the solar system will disturb the planets in their orbits. As the star approaches, its gravitational pull slowly increases, constantly changing in direction, making the calculation of the perturbation of a planet's orbit tricky. Instead, why not just assume that the planet appears from nowhere, popping into existence at a distance from the solar system equal to its closest approach in the real situation. Keep it there for a time roughly comparable to the time needed to pass the solar system, and then make it disappear again. This is now an easy problem to solve: the gravitational pull of the star is always from the same direction and always has the same strength, and the answer won't be too far wrong.

Exercise 4:
Roughly how far must you look in any particular direction before your line of sight passes through a galaxy?

Hint 4: Use Ratios

Ratios are wonderful things: they avoid the need to work out constants and fiddle with many details. For example, the gravity on the surface of a planet of radius r and density ρ is

$$g = 4G\pi r\rho \propto r\rho$$

Now, we know g on earth (=9.81 m s^{-2}). What is g on Mars? Well, Mars is rocky, just like the Earth, so its density is going to be about the same. Its radius is 1/3 that of the Earth, so on Mars, g ~10/3 ~ 3. We never needed to know G, or the density and radius of either planet.

Exercise 5:
If the maximum possible height of a mountain is set by the pressure at which the rocks at its base become plastic, and Mt Everest is roughly at the maximum height mountains can have on Earth, estimate the maximum possible height of mountains on Mars.

Exercise 6:
Prove that all four-legged animals can jump to the same height.

Hint 5: Use Conservation Laws

One of the many wonderful things about Physics is all the lovely conservation laws: conservation of energy, mass, momentum, angular momentum, etc. By judicious use of these laws, you can get an approximate answer to many problems, while leaving out all the messy details.

For example, as a giant molecular gas cloud shrinks down to form stars and planets, the details of gas flows, turbulence, shocks and accretion are so complex that not even the world's fastest supercomputers come close to simulating it. Nonetheless, somehow all this messy physics must produce a final solar system with the same angular momentum as the cloud had at the beginning!

Exercise 7:
Deep in space, out near Pluto, lies the spaceship Canberra, Australia's first interstellar probe. Its weight, including the 27 astronauts, 46 sheep, 15 kangaroos, and 45 tonnes of meat pies, is 1327 tonnes. It has a nuclear reactor on board, which can generate a total energy output of 10^{18} J, in the process using up its entire fuel (50 kg of anti-matter, in the form of anti-tim-tams). This energy will be used to accelerate 40 tonnes of xenon gas, which will be fired out backwards to provide the rocket thrust.
How long will it take the Canberra to arrive at Alpha Centauri, and will they have run out of meat pies by then?

Hint 6: The Method of Dimensions

The laws of physics are supposedly universal: they work for everyone. This means that every valid equation should work equally well, regardless of the units you use (as long as they are self consistent). Thus

$$F = ma$$

should work regardless of whether you use Newtons, Kilograms and metres per second, or some other units (mass in Australian standard sheep, speed in furlongs per fortnight?).

This means that the units of both sides of any valid equation must always be the same. You can often use this fact to work out the form of an equation, without any knowledge of the physics. Play around with the various plausible terms in the equation (which you can usually guess) until you come up with something with the same units (dimensions) on both sides of the equals sign. This will hopefully be correct, apart from some dimensionless constant (like 2 or π).

One hint on using this method: if you are doing a calculation that will have a numerical answer, it is sometimes tempting to substitute numbers in place of symbols early in the analysis. This is bad for two reasons: firstly, it makes it impossible to use the method of dimensions to check your results. Secondly, if you find you got a silly answer, it is hard to see where you went wrong, and to recalculate things. It is almost always best to keep things as symbols right through to the end of the algebra, and only then to substitute in numbers.

Exercise 8:

A starship is ploughing through interstellar space. How much force (per unit area) does the interstellar gas exert on the spacecraft, slowing it down? This pressure might depend on the density ρ of the interstellar gas, the velocity v of the spacecraft, on the time t it has been flying, on the mass m of the individual gas particles, etc.

Ignoring all dimensionless constants, use the method of dimensions to derive an expression for the drag pressure.

Hint 7: Plausibility Checking

There are several ways to check your results without re-doing a whole tedious and complex calculation. The method of dimensions, discussed above, is one such method. Another is to check that your solution gives a correct result in some situation where you know the answer. For example, if you derive an equation which tells you the thickness of the atmosphere on a planet of a given mass and composition, use it to calculate the thickness of the atmosphere of the Earth, and see if it comes out right.

Another very powerful method is to check that the functional form of the equation is correct. For example, say that you derived an equation relating the mass *m* of a star to its luminosity *L*. Imagine that the equation you derived was

$$l \propto 1.05 + m^3 + 4\pi / (3 - m)$$

with *m* measured in units of solar masses and *l* in units of solar luminosities.

