**Assignment 2** Semester 1, 2007

1. The Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n$$

for a polytrope of index $n$ in which $P = \mathrm{K}\rho^{1+\frac{1}{n}}$ and $\rho = \rho_c\Theta^n$, $\rho_c$ being the central density. Show that the total mass of the polytrope is given by

$$M = \left[\frac{(n+1)P_c}{(4\pi)^{\frac{1}{3}}\mathrm{G}\rho_c^{\frac{4}{3}}}\right]^{\frac{3}{2}}\left[-\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_s},$$

where $\xi_s$ is the value of $\xi$ at the stellar surface.

2. As a crude approximation, we can replace an equation such as

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

by a dimensional relation

$$\frac{P_c}{R} \sim \frac{GM}{R^2}\frac{M}{R^3} \quad ,$$

where $P_c$ is the central pressure and $R$ and $M$ are the stellar radius and mass, respectively. Use approximations of this type to show that for a sequence of stars in hydrostatic equilibrium, consisting of a perfect gas, with a similar average opacity throughout, and in which energy is transported by radiation

$$T_c \sim \frac{M}{R}, \quad \text{and}$$

$$L \sim M^3.$$

Here, $T_c$ is the central temperature. (Note. The equation $L \sim M^3$ is not too bad an approximation for main-sequence stars with $M > M_\odot$: they actually follow a relation $L \sim M^{3.8}$.)

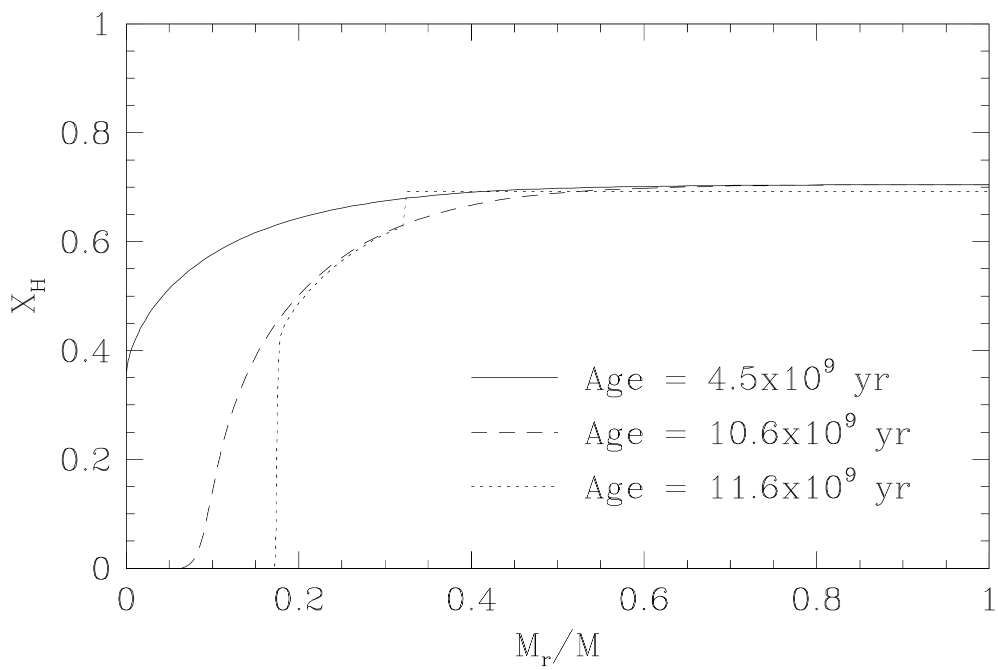
The above approximations do not use any information about the *source* of the luminosity (nuclear fusion)! Given that most of the stellar luminosity in a main-sequence star is produced by nuclear fusion near the centre of the star, and that the rate of nuclear energy production increases very rapidly with central temperature $T_c$, use the results above to describe how a newly-formed star comes to a state of thermal and nuclear-burning equilibrium after it condenses out from a molecular cloud.

