We now come to our first major astrophysical application of the formalism we have developed: *brehmsstrahlung* ("braking radiation" in German). This is the process whereby free charges in an ionised plasma, which accelerate when the scatter off one another, emit radiation. Because the emission is coming from charges that are free before the emission process, and remain free after it, this process is also called *free-free emission*.

We will use the formalism we have developed in the last class for emission from accelerating charges to calculate the rate at which plasmas radiate. This is a major emission process across a wide range of wavelengths, from the radio to the X-ray, depending on the temperature of the emitting plasma. In plasma that is sufficiently hot, for example the intracluster medium of galaxy clusters, this process can produce copious high energy radiation.

I. Elementary considerations

Before starting on a calculation, we begin with some elementary considerations. We have a plasma consisting of electrons and ions, all of which are moving and colliding with one another, accelerating as they do so. However, we do not need to consider all possible combinations of particles; in fact we need only consider emission by electrons scattering off ions.

To see why, recall that we have shown that, to leading order, the emission of radiation depends on the time variation of the dipole moment of the charge distribution. For a collision between two identical particles, ions or electrons, the dipole moment in the centre of mass frame is $q(\mathbf{x}_1 + \mathbf{x}_2)$, where q is the charge and $\mathbf{x}_1$ and $\mathbf{x}_2$ are the particle positions. However, for two particles of equal mass the quantity $(\mathbf{x}_1 + \mathbf{x}_2)/2$ is just the centre of mass position, which is constant unless the particles are being acted upon by an outside force. Thus in interactions between like particles, the time variation of the dipole moment is exactly zero, and there is no radiation in the dipole approximation.

Next recall that the radiated power depends on the square of the acceleration of the emitting particle. This immediately tells us that, in an electrically-neutral plasma where there is at least one electron per positively charged particle, the electrons will completely dominate the emission. This is because, in an encounter between an electron and an ion, the electron acceleration will be larger by a factor of $m_{\mathrm{ion}}/m_e > 1000$. Thus the electron will radiate at least $\approx 10^6$ times more energy than the ion − more if the ion is heavier than a single hydrogen nucleus. Thus we need only consider electron-ion encounters, and we need only consider the emission from the electron.

Our final elementary consideration involves the likely speeds of encounters. Consider a plasma of electron density n_e. The mean distance between electrons and ions is $\approx n_e^{-1/3}$, so the mean potential energy between an electron and its nearest ion is $Z_i e^2 n_e^{1/3}$, where Z_i is the ion charge in units of elementary charges. This can be compared to the average kinetic energy of a free electron, which in a plasma where the electrons are at temperature T is just $k_B T$. The ratio of these two energies is

$$\frac{Z_i e^2 n_e^{1/3}}{k_B T} = 1.7 \times 10^{-7} \left(\frac{n_e}{\mathrm{cm}^{-3}}\right)^{1/3} \left(\frac{T}{10^4\,\mathrm{K}}\right)^{-1} \tag{1}$$

For most astrophysical plasmas, which have n_e of at most a few thousand, this is a tiny number. (Exceptions are mainly places like planetary or stellar interiors.) Thus we learn that the typical electron encountering the typical ion at the typical distance has a huge amount of kinetic energy compared to the potential energy. It is therefore likely to be deflected only a very small amount by an encounter. This motivates us to treat the problem in the limit where the scattering angle of the electrons encountering ions is small.

II. Encounter between a single electron and ion

A. Encounters at fixed velocity and impact parameter

Consider an electron that approaches an ion at velocity v with impact parameter b; the impact parameter is defined as the distance of closest approach the electron would make with the ion if it were to travel in a straight line, not being deflected at all by the Coulomb force. However, since we are treating the Coulomb force as a small perturbation on the electron's path due to the large ratio of kinetic to potential energy, b is also in fact the closest approach in our approximation. That is, we assume that the electron's trajectory is nearly a straight line, since typically its kinetic energy is $\sim 10^6$ times is potential energy. We will also neglect the motion of the ion, since it is more massive than the electron by at least a factor of 1000.

