Our goal in this lecture is to use the formalism of relativistic electromagnetism built up in the previous class to derive some basic results about radiation emitted by relativistically-moving systems. We will start with the simplest possible case, of a single non-accelerating charge, and then consider accelerating charges and derive a relativistic generalisation of the Larmor formula and related results describing motion by accelerating relativistic charges. We will conclude by generalising our statistical treatment of radiation transfer to the relativistic case, by introducing the Thomas transformations.

I. Uniformly moving charge

To get some practice using our newfound tools, we being by considering the simplest possible case: a single point charge q moves at constant velocity v along the x axis, passing the origin at time $t = 0$. What electric and magnetic fields will an observer measure at any given time?

A. Transformation laws for electric and magnetic fields

To solve this problem, we will first use the formalism we developed in the last class to write down the rules for how electric and magnetic fields transform between reference frames. We want these rules because the electric and magnetic fields in the frame co-moving with the charge are trivial: the electric field is just a $1/r^2$ radial vector, while the magnetic field is zero. If we can transform these to the frame in which the charge is moving, we are done.

Fortunately, this is easy. Recall last time that we showed that the electric and magnetic fields are part of a rank 2 antisymmetric tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{1}$$

This transforms from frame to frame following the general rule for all tensors:

$$F'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\sigma \tilde{\Lambda}_\nu{}^\tau F_{\sigma\tau}, \tag{2}$$

where for a boost in the x direction the Lorentz transformation takes the form

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \tilde{\Lambda}_\mu{}^\nu = \eta_{\mu\tau}\Lambda^\tau{}_\sigma \eta^{\sigma\nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

and $\eta_{\mu\tau}$ is the metric tensor. Thus calculation of $F'_{\mu\nu}$ is literally nothing more than a matter of doing some matrix multiplication. Sparing you the arithmetic (which is best done with something like mathematica, at least if you're as bad at arithmetic as I am), the result is

$$F'_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -\gamma(E_y - \beta B_z) & -\gamma(E_z + \beta B_y) \\ E_x & 0 & \gamma(B_z - \beta E_y) & -\gamma(B_y + \beta E_z) \\ \gamma(E_y - \beta B_z) & -\gamma(B_z - \beta E_y) & 0 & B_x \\ \gamma(E_z + \beta B_y) & \gamma(B_y + \beta E_z) & -B_x & 0 \end{pmatrix}. \tag{4}$$

Comparing to Equation 1, we can see that the x components of $\mathbf{E}$ and $\mathbf{B}$ have been left unchanged, while the y and z components have been mixed together. One can readily verify that this transformation can be written out in traditional three-vector form, and for arbitrary direction of boost, as

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \tag{5}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \boldsymbol{\beta} \times \mathbf{B}) \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \boldsymbol{\beta} \times \mathbf{E}) \tag{6}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\mathbf{E}_{\parallel}$ and $\mathbf{E}_{\perp}$ represent the parts of the electric field vector $\mathbf{E}$ that are parallel and perpendicular to the boost direction, respectively, and similarly for $\mathbf{B}_{\parallel}$ and $\mathbf{B}_{\perp}$.

The inverse transforms can be obtained just by exchanging the primed and unprimed quantities and replacing $\boldsymbol{\beta}$ with $-\boldsymbol{\beta}$.

B. Application to a single moving charge

Having written down the general transformation rules, it is now trivial to apply them to our example of a single moving charge. In the charge's rest frame the fields are

$$\mathbf{E} = \frac{q}{r'^3}\mathbf{r}' \qquad \mathbf{B}' = 0, \tag{7}$$

where $\mathbf{r}' = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z'\hat{\mathbf{z}}'$, and $r' = |\mathbf{r}'|$. Plugging into the transformation laws, the fields in the observer's frame are

$$E_x = \frac{q}{r'^3}x' \qquad B_x = 0 \tag{8}$$

$$E_y = \gamma\frac{q}{r'^3}y' \qquad B_y = -\gamma\beta\frac{q}{r'^3}z' \tag{9}$$

$$E_z = \gamma\frac{q}{r'^3}z' \qquad B_z = \gamma\beta\frac{q}{r'^3}y' \tag{10}$$

Changing from the primed to the unprimed position coordinates, and noting that $x' = \gamma(x - vt)$ while y and z are the same in both frames, we have

$$E_x = \gamma\frac{q}{r^3}(x - vt) \qquad B_x = 0 \tag{11}$$

$$E_y = \gamma\frac{q}{r^3}y \qquad B_y = -\gamma\beta\frac{q}{r^3}z \tag{12}$$

$$E_z = \gamma\frac{q}{r^3}z \qquad B_z = \gamma\beta\frac{q}{r^3}y, \tag{13}$$

where $r = [\gamma^2(x - vt)^2 + y^2 + z^2]^{1/2}$. We have therefore found the electric and magnetic fields at arbitrary positions and times, as desired.