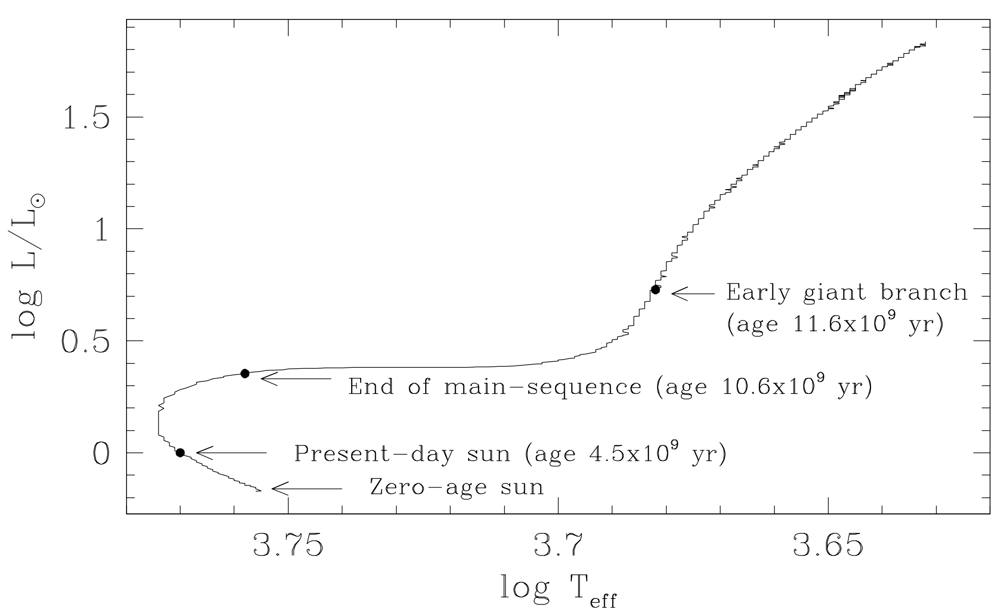
3. This example explores the evolution of the sun. The attached figure shows the evolution of the sun in the theoretical Hertzsprung-Russell diagram, as well as a plot of the hydrogen abundance $X_H$ against mass fraction $M_r/M$ in the sun.

(a) Use the plot to make a rough estimate the mass of hydrogen that has been converted to helium in the sun during its lifetime, from birth to the present day. Given that the mass of proton is 1.00870 atomic mass units and the mass of a helium nucleus is 4.0026 atomic mass units, estimate the sun's average rate of energy production by nuclear reactions (in solar luminosities) over its lifetime. Take the age of the sun as $4.5 \times 10^9$ yr. Does nuclear energy production seem to successfully account for the energy output of the sun?

(b) Comment on the changes in the abundance profiles as the sun ages. In particular, for each profile, where is hydrogen burning occurring in each model, why is there a discontinuity in the abundance profile near mass-fraction 0.31 when the sun is near the base of the red giant branch, and why is the surface hydrogen abundance slightly reduced in the giant branch model.

Some constants you may need are: $M_\odot = 1.989 \times 10^{33}$ g; $L_\odot = 3.9 \times 10^{33}$ erg s$^{-1}$; speed of light $c = 3 \times 10^{10}$ cm s$^{-1}$; atomic mass unit $= 1.66 \times 10^{-24}$ g.



**The CNO bi-cycles**

The first way to process H into $^4$He is by the PP chains, while a second way is through a series of nuclear reactions involving different isotopes of carbon, nitrogen and oxygen. The reactions are as follows:
The CN cycle

$$
\begin{aligned}
{}^{12}\mathrm{C} + p &\longrightarrow {}^{13}\mathrm{N} + \gamma && (1)\\
{}^{13}\mathrm{N} &\longrightarrow {}^{13}\mathrm{C} + \mathrm{e}^+ + \nu_e && (2)\\
{}^{13}\mathrm{C} + p &\longrightarrow {}^{14}\mathrm{N} + \gamma && (3)\\
{}^{14}\mathrm{N} + p &\longrightarrow {}^{15}\mathrm{O} + \gamma && (4)\\
{}^{15}\mathrm{O} &\longrightarrow {}^{15}\mathrm{N} + \mathrm{e}^+ + \nu_e && (5)\\
{}^{15}\mathrm{N} + p &\longrightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{H}e && (6)
\end{aligned}
$$