Let $t = 0$ be the time when the electron is at its point of closest approach to the ion, so vt is the distance of the electron from the point of closest approach. Without loss of generality we will place the electron and ion in the xy plane, and let the electron move parallel to the x axis. The position and acceleration of the electron as a function of time are therefore

$$\mathbf{x} = vt\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}} \tag{2}$$

$$\ddot{\mathbf{x}} = -\frac{Z_i e^2}{m_e(b^2+v^2t^2)}\left(\frac{vt}{\sqrt{b^2+v^2t^2}}\hat{\mathbf{x}} + \frac{b}{\sqrt{b^2+v^2t^2}}\hat{\mathbf{y}}\right) \tag{3}$$

We are interested in the spectrum of radiation emitted as a result of this encounter. Per our previous discussion of the spectrum of Larmor radiation, this depends on the Fourier transform of $\ddot{\mathbf{d}}$, where $\mathbf{d} = e\mathbf{x}$ is the dipole moment of the system (assuming the ion remains at the origin at all times). This is

$$\tilde{\ddot{\mathbf{d}}}(\omega) = -\omega^2\tilde{\mathbf{d}}(\omega) = \frac{e}{2\pi}\int_{-\infty}^{\infty}\ddot{\mathbf{x}}(t)e^{i\omega t}\,dt, \tag{4}$$

where $\tilde{}$ indicates the Fourier transform of a quantity. Substituting the acceleration $\ddot{\mathbf{x}}$ into the integral, we see that the $\hat{\mathbf{x}}$ component vanishes by symmetry, and the $\hat{\mathbf{y}}$ component is

$$\tilde{\mathbf{d}}(\omega) = \hat{\mathbf{y}}\frac{Z_i e^3}{2\pi\omega^2 m_e b^2}\int_{-\infty}^{\infty}\frac{1}{[1+(vt/b)^2]^{3/2}}e^{i\omega t}\,dt \tag{5}$$

The integral is one of those ones that you look up in an integral table or get mathematica to do, and it turns out to evaluate to

$$\tilde{\mathbf{d}}(\omega) = \hat{\mathbf{y}}\frac{Z_i e^3}{\pi\omega m_e v^2}K_1(\omega b/v), \tag{6}$$

where K_1 is a modified Bessel function. The Bessel function is a bit of a pain to deal with, so we will just think about its limiting behaviour. For $\omega \ll v/b$, meaning that the argument of K_1 is small, we can replace the $K_1(\omega b/v)$ with its first order Taylor expansion, which is just $v/b\omega$. In the opposite limit, $\omega \gg v/b$, the Bessel function goes to zero very quickly. We therefore have approximately

$$\tilde{\mathbf{d}}(\omega) = \hat{\mathbf{y}} \begin{cases} Z_i e^3/\pi\omega^2 m_e b v, & \omega \ll v/b \\ 0, & \omega \gg v/b \end{cases} \tag{7}$$

Having obtained the Fourier transform of the dipole moment, and shown that it is always in the same direction, we can now simply substitute into the result we derived previously for the frequency-dependence of the radiated power. The total power radiated per unit angular frequency is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\tilde{d}(\omega)|^2 = \frac{8Z_i^2 e^6}{3\pi c^3 m_e^2 v^2 b^2}. \tag{8}$$

for $\omega \ll v/b$, and close to zero for $\omega \gg v/b$. Note that the spectrum is flat, i.e., $dW/d\omega$ is close to independent of ω until $\omega \approx v/b$. The quantity v/b is called the cutoff frequency. Intuitively, this makes sense. The frequency is the inverse of the timescale on which the electric dipole moment is changing. The change is occurring fastest when the electron is closest to the ion, and at this point the timescale on which it is changing is of order the electron-ion distance divided by the velocity, i.e., b/v. The highest frequency of radiation emitted is just the inverse of this.