For simplicity we can choose to set up our coordinate system so that the observer at $x = 0$, $z = 0$ and $y = b$, so b is the impact parameter between the particle and the observer, in which case we have

$$E_x = -\gamma\frac{q}{r^3}vt \qquad B_x = 0 \tag{14}$$

$$E_y = \gamma\frac{q}{r^3}b \qquad B_y = 0 \tag{15}$$

$$E_z = 0 \qquad B_z = \gamma\beta\frac{q}{r^3}b. \tag{16}$$

with $r = (\gamma^2 v^2 t^2 + b^2)^{1/2}$. We therefore see that a particle passing by an observer generates an electric field that has a component along the particle's direction of motion and along the direction from the particle's path to the observer. The fields have significant magnitude only during the brief interval when $|t| \lesssim b/\gamma v$, so that the particle is relatively near the observer; at earlier or later times, the fields decrease as $|t|^{-3}$. During the period when the fields are strong $|E_y/E_x| \sim b/vt \gtrsim \gamma$, so if the particle is highly relativistic, $\gamma \gg 1$, the field is mainly in the direction transverse to the particle's motion.

The magnetic field that circles around the direction of the particle's motion, and, for ultrarelativistic motion ($\gamma \gg 1$, $\beta \approx 1$), the circling magnetic field is equal in magnitude to the transverse electric field. In retrospect this geometry is exactly what we might have expected, since the moving particle represents a current, and thus we expect a magnetic field circling the current. Indeed, one can also solve this problem (with considerably less insight and considerably more vector algebra) by staying in the observer's frame and using the Liénard-Wiechert potentials, and when one does so, the magnetic field appears precisely as a result of the current associated with the moving particle.

It is interesting to note that the configuration we have just described – an electric field in the y direction and a magnetic field of equal magnitude in the z direction – is *exactly* what a plane wave travelling in the x direction looks like. The configuration we have found is not in fact a plane wave, because it falls off rapidly away from the particle. However, to an observer who sees the particle go by, it looks as if the particle is accompanied by a "cloak" of propagating radiation. In quantum field theory, this cloak is interpreted as a sea of virtual photons.

II. Emission from relativistic accelerating charges

Having practiced our ability to use Lorentz transformations to calculate the fields of non-accelerating moving charges, we are now ready to tackle the more general problem of accelerating charges. We wish to compute the power radiated by a charge q moving with a velocity $\mathbf{v}$ relative to the observer that undergoes an acceleration $\mathbf{a}$.

A. Total power emitted

We begin by computing the total power emitted, leaving the question of angular distribution for later. The basic approach we will use is to work in a frame in which the particle in question instantaneously has zero velocity, and thus is non-relativistic. In this frame we can use our ordinary, non-relativistic Larmor formula to calculate the radiation. We can then boost back into the observer's frame to figure out what the radiation looks like to the observer.

Let the primed frame be the one in which the particle in question instantaneously has zero velocity, and let dW' be the amount of energy the particle emits during a time dt' measured in this frame. Since the particle is not moving relativistically in this frame, the total power emitted is given by the usual Larmor formula,

$$\frac{dW'}{dt'} = \frac{2q^2}{3c^3}|\mathbf{a}'|^2, \tag{17}$$

where $\mathbf{a}'$ is the particle acceleration measured in the primed frame. To figure out the corresponding radiated power in the observer's frame, in which the particle has velocity $v \neq 0$, we must Lorentz transform the energy and the time.