or the ON cycle, which starts with

$$ {}^{14}\mathrm{N}(p,\gamma)^{15}\mathrm{O}(\beta^+\nu)^{15}\mathrm{N} \qquad (4,5') $$

then follows

$$
\begin{aligned}
{}^{15}\mathrm{N} + p &\longrightarrow {}^{16}\mathrm{O} + \gamma && (7)\\
{}^{16}\mathrm{O} + p &\longrightarrow {}^{17}\mathrm{F} + \gamma && (8)\\
{}^{17}\mathrm{F} &\longrightarrow {}^{17}\mathrm{O} + \mathrm{e}^+ + \nu_e && (9)\\
{}^{17}\mathrm{O} + p &\longrightarrow {}^{14}\mathrm{N} + {}^{4}\mathrm{H}e. && (10)
\end{aligned}
$$

Note that we are ignoring the extra cycles that involve $^{18}$O and $^{19}$F.

**Part A:**
By summing the reacting and the product particles for the CN and ON cycle, show that the C, N and O nuclei only play the role of catalysts.
Compute the Gamow peak and width for the $^{12}$C(p,$\gamma$)$^{13}$N reaction using

$$ E_0 = 0.122 \left( Z_a^2 Z_X^2 A T_9^2 \right)^{1/3} \quad \text{(MeV)} \qquad (11) $$

and

$$ \Delta = 0.237 \left( Z_a^2 Z_X^2 A T_9^5 \right)^{1/6} \quad \text{(MeV)} \qquad (12) $$

where $Z_a$ and $Z_X$ are the element number, $A$ the reduced mass

$$ A = \frac{A_a A_X}{A_a + A_X} \qquad (13) $$

where $A_a$, $A_X$ are the atomic masses. Compute $E_0$ and $\Delta$ for i) 15 million K, appropriate for the centre of the Sun, and ii) for 25 million K, characteristic of CNO burning on the upper main sequence.
Comment on your answers, in particular compared to the lowest experimental energy measurements available. **Hint:** Look at values in the recent NACRE compilation article, Angulo, C., 1999, Nuclear Physics A, 656, p3-187.

**Part B:**

We can define lifetime of a nuclear species in a given environment

$$\tau_a(X) = (\lambda_{aX} N_a)^{-1}, \tag{14}$$

where $\tau_a(X)$ is the lifetime of species X against reactions with species $a$, and $\lambda_{aX} =< \sigma v >$, the reaction rate cross section, and $N_a$ the number density of species $a$, given by $N_a = (X(a)\rho)/Am_u$, where $X(a)$ is the mass fraction of $a$, $\rho$ is the density, $A$ the atomic mass and $m_u$ the atomic mass unit, given by 1.66 $\times 10^{-24}$ grams.

If we assume that the radioactive nuclides $^{13}$N, $^{15}$O, $^{17}$F beta decay essentially instantaneously to $^{13}$C, $^{15}$N, $^{17}$O, then we can write a simple set of differential equations that govern the abundances of the C, N and O nuclei as a function of time. These equations are:

$$\frac{d^{12}\mathrm{C}}{dt} = -\frac{^{12}\mathrm{C}}{\tau_{12}} + \alpha\frac{^{15}\mathrm{N}}{\tau_{15}} \tag{15}$$

$$\frac{d^{13}\mathrm{C}}{dt} = \frac{^{12}\mathrm{C}}{\tau_{12}} - \frac{^{13}\mathrm{C}}{\tau_{13}} \tag{16}$$

$$\frac{d^{14}\mathrm{N}}{dt} = \frac{^{13}\mathrm{C}}{\tau_{13}} - \frac{^{14}\mathrm{N}}{\tau_{14}} + \frac{^{17}\mathrm{O}}{\tau_{17}} \tag{17}$$