In principle we could also go on to determine the angular distribution of the emission, since we have the machinery to do so. However, since the astrophysical approximation in which we are finally interested is one where there are many particles moving in random directions, the average emission is going to wind up being isotropic. For this reason, a calculation of the total power radiated in all directions will suffice.

B. Integration over impact parameter

Now let us consider a somewhat more general situation. Instead of firing a single electron at the ion target, we fire a beam of electrons with density n_e at velocity v. How much bremmstrahlung per unit time does this electron beam produce?

The electron flux in this case is $n_e v$, and we can envision a bullseye painted around the target ion; each concentric ring around the ion at a distance b from it has an area $2\pi b\, db$. If the electrons in the beam are arranged randomly, then the rate at which electrons pass through each concentric ring around the target ion is the flux times the area, $2\pi b n_e v\, db$. Each encounter produces emission at a rate given by Equation 8. To figure out the total rate of emission, we just have to integrate this over impact parameter:

$$\frac{dW}{d\omega\, dt} = \int 2\pi b n_e v \frac{dW}{d\omega}\, db = n_e \frac{16 Z_i^2 e^6}{3c^3 m_e^2 v} \int \frac{db}{b}. \tag{9}$$

To make further progress we will need to make some approximations. Clearly the integral over b, as written, will diverge if we attempt to integrate from $b = 0$ to ∞. We must truncate the integral at both low and high b. The truncation at high b is obvious, since we have already seen that the emission goes to zero for $\omega \gg v/b$. This suggests that we take $b_{\max} = v/\omega$.

How about on the other end? A reasonable approach is to note that radiation is only expected if the system has at least one quantum of angular momentum, or at least that quantum mechanical effects will clearly become important once the angular momentum of the electron approaches this value. The condition for the system to have only a single quantum of angular momentum is $m_e v b = \hbar$, which suggests the choice $b_{\min} = h/m_e v$.

With these approximations, we have

$$\frac{d^2W}{d\omega\,dt} = n_e \frac{16 Z_i^2 e^6}{3c^3 m_e^2 v} \ln \frac{b_{\max}}{b_{\min}}. \tag{10}$$

The combination $\ln(b_{\max}/b_{\min})$ must be derived from a quantum mechanical calculation. Fortunately, the results are not hugely sensitive to it, since they just appear in the logarithm. We usually parameterise the results of the quantum mechanical combination in terms of a quantity known as the Gaunt factor (named after the physicist John Arthur Gaunt who first calculated it), defined relative to what we have written down as

$$g_{\mathrm{ff}}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln \frac{b_{\max}}{b_{\min}}, \tag{11}$$

so that the total emission rate is

$$\frac{d^2W}{d\omega\,dt} = n_e \frac{16\pi Z_i^2 e^6}{3\sqrt{3} c^3 m_e^2 v} g_{\mathrm{ff}}(v,\omega). \tag{12}$$

Tabulated Gaunt factors are easy to look up in the literature, e.g., van Hoof et al. (2014, MNRAS, 444, 420).

III. Radiation by a thermal distribution of particles

We have now computed the spectrum of emission from a single electron encountering a single ion at a particular impact parameter and velocity, and from a beam of electrons of fixed velocity striking an ion. We now proceed to the next level of generality, which is the case where we have not a single ion but a population of them, and not a single electron velocity, but a thermal distribution of electron velocities. This situation is called thermal bremmstrahlung, and it is one of the most ubiquitous astrophysical emission processes.

A. Free-free emissivity

Consider a plasma with electron density n_e and ion density n_i. The electrons are at temperature T.[1] Due to the ions' much greater mass, we will treat them as stationary in comparison to the electrons.