The latter is straightforward: the time interval dt measured by the observer and the time interval dt' measured in the particle's rest frame obey the usual relationship

$$dt = \gamma\, dt'. \tag{18}$$

The former is almost as easy. The radiated power is carried by photons, each of which has a four-momentum P^μ. The sum of the four-momenta of all the photons emitted during time dt' is also a four-vector, since it is just a sum of four-vectors. Thus we can write the total four-momentum of the radiation field as P'^μ. Since this total four-vector describes the total energy emitted dW', we must have $P'^0 = dW'$. Now recall that the non-relativistic Larmor formula has an angular distribution $dP/d\Omega \propto \sin^2\theta$, where θ is the angle relative to the direction of acceleration. This distribution is symmetric, so that if we integrate over 4π sr, we get zero, meaning that the radiation field carries zero total momentum *in the frame in which the particle is at rest*. Thus the four-vector describing the total radiation field must be

$$P'^\mu = \begin{pmatrix} dW' \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{19}$$

Boosting this to the unprimed frame, the time-like component is just $P^0 = dW = \gamma\, dW'$.

Thus we learn that

$$\frac{dW}{dt} = \frac{dW'}{dt'} = \frac{2q^2}{3c^3}|\mathbf{a}'|^2. \tag{20}$$

An interesting corollary of this argument is that the total power radiated over all angles is the same in any frame; Lorentz transformation redistributes power with respect to angle, but does not change the total amount of power. The power radiated is thus a Lorentz invariant.

The final step in our argument is to replace the acceleration $\mathbf{a}'$ measured in the particle's rest frame with something measured in the observer's frame. To do so, we note that, since the total power is a Lorentz invariant, there must be a way to write it in covariant form. Recognising the correct covariant form is made a bit easier by noting that the 0 component of the four-acceleration, a^0, must be identically zero in the particle's rest frame. The way we can see this is to note that the magnitude of the particle's four-velocity is a constant, since the magnitude of any four-vector is a constant. However, this means that

$$\frac{d}{d\tau}\left(\eta_{\mu\nu} U^\mu U^\mu\right) = \eta_{\mu\nu}\left(a^\mu U^\nu + U^\mu a^\nu\right) = 0 \tag{21}$$

in any frame. Since $U'^\mu = (c, 0, 0, 0)$, the only way that this requirement can be satisfied is if $a'^0 = 0$. This allows us to recognise that the quantity $|\mathbf{a}'|^2$ that appears in the formula for total power must simply be $|\vec{a}|^2 = a^\mu a_\mu$, which is a Lorentz invariant. Thus we have succeeded in finding a covariant formula for total power radiated by an accelerating charge:

$$\frac{dW}{dt} = \frac{2q^2}{3c^3} a^\mu a_\mu, \tag{22}$$

where a^μ is the four-acceleration. This can be computed in any frame, since $a^\mu a_\mu$ is the same in all frames.

With a bit of algebra on a^μ, which we will not do at this point but which is fairly straightforward, one can show that the ordinary three-accelerations in the observer frame and the particle rest frame are related by

$$a'_\parallel = \gamma^3 a_\parallel \qquad a'_\perp = \gamma^2 a_\perp, \tag{23}$$

where $\parallel$ and $\perp$ denote the components of acceleration parallel and perpendicular to the boost direction. Thus

$$\frac{dW}{dt} = \frac{2q^2}{3c^3}|\mathbf{a}'|^2 = \frac{2q^2}{3c^3}\left(a_\parallel^2 + a_\perp^2\right) = \frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2 + \gamma^2 a_\parallel^2\right). \tag{24}$$

B. Transformations of angular power

We next attempt to compute the angular distribution of the radiated energy, using much the same approach of working in the frame co-moving with the particle at the moment it is accelerated and then transforming to the observed frame. We will first approach the problem generically, without worrying what the angular distribution looks like in the rest frame, and we will then specialise to the case of Larmor-like radiation due to particle acceleration.

For this problem we set up our coordinate system so that, in the observer's frame, the particle's velocity is in the $\hat{\mathbf{z}}$ direction, and we define the angles θ and ϕ as usual in polar coordinates, so that θ is the angle between any particular direction of interest and the z axis. As a shorthand, we will denote the cosine of this angle as $\mu = \cos\theta$. The corresponding values measured in the particle's rest frame are θ' and $\mu' = \cos\theta'$.