$$\frac{d^{15}\mathrm{N}}{dt} = \frac{^{14}\mathrm{N}}{\tau_{14}} - \frac{^{15}\mathrm{N}}{\tau_{15}} \tag{18}$$

$$\frac{d^{16}\mathrm{O}}{dt} = \gamma\frac{^{15}\mathrm{N}}{\tau_{15}} - \frac{^{16}\mathrm{O}}{\tau_{16}} \tag{19}$$

$$\frac{d^{17}\mathrm{O}}{dt} = \frac{^{16}\mathrm{O}}{\tau_{16}} - \frac{^{17}\mathrm{O}}{\tau_{17}}, \tag{20}$$

where, in the short-hand notation used here, $\tau_{12}$ refers to the lifetime of $^{12}$C against reactions with protons, and $^{12}$C refers to the number density of $^{12}$C atoms in the plasma. The constants $\alpha$ and $\gamma$ are branching ratios from the CN cycle into the ON cycle.

Compute the equilibrium values $(^{12}\mathrm{C}/^{13}\mathrm{C})_\mathrm{e}$, $(^{12}\mathrm{C}/^{14}\mathrm{N})_\mathrm{e}$, and $(^{16}\mathrm{O}/^{14}\mathrm{N})_\mathrm{e}$, assuming a temperature of 25 million K (MK) and $\rho X_\mathrm{H} = 50$. Assume that the $^{14}$N/$^{15}$N ratio has reached equilibrium. Comment on your answers.

**Hint**: Use Table 5–3 from Clayton's textbook, where he provides

$$\log(\frac{\rho X_\mathrm{H}}{100}\tau) \tag{21}$$

as a function of $T_6$, the temperature in units of $10^6$ K, and where the lifetimes $\tau$ are expressed in years. The branching ratio $\gamma$ has a weak temperature dependence, and multiplication factors at different temperatures and densities are also provided in Table 5–3. Table 5–3 is attached to the end of this assignment sheet.

**Part C:**

The variation of the abundance $n_i$ of a species $i$ at a given time $t$, $dn_i(t)/dt$, can be approximated by the exponential law,

$$\frac{dn_i}{dt} = ke^{-t/\tau} \tag{22}$$

where $\tau$ is the lifetime, defined in Part B, and $k = -1/\tau$. This differential equation has the solution $n(t) = Ce^{-t/\tau}$, where $C$ is some constant.

Using the lifetimes computed previously in Part B for 25MK and $\rho X_{\mathrm{H}} = 50$, how long does it take the initial abundances of $^{12}$C to be reduced by a factor of 10? What about $^{16}$O? Comment on the difference between the two times? What about at 75 million K and $\rho X_{\mathrm{H}} = 50$?

[Note: The CN cycle reaches equilibrium in a time of the order of $\tau_{12}$, which is much faster than any significant interchange of nuclei between the CN cycle and the ON cycle. Those two interchange times are characteristically $10^3\tau_{14}$ and $\tau_{16}$.]