In a thermal plasma, the distribution of electron velocities just follows the usual Boltzmann distribution

$$\frac{dp}{dv} = 4\pi \left(\frac{m_e}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{m_e v^2}{2 k_B T}\right). \tag{13}$$

Note that the vector of $4\pi v^2$ is a phase space factor: the probability of an electron being at a particular vector velocity $\mathbf{v}$ is proportional to $\exp(-m_e|\mathbf{v}|^2/2k_BT)$, so the probability of being at a scalar velocity from v to $v + dv$ is the probability density

[1] In an astrophysical plasma the electrons and ions are frequently at different temperatures, since energy exchange between the two populations is slow due to their difference in mass. However, the two populations internally relax to a Maxwellian velocity distribution much faster than they exchange energy with each other. Thus it is a common situation to have two distinct temperatures, one for the electrons and one for the ions. In our application only the electron temperature matters.

multiplied by the volume of points in velocity space for which the scalar velocity is in the desired range; this is $4\pi v^2\, dv$.

To get the emission rate per unit volume, we just have to multiply by the density of ion targets per unit volume, and integrate over the Maxwellian velocity distribution. Thus

$$\begin{aligned}
\frac{d^3W}{d\omega\, dt\, dV} &= n_i \int \frac{dp}{dv} \frac{d^2W}{d\omega\, dt}\, dv && (14)\\
&= \frac{64\pi^2 Z_i^2 e^6}{3\sqrt{3}c^3 m_e^2} \left(\frac{m_e}{2\pi k_B T}\right)^{3/2} n_e n_i \int v \exp\left(-\frac{m_e v^2}{2k_B T}\right) g_{\mathrm{ff}}(v,\omega)\, dv && (15)\\
&= \frac{32\pi Z_i^2 e^6}{3 m_e c^3} \sqrt{\frac{2\pi}{3 m_e k_B T}} n_e n_i \int x e^{-x^2} g_{\mathrm{ff}}(x,\omega)\, dx, && (16)
\end{aligned}$$

where in the last step we have set $x = v/\sqrt{2k_B T/m_e}$.

We must now consider limits once again. One might be tempted to integrate this from $x = 0$ to ∞. The upper limit is fine. However, the lower limit ignores an important quantum mechanical effect, which we can again approximately incorporate. This effect is that, if v is small enough, then the electron will not have enough energy to produce a photon of some particular angular frequency of interest. Thus we expect no emission for impact velocities v that are too small.

Crudely accounting for this, we set

$$x_{\mathrm{min}} = \frac{\hbar\omega}{k_B T}, \tag{17}$$

which amounts to setting x_{min} to the value such that $h\nu = m_e v_{\mathrm{min}}^2/2$. Integrating from x_{min} to ∞, and making a change of variables from ω to $\nu = 2\pi\omega$, we arrive at our final result

$$\frac{d^3W}{dV\, dt\, d\nu} = \frac{32\pi Z_i^2 e^6}{3 m_e c^3} \left(\frac{2\pi}{3 m_e k_B T}\right)^{1/2} n_e n_i \bar{g}_{\mathrm{ff}}(T,\nu) e^{-h\nu/k_B T} \tag{18}$$

where $\bar{g}_{\mathrm{ff}}(T,\nu)$ is the average of $g_{\mathrm{ff}}(v,\nu)$ weighted over the Boltzmann velocity distribution at temperature T; this number is usually of order unity in most astrophysical situations. Evaluating numerically,

$$\frac{d^3W}{dV\, dt\, d\nu} = 6.8 \times 10^{-38} \bar{g}_{\mathrm{ff}} Z_i^2 \left(\frac{T}{\mathrm{K}}\right)^{-1/2} \left(\frac{n_e n_i}{\mathrm{cm}^{-6}}\right) e^{-h\nu/k_B T} \text{ erg cm}^{-3}\text{ s}^{-1}\text{ Hz}^{-1}, \tag{19}$$

where from this point on we omit the dependence of $\bar{g}_{\mathrm{ff}}$ and T and ν for compactness.