Let dW' be the energy emitted by the particle into some solid angle $d\Omega' = \sin\theta'\, d\theta'\, d\phi' = d\mu' d\phi'$ during a time interval dt', all measured in the particle's rest frame. The four-vector describing the energy and momentum of this radiation is

$$P'^\mu = \frac{dW'}{c}\begin{pmatrix} 1 \\ \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \mu' \end{pmatrix}. \tag{25}$$

Here component P'^0 is the energy carried by the radiation field, and the three space-like components are the momenta (total momentum = energy over c) in each of the three cardinal directions. The transformation matrix required to boost back into the observer's frame is

$$\tilde{\Lambda}^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \tag{26}$$

so the four-vector that the observer sees is

$$P^\mu = \tilde{\Lambda}^\mu{}_\nu P'^\nu = \frac{dW'}{c}\begin{pmatrix} \gamma + \beta\gamma\mu' \\ \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \beta\gamma + \gamma\mu' \end{pmatrix}. \tag{27}$$

The time-like component of this is the energy over c, so the energy dW' in the emitter's rest frame corresponds to an energy

$$dW = \gamma(1 + \beta\mu')dW' \tag{28}$$

in the observer's frame.

The solid angle $d\Omega$ into which this energy is carried in the observer's frame is also not the same as the solid angle $d\Omega'$ viewed in the particle's rest frame. The angles that describe rotation about the direction of motion are the same, $\phi = \phi'$, but the angles measured along the direction of motion are related by the angle transformation formula we derived in the previous class:

$$\mu = \frac{\mu' + \beta}{1 + \beta\mu'}. \tag{29}$$

Taking the derivative of both sides,

$$d\mu = \frac{d\mu'}{1 + \beta\mu'} - \frac{\beta(\beta + \mu')\,d\mu'}{(1 + \beta\mu')^2} = \frac{d\mu'}{\gamma^2(1 + \beta\mu')^2} \tag{30}$$

Thus the solid angles are related by

$$d\Omega = \frac{d\Omega'}{\gamma^2(1 + \beta\mu')^2}. \tag{31}$$

Putting this together with the energy, we have

$$\frac{dW}{d\Omega} = \gamma^3(1 + \beta\mu')^3 \frac{dW'}{d\Omega'}. \tag{32}$$

We have therefore figured out how to transform the angular distribution of emitted radiation energy between the frames.

To turn this into a power radiated, we need to divide by the time interval over which this energy was produced. In the rest frame of the emitting particle, the time interval is of course dt', so the angular distribution of power radiated is

$$\frac{dP'}{d\Omega'} = \frac{dW'}{d\Omega'\,dt'}. \tag{33}$$

In the observer's frame the choice might seem equally obvious: we just divide by the time interval over which the radiation was emitted as measured in the observer's frame, which is $dt = \gamma\,dt'$. This gives us one definition of the angular distribution of radiated power:

$$\frac{dP_e}{d\Omega} = \frac{dW}{d\Omega\,dt} = \gamma^2(1 + \beta\mu')^3 \frac{dP'}{d\Omega'} = \frac{1}{\gamma^4(1 - \beta\mu)^3} \frac{dP'}{d\Omega'}, \tag{34}$$

where the last step follows by using the angle transformation formula Equation 29 to replace the μ' with a μ, so everything is given in the observer's frame.

However, this is *not* the power that the observer will actually measure with a telescope! Why not? Because we have not yet accounted for the fact that the source moved some distance during the time it emitted the radiation, and we have to account for the light travel time effects.

The issue is easiest to visualise if we consider an observer at $z = +\infty$, and thus off in the same direction as that in which the emitting particle is moving. Suppose the emitting particle is moving at $\beta = 0.99$ and emits a very sharp pulse of radiation over a time interval $dt = 1$ s as measured in the observer's frame. If the emitting particle were at rest, then the leading edge of the pulse would be a distance $c\,dt = 1$

light-second ahead of the trailing edge, and so it would take 1 second for the entire pulse to reach the observer's detector, and the observer would see a pulse lasting a time $dt = 1$ second.

However, because the emitter is moving, by the time the pulse ends it has moved a distance $\beta c\, dt = 0.99$ light seconds toward the observer, and so it is only 0.01 light-seconds behind the leading edge of the pulse. Thus the pulse that reaches the observer is only 0.01 light-seconds long, and lasts only 0.01 seconds, even though it was emitted over 1 second. The total energy received is the same, but because it arrives in a time interval that is 100 times shorter, the instantaneous power is 100 times larger.