**Table 5-3 Dependence of** $\log\,(\tau_p X_H/100)$ **on temperature**†

| *Temperature,* $T_6$ | *Reaction*‡ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $C^{12}(p,\gamma)N^{13}$ | $C^{13}(p,\gamma)N^{14}$ | $N^{14}(p,\gamma)O^{15}$ | $N^{15}(p,\alpha)C^{12}$ | $10^4\gamma$ | $O^{16}(p,\gamma)F^{17}$ | $O^{17}(p,\alpha)N^{14}$ |
| 5 | 16.32 | 15.73 | 19.79 | 15.53 | 4.649 | 22.95 | 21.92 |
| 6 | 14.32 | 13.73 | 17.57 | 13.29 | 4.598 | 20.51 | 20.02 |
| 7 | 12.72 | 12.13 | 15.79 | 11.50 | 4.551 | 18.56 | 18.26 |
| 8 | 11.41 | 10.81 | 14.32 | 10.03 | 4.508 | 16.95 | 16.50 |
| 9 | 10.29 | 9.69 | 13.08 | 8.78 | 4.468 | 15.59 | 15.10 |
| 10 | 9.33 | 8.73 | 12.02 | 7.70 | 4.431 | 14.42 | 14.05 |
| 11 | 8.50 | 7.90 | 11.09 | 6.76 | 4.396 | 13.39 | 13.15 |
| 12 | 7.75 | 7.15 | 10.26 | 5.93 | 4.363 | 12.49 | 12.38 |
| 13 | 7.09 | 6.49 | 9.52 | 5.18 | 4.332 | 11.68 | 11.68 |
| 14 | 6.49 | 5.89 | 8.86 | 4.51 | 4.303 | 10.95 | 11.02 |
| 15 | 5.95 | 5.35 | 8.26 | 3.90 | 4.275 | 10.29 | 10.32 |
| 16 | 5.45 | 4.85 | 7.71 | 3.34 | 4.248 | 9.68 | 9.55 |
| 17 | 5.00 | 4.39 | 7.20 | 2.83 | 4.223 | 9.13 | 8.70 |
| 18 | 4.58 | 3.97 | 6.73 | 2.35 | 4.198 | 8.61 | 7.86 |
| 19 | 4.18 | 3.58 | 6.30 | 1.91 | 4.175 | 8.14 | 7.01 |
| 20 | 3.82 | 3.21 | 5.89 | 1.50 | 4.152 | 7.69 | 6.18 |
| 22 | 3.16 | 2.55 | 5.16 | 0.75 | 4.110 | 6.89 | 4.78 |
| 24 | 2.57 | 1.97 | 4.51 | 0.09 | 4.071 | 6.18 | 3.63 |
| 25 | 2.30 | 1.70 | 4.21 | −0.21 | 4.052 | 5.85 | 3.10 |
| 26 | 2.05 | 1.44 | 3.93 | −0.50 | 4.034 | 5.54 | 2.62 |
| 28 | 1.58 | 0.97 | 3.41 | −1.03 | 4.000 | 4.97 | 1.75 |
| 30 | 1.15 | 0.54 | 2.93 | −1.51 | 3.967 | 4.45 | 1.05 |
| 35 | 0.23 | −0.38 | 1.91 | −2.55 | 3.893 | 3.33 | −0.42 |
| 40 | −0.53 | −1.14 | 1.07 | −3.42 | 3.829 | 2.41 | −1.50 |
| 45 | −1.18 | −1.78 | 0.36 | −4.14 | 3.771 | 1.64 | −2.33 |
| 50 | −1.73 | −2.33 | −0.25 | −4.77 | 3.719 | 0.97 | −2.99 |
| 55 | −2.21 | −2.82 | −0.78 | −5.32 | 3.673 | 0.39 | −3.53 |
| 60 | −2.64 | −3.24 | −1.25 | −5.81 | 3.630 | −0.12 | −3.97 |
| 65 | −3.02 | −3.63 | −1.67 | −6.24 | 3.590 | −0.58 | −4.33 |
| 70 | −3.37 | −3.97 | −2.05 | −6.63 | 3.554 | −0.99 | −4.65 |
| 75 | −3.68 | −4.28 | −2.39 | −6.99 | 3.521 | −1.37 | −4.91 |
| 80 | −3.97 | −4.57 | −2.71 | −7.32 | 3.489 | −1.71 | −5.14 |
| 85 | −4.23 | −4.83 | −2.99 | −7.62 | 3.460 | −2.02 | −5.35 |
| 90 | −4.48 | −5.08 | −3.26 | −7.90 | 3.433 | −2.31 | −5.52 |
| 95 | −4.70 | −5.30 | −3.51 | −8.15 | 3.407 | −2.58 | −5.68 |
| 100 | −4.91 | −5.51 | −3.74 | −8.39 | 3.383 | −2.83 | −5.82 |

† Adapted from G. R. Caughlan and W. A. Fowler, *Astrophys. J.*, **136**:453 (1962). By permission of The University of Chicago Press. Copyright 1962 by The University of Chicago.

‡ The lifetimes against protons are expressed in years, and the density $\rho$ is in grams per cubic centimeter.