Since this emission is isotropic, the corresponding emission coefficient is just this divided by 4π sr:

$$j_{\nu,\mathrm{ff}} = \frac{8 Z_i^2 e^6}{3 m_e c^3} \left(\frac{2\pi}{3 m_e k_B T}\right)^{1/2} n_e n_i \bar{g}_{\mathrm{ff}} e^{-h\nu/k_B T}. \tag{20}$$

B. Free-free absorption and optical depth effects

Our calculation of the free-free absorption coefficient allows us to immediately calculate the rate for the inverse process, free-free absorption, using Kirchoff's Law of thermal emission. For a medium in thermal equilibrium, as we have assumed, the emission and absorption coefficients are related by

$$\alpha_{\nu,\mathrm{ff}} = \frac{j_{\nu,\mathrm{ff}}}{B_\nu(T)} = \frac{4Z_i^2 e^6}{3m_e hc}\left(\frac{2\pi}{3m_e k_B T}\right)^{1/2} n_e n_i \bar{g}_{\mathrm{ff}} \frac{1-e^{-h\nu/k_B T}}{\nu^3}. \tag{21}$$

Evaluating numerically,

$$\alpha_{\nu,\mathrm{ff}} = 3.7\times 10^8 \bar{g}_{\mathrm{ff}} Z_i^2 \left(\frac{T}{K}\right)^{-1/2}\left(\frac{n_e n_i}{\mathrm{cm}^{-6}}\right)\left(\frac{\nu}{\mathrm{Hz}}\right)^{-3}\left(1-e^{-h\nu/k_B T}\right)\ \mathrm{cm}^{-1}. \tag{22}$$

Physically, we can envision the absorption process as a situation where an electromagnetic wave passing over an electron-ion pair produces an electric field that accelerates the electron as it approaches the ion, and then oscillates to reverse direction so that it also accelerates the electron once it has moved past the ion. As a result the electron gains energy from the wave. The as the electron accelerates under this influence, it generates an electric field that is opposite in direction to the incoming field, so that the intensity of the wave is diminished after it goes by.

Let us consider how absorption might affect the spectrum we see from a source producing free-free emission. Since there will only be substantial emission as frequencies such that $h\nu/k_B T \lesssim 1$, let us specialise to to the case $h\nu \ll k_B T$, so we can Taylor expand the exponential term as $e^{-x} \approx 1 - x$. This gives

$$\alpha_{\nu,\mathrm{ff}} \approx \frac{4Z_i^2 e^6}{3m_e c k_B T}\left(\frac{2\pi}{3m_e k_B T}\right)^{1/2} n_e n_i \bar{g}_{\mathrm{ff}} \nu^{-2} \tag{23}$$

$$\approx 0.018 \bar{g}_{\mathrm{ff}} Z_i^2 \left(\frac{T}{K}\right)^{-3/2}\left(\frac{n_e n_i}{\mathrm{cm}^{-6}}\right)\left(\frac{\nu}{\mathrm{Hz}}\right)^{-2}\ \mathrm{cm}^{-1}. \tag{24}$$

The absorption coefficient has a ν^{-2} dependence, and so it is higher at lower frequencies. This means that, at sufficiently low frequency, any astrophysical plasma that is emitting bremsstrahlung will become optically thick to its own radiation. At these frequencies, the spectrum must cease being flat, and must instead approach the Planck spectrum, as all emitting and absorbing media in thermal equilibrium do when they become sufficiently opaque. At frequencies $h\nu/k_B T \ll 1$, the Planck function has $B_\nu \propto \nu^2$, and thus the free-free spectrum should go from being flat, $F_\nu \propto \nu^0$, so rising, $F_\nu \propto \nu^2$.