With a little geometry you should be able to convince yourself that the time interval over which the radiation is received, dt_r, is in general related to the time interval over which it was emitted, dt, by

$$dt_r = (1 - \beta\mu)\, dt. \tag{35}$$

The $1-\beta\mu$ factor just comes from projecting the component of the emitter's velocity along the line of sight to the observer; the example we just walked through verbally is $\mu = 0$. This gives us an alternative definition of the angular distribution of power,

$$\frac{dP_r}{d\Omega} = \frac{1}{\gamma^4(1-\beta\mu)^4}\frac{dP'}{d\Omega'}, \tag{36}$$

which is the same as Equation 34, but with an extra factor of $1 - \beta\mu$ in the denominator to account for the light travel time effect.

C. Angular distribution of power from an accelerated particle

Now that we have figure out how to transform angular power distributions between frames, we are ready to plug in the Larmor formula for $dP'/d\Omega'$:

$$\frac{dP'}{d\Omega'} = \frac{q^2 a'^2}{4\pi c^3}\sin^2\psi', \tag{37}$$

where a' is the magnitude of the (three-vector, not four-vector) acceleration and ψ' is the angle between the acceleration and the direction of radiation emitted; both of these quantities are measured in the particle's rest frame, as indicated by the primes.

The corresponding angular distribution in the observed frame (using the power received definition) is

$$\frac{dP_r}{d\Omega} = \frac{q^2 a'^2}{4\pi c^3\gamma^4(1-\beta\mu)^4}\sin^2\psi'. \tag{38}$$

We now want to transform all the primed quantities in this expression into unprimed ones, so we have everything in terms of quantities measured in the observed frame. Using the relationship for transforming three-vector accelerations between frames given above (Equation 23), we can get rid of the a', obtaining

$$\frac{dP_r}{d\Omega} = \frac{q^2(\gamma^2 a_\parallel^2 + a_\perp^2)}{4\pi c^3(1-\beta\mu)^4}\sin^2\psi'. \tag{39}$$

Transforming the ψ' between frames in general, when the velocity vector, acceleration vector, and direction of radiation can all be oriented arbitrarily relative to one another, leads to an expression that is too unwieldy to be worth writing down. However, to get a feel for the physical nature of the result it suffices to just consider two limiting cases. First, suppose that the acceleration is parallel to the particle's velocity, so the particle is just speeding up or slowing down. In this case $a_\perp = 0$ and $\psi' = \theta'$,[1] and we can use our angle transformation formula (Equation 29). Rearranging this to solve for μ' gives

$$\mu' = \frac{\mu - \beta}{1 - \beta\mu} \tag{40}$$

$$\sqrt{1 - \sin^2\theta'} = \frac{\sqrt{1 - \sin^2\theta} - \beta}{1 - \beta\mu} \tag{41}$$

$$\sin^2\theta' = \frac{\sin^2\theta}{\gamma^2(1 - \beta\mu^2)^2}. \tag{42}$$

Plugging this in, the angular distribution of the radiation in the case of parallel acceleration is

$$\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3} a_\parallel^2 \frac{\sin^2\theta}{(1 - \beta\mu)^6}. \tag{43}$$

Now consider the case where the acceleration is perpendicular to the velocity. Recall that we set up our coordinate system so that the velocity vector points in the $\hat{\mathbf{z}}'$ direction. Without loss of generality we can rotate our coordinate system around the $\hat{\mathbf{z}}'$ axis so that the acceleration vector points in the $\hat{\mathbf{x}}'$ direction. In this coordinate system, the direction of radiation propagation is

$$\hat{\mathbf{n}}' = (\sin\theta' \cos\phi', \sin\theta' \sin\phi', \cos\theta'), \tag{44}$$

where θ' and ϕ' are the usual spherical polar coordinates. Then we can get the angle ψ' between the acceleration vector and the radiation direction by noting that

$$\cos\psi' = \hat{\mathbf{n}}' \cdot \hat{\mathbf{x}}' = \sin\theta' \cos\phi'. \tag{45}$$