Do you think this could be correct? If you look at the equation, you can easily see two problems. Firstly, what happens if *m* is very small? Say a star with a mass on 1kg? It clearly should not be very bright. But the equation above says that even if *m=0*, the luminosity is still more than 5 solar luminosities. This clearly makes no sense: if this equation were correct, then even pebbles would outshine the sun. Secondly, what happens if *m=3*? The last term in the equation goes to infinity. This is saying that stars with three times the mass of the Sun are infinitely luminous. This doesn't seem to make much sense. Worse: it gives the wrong answer when *m = 1* (the Sun should presumably have a luminosity of one solar luminosity…) and for *m* just larger than three, it gives a negative answer (how can anything have a negative luminosity?).

So: always check the functional form of your answer to make sure that it is behaving in the correct way (another good reason for leaving your algebra in the form of symbols right until the end). If they don't seem to make sense, go back and think very hard about what you've done. Sometimes the calculation is right, and the answer that you thought made no sense is actually telling you something revolutionary about the universe (this is how the Hawking radiation from black holes was discovered), but usually it is telling you that you've stuffed up your calculation somewhere…

Exercise 9:

Our view of many parts of the universe is obscured by interstellar dust: tiny grains of rocky or icy materials floating in interstellar space. It is therefore very important to work out by how much this dust obscures our view.

Consider a dust cloud of thickness d, composed of dust grains, each of radius r. There are n dust grains per unit volume, throughout the cloud. Behind the cloud is a star, emitting light of intensity I_0. This light hits the cloud, and some of it is absorbed: only a smaller intensity I makes it through the dust cloud, and is detected by our telescopes.

One astronomer claims to have calculated how I and I_0 are related. The claim is that

$$\frac{I}{I_0} = 1 - dnr^2$$

Using the methods described in this section, work out whether this equation can be correct or not.

Exercise 10:

Can the following equations be correct? Without actually deriving anything, assess the plausibility of the following equations, and give your considered opinion on whether they can possibly be correct. Try not to make the problems any harder than they have to be!

I. *White dwarf stars are supported by the pressure ofrelativistic electrons within them. It has been suggested that the momentum p of these electrons is given by the equation $p^2 = \frac{E^2}{c^2} - m_e^2 c^2$ where E is the mean electron energy, m_e the electron mass, and c the speed of light. Is this result plausible?*

II. *If a galaxy is observed at some redshift z, it has been suggested that the time that its light has been travelling towards us t is given by the equation $t = \frac{1}{H_0}\left\{z - \frac{4}{3} q_0 z^2\right\}$ where H_0 is Hubble's constant and q_0 is the deceleration parameter (half the ratio of the mass density in the universe to the critical mass density of the universe). Is this equation plausible?*

III. *A black hole of mass M is passing through a gas cloud of density ρ at a relative speed v. It has been claimed that any gas that passes within radius r of the black hole will be sucked into a vortex of gas around it, and eventually consumed. The radius is given by the equation $r = \frac{2GM\rho}{v^2}$ where G is the gravitational constant. Do you believe this result?*

IV. *A quasar at redshift z has a total power-output of L. The distance to the quasar, as measured along the light path, is D. If we point a telescope at it, we detect an amount of power per unit area (of the telescope primary mirror) F, given by the equation $F = \frac{L}{4\pi D^2 (1+z)^2}$ Is this equation plausible?*

Advanced Exercises

Using all the tricks you've learned (as well as any dirty tricks you can dream up by yourselves) have a go at these three difficult problems.

1. *A group of aliens from Alpha Centauri were enraged by the quality of the TV broadcasts that the Earth is beaming into space. They sent a spacecraft here to punish us for our crass taste in soap operas. The spacecraft has just deployed a giant orbiting sun-screen that blocks all sunlight from reaching the Earth. How long do we have to improve the quality of our programs before thePacific Ocean freezes?*

2. *How often does a passing star come sufficiently close to the Sun that it disrupts the orbit of the planets?*

3. *An asteroid, one kilometre in diameter, lands in the North Atlantic. How high will the tidal waves be?*

Conclusions

I hope I've convinced you that the art of doing “back-of-the-envelope” calculations is a very valuable one. They cannot take the place of proper calculations, of the sort you are doing in your physics and maths assignments, but most professional astronomers spend more of their time on these guesstimates than they do on full-blown calculations.

Doing these approximate calculations is admittedly very hard: you have to really understand the physics, and rote learning of techniques and equations won't help you with these. You will get plenty of chances to practice, however, as all parts of the undergraduate astrophysics curriculum involve these guesstimates (including all PHYS1011 assignments).