Thus

$$\sin^2\psi' = 1 - \cos^2\psi' = 1 - \sin^2\theta' \cos^2\phi' = 1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2(1 - \beta\mu)^2}, \tag{46}$$

where in the last step we used Equation 42. We therefore have

$$\frac{dP_r}{d\Omega} = \frac{q^2}{4\pi c^3} a_\perp^2 \frac{1}{(1 - \beta\mu)^4} \left[1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2(1 - \beta\mu)^2}\right]. \tag{47}$$

The main thing to notice about Equation 43 and Equation 47 is that they both involve high powers of $1 - \beta\mu$ in the denominator. For highly-relativistic emitters, $\beta \approx 1$, this term gets very close to zero near $\mu = 0$, i.e., along the direction of the particle's motion. Thus the radiation is very highly beamed along the direction the particle is moving. To be precise, we can expand μ and β in a Taylor series about $\theta = 0$ and $\gamma^{-1} = 0$:

$$\mu = \cos\theta = 1 - \frac{\theta^2}{2} + O(\theta^4) \tag{48}$$

$$\beta = \left(1 - \gamma^{-2}\right)^{1/2} = 1 - \frac{1}{2}\gamma^{-2} + O(\gamma^{-4}) \tag{49}$$

[1] Or $\psi' = -\theta'$. However, since the results depend only on $\sin^2\psi'$ this difference has no effect. Physically, the radiation pattern is symmetric about the direction of acceleration.

Thus the combination $1 - \beta\mu$ expands to

$$1 - \beta\mu = \frac{1 + \gamma^2\theta^2}{2\gamma^2} + \cdots . \tag{50}$$

We clearly see that, as long as $\theta \lesssim 1/\gamma$, this combination will be of order $1/\gamma^2$, while for $\theta \gg 1/\gamma$ it is of order θ^2 (which is of order unity in most directions).

Examining the angular power distribution formulae, Equation 43 and Equation 47, we see that for a parallel acceleration the power varies with angle as $(1 - \beta\mu)^{-6}$, while for a perpendicular acceleration it varies as $(1 - \beta\mu)^{-4}$. Our series expansion shows that, for $\gamma \gg 1$, this translates to a factor of γ^{12} and γ^8, respectively, when $\theta \lesssim 1/\gamma$. This is a big effect: the power beamed into angles $\theta \lesssim 1/\gamma$ winds up many, many orders of magnitude brighter than the power at other angles, even for γ of a few. Relativistic emitters effectively beam all their radiation into a narrow cone with opening angle $\theta \sim 1/\gamma$.

III. Thomas transformations

We have now seen how relativistic beaming affects the total power and its angular distribution for a single particle. Our final goal for today is to extend that treatment to our statistical theory of radiation transfer, which involves the intensity, emissivity, and absorption coefficient that describe the interaction of radiation with a population of particles. How do these quantities Lorentz transform? The answer to this question is a set of transformation rules known as the Thomas transforms, after L. H. Thomas, the mathematician / physicist who first derived them in 1930.

A. Transformation rules

Thomas's argument begins with the following observation: we can disagree about energies, times, distances, etc. between reference frames, but we had better agree on numbers of discrete objects in all frames. Thus the proper way to figure out how to transform radiation intensities is just to count photons. This observation suggests the following argument. Suppose in one frame we have a unit area dA that lies in the xy plane, so its normal vector is $\hat{\mathbf{z}}$. Suppose the radiation intensity at the slab in some particular direction $\hat{\mathbf{n}}$ is I_ν. How many photons in some frequency interval $d\nu$ will cross the slab per unit time dt traveling in a direction within a solid angle $d\Omega$ about $\hat{\mathbf{n}}$?

We can write down the answer immediately:

$$N = \frac{I_\nu}{h\nu} \, d\nu \, d\Omega \, dt \, (\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}) dA = \frac{I_\nu}{h\nu} \, d\nu \, d\Omega \, dt \, \mu \, dA \tag{51}$$

The term I_ν is the energy carried in the beam, and each photon has energy $h\nu$, so $I_\nu/h\nu$ is the number of photons carried in the beam per unit area per unit solid angle per unit frequency per unit time. To get a total number, we just multiply by the frequency interval $d\nu$, the solid angle $d\Omega$, the time interval dt, and the component of the area that is normal to the beam, which is $(\hat{\mathbf{n}} \cdot \hat{\mathbf{z}})dA$, which in the final line we have written as $\mu \, dA$, where θ is the usual polar angle, and $\mu = \cos\theta$ is the z component of $\hat{\mathbf{n}}$.

Now let us repeat the counting exercise for an observer who is moving with velocity v in the $\hat{\mathbf{z}}$ direction. The number this observer will measure is

$$N' = \frac{I'_{\nu'}}{h\nu'} \, d\nu' \, d\Omega' \, dt' \, \mu' \, dA' + \frac{I'_{\nu'}}{h\nu' c} \, d\nu' \, d\Omega' (v \, dt' \, dA') \tag{52}$$

The first term on the right hand side is the same as for the observer who sees the slab at rest, just will all quantities transformed into the new frame. The second term is an additional on that accounts for the fact that the observer in the primed frame sees the slab as moving at velocity $-v\hat{\mathbf{z}}$, and thus sees the slab sweep up a bunch of additional photons. The number of additional photons swept up must be the volume that the slab sweeps up, $v\,dt'\,dA'$, multiplied by the number density of photons in the right range of frequency and angle, $(I'_{\nu'}/h\nu' c)d\nu'\,d\Omega'$.

The fact that both observers have to agree on how many photons the slab swept up tells us that $N' = N$, and thus

$$I_\nu = I'_{\nu'} \frac{\nu}{\nu'} \frac{d\nu'}{d\nu} \frac{dA'}{dA} \frac{d\Omega'}{d\Omega} \frac{dt'}{dt} \left(\frac{\mu' + \beta}{\mu} \right). \tag{53}$$

Now to figure out the transformation we just need to use our various transformation rules for the various quantities, all of which we know. Taking them in order:

- The frequencies ν and ν' are related by the relativistic Doppler effect formula that we derived in the previous class:

$$\frac{\nu}{\nu'} = \frac{d\nu}{d\nu'} = \frac{1}{\gamma(1-\beta\mu)} \tag{54}$$

- dA and dA' are the same, because the slab is perpendicular to the boost direction, and lengths perpendicular to the boost direction are unaffected by the boost. Thus $dA'/dA = 1$.
- We previously showed that solid angles in the two frames are related by (Equation 31):

$$\frac{d\Omega'}{d\Omega} = \gamma^2(1-\beta\mu')^2 = \frac{1}{\gamma^2(1-\beta\mu)^2} \tag{55}$$

- The times are related by the usual $dt' = \gamma\,dt$.
- Using our angle transformation formula (Equation 29) to write μ' in terms of μ and β, with a little bit of algebra we have

$$\frac{\mu' + \beta}{\mu} = \frac{1}{\gamma^2(1-\beta\mu)}. \tag{56}$$

At this point it is just a matter of collecting the various factors of γ and $(1-\beta\mu)$. The final result is

$$I_\nu = I'_{\nu'} \frac{1}{\gamma^3(1-\beta\mu)^3} = \left(\frac{\nu}{\nu'}\right)^3 I'_{\nu'} \tag{57}$$

The implication here is that the quantity

$$\mathcal{I}_\nu = \frac{I_\nu}{\nu^3} \tag{58}$$

is a Lorentz invariant, known as the invariant intensity.

Now that we know how to transform intensities, the remaining terms that appear in the transfer equation are easy to back out. The source function is easiest: since the intensity becomes equal to the source function S_ν in a uniform medium, and all observers will agree that a medium is uniform regardless of their frame, the source function must transform exactly like the intensity:

$$S_\nu = \left(\frac{\nu}{\nu'}\right)^3 S'_{\nu'}, \tag{59}$$

and S_ν/ν^3 is a Lorentz invariant.

For the emissivity, we can use the same strategy we used for the intensity: all observers must agree on the number of photons emitted from some particular volume $dx\,dy\,dz$ of matter in a given time dt and in a given frequency range $d\nu$ and into a given range of solid angle, so we must have

$$\frac{j_\nu}{h\nu} d\nu\, d\Omega\, dt\, dx\, dy\, dz = \frac{j'_{\nu'}}{h\nu'} d\nu'\, d\Omega'\, dt'\, dx'\, dy'\, dz' \tag{60}$$

Inserting the various ratios already derived, and using the fact that $dx = dx'$, $dy = dy'$, and $dz' = dz/\gamma$, we immediately get

$$j_\nu = \left(\frac{\nu}{\nu'}\right)^2 j'_{\nu'}, \tag{61}$$

so j_ν/ν^2 is a Lorentz invariant.

Finally, all observers must agree on the number of photons absorbed by a particular volume over a given time, frequency range, and solid angle, from a particular radiation field, so

$$\frac{\alpha_\nu I_\nu}{h\nu} d\nu\, d\Omega\, dt\, dx\, dy\, dz = \frac{\alpha'_{\nu'} I'_{\nu'}}{h\nu'} d\nu'\, d\Omega'\, dt'\, dx'\, dy'\, dz', \tag{62}$$

and again it is just a matter of substitution and cancelling to obtain

$$\alpha_\nu = \left(\frac{\nu'}{\nu}\right) \alpha'_{\nu'}, \tag{63}$$

so $\nu\alpha_\nu$ is the Lorentz invariant quantity.

This completes the set of Thomas transformations. The Thomas transformations are invaluable any time we wish to solve a radiative transfer problem for a relativistic system, because they allow us to write the emission and absorption terms in the rest frame of the matter, where they are usually simple, to the frame in which we want to solve the problem or observe the radiation.

B. Implications for thermal equilibrium radiation fields

We conclude today with one very important implication of the Thomas transformations – important for both practical astrophysics applications, and for saying something profound about physics. Suppose that we have a thermal equilibrium radiation field, for which the intensity is just given by the Planck function

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1}. \tag{64}$$

What does this look like to an observer in a different reference frame?

We can answer this question immediately using the Thomas transformations:

$$I'_{\nu'} = \left(\frac{\nu'}{\nu}\right)^3 I_\nu = \frac{2h\nu^3}{c^2} e^{h\nu/k_B T} - 1. \tag{65}$$

Using the Doppler formula to transform ν to ν', we have

$$\nu = \nu'\gamma(1 - \beta\mu'), \tag{66}$$

where μ' is the cosine of the angle between the direction of motion and the direction from which the radiation is coming, as viewed in the frame of the moving observer; radiation coming from the direction that the observer is moving corresponds to $\mu' = 0$. Thus

$$I'_{\nu'} = \frac{2h\nu^3}{c^2} e^{h\nu'\gamma(1-\beta\mu')/k_B T} - 1 = \frac{2h\nu^3}{c^2} e^{h\nu'/k_B T'} - 1 = B_{\nu'}(T') \tag{67}$$

where

$$T' \equiv \frac{T}{\gamma(1-\beta\mu')}. \tag{68}$$

Thus the spectrum that the observer sees is just another blackbody, but shifted to a different temperature T'. Radiation that is coming from the direction the observer is moving, $\mu' = 1$, appears to be shifted to a higher temperature, and radiation from the direction opposite the motion, $\mu' = -1$, appears to have lower temperature.

One astrophysical implication of this is that, for small sources that are moving at a single speed with respect to the observer, and that have spectra that are close to thermal equilibrium, there is a degeneracy between temperature and velocity: you can't tell the difference between a source at a lower temperature that is standing still, and a source at a higher temperature moving toward you. This statement only applies to emitters in thermal equilibrium; any disequilibrium will remove the degeneracy, which is why in practice we can measure the temperatures of most astrophysical objects. However, this is only possible because they are at least slightly out of equilibrium.

On the other hand, for an astrophysical source that covers the entire sky, i.e, the cosmic microwave background (CMB) radiation, this effect can be used to deduce the motion of the Earth relative to the rest frame of the CMB. The CMB in the direction in which we're moving appears to have a higher temperature, and the CMB in the opposite direction appears to have a lower temperature. By measuring the temperature variation, we can immediately back out the velocity of the Earth.

The physics point to make is that we should have been able to guess this result just from Kirchhoff's original argument about thermal radiation. After all, if we ask "Is this system in thermal equilibrium or not?", all observers should give the same answer regardless of their reference frame. Thus the argument that the radiation field inside a box in thermal equilibrium must depend only on its temperature must still apply in any reference frame: two observers in different reference frames don't necessarily have to agree on what the temperature is, but they have to agree that there is a temperature, and that the radiation spectrum is a function of that temperature alone.