How low in frequency do we have to go before this effect becomes significant? The condition for a medium to be opaque is $\alpha_\nu L \approx 1$, where L is the length of the line of sight passing through the medium. The frequency at which the medium becomes opaque is therefore

$$\nu \approx \frac{2Z_i e^3}{\sqrt{3m_e c k_B T}}\left(\frac{2\pi}{3m_e k_B T}\right)^{1/4}\sqrt{n_e n_i L \bar{g}_{\mathrm{ff}}} \tag{25}$$

$$\approx 230\, Z_i \left(\frac{T}{\mathrm{K}}\right)^{-3/4}\left(\frac{n_e n_i}{\mathrm{cm}^{-6}}\right)^{1/2}\left(\frac{L}{\mathrm{pc}}\right)^{1/2} \bar{g}_{\mathrm{ff}}^{1/2}\ \mathrm{MHz}. \tag{26}$$

This suggests that, for plasma at interstellar or intergalactic densities, $n_e \lesssim 1\ \mathrm{cm}^{-3}$, free-free emission will start to become optically thick, and thus the spectrum will approach a blackbody spectrum, at radio wavelengths. However, notice that free-free

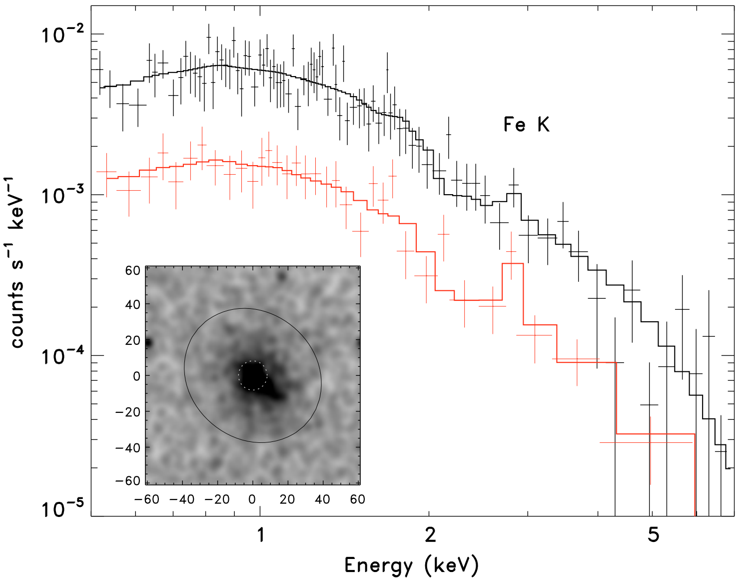

Figure 1: X-ray image (inset) and spectra (red and black points with error bars, averages over two different radii) of the galaxy cluster XMMU J2235.3-2557, observed with *Chandra* (Rosati et al., 2009, A&A, 508, 583). The lines are model fits to the observed spectra. The slight bump near 3 keV is a emission line of highly-ionised iron.

absorption is also the main opacity source for the Sun, which has a mean electron density closer to $n_e \sim 10^{24}$ cm^{-3}, and higher in the core. This places the frequency at which the optical depth approaches unity far, far into the high energy part of the electromagnetic spectrum. This is why, despite the fact that the Sun is mostly emitting and absorbing radiation via bremmstrahlung, it has a spectrum that is close to black body.

IV. Sample application: galaxy clusters

Due to the presence of the cutoff frequency, detection of bremmstrahlung emission provides an extremely useful tool to characterise the temperatures of astrophysical plasmas – a cutoff in the spectrum can be directly translated into a temperature. One important application of this is to measuring the temperatures, and as we shall see therefore the masses, of galaxy clusters.

As motivation, we can examine an observed X-ray spectrum (Figure 1). We see the characteristics we expect for a spectrum dominated by thermal bremsstrahlung: a relatively flat spectrum out to some energy, followed by a steep decline at higher energies. Such a spectrum allows us to almost just read off the value of k_BT, since the exponential decline is just due to the $e^{-h\nu/k_BT}$ term. In this case, one a simple by-eye it would suggest $k_BT \approx 2-3$ eV. The galaxy cluster is at redshift $z = 1.39$, so we have to multiply this energy by $1+z = 2.39$, giving an energy of $\approx 5-7$ keV. A detailed fit, shown in the plot, gives 8.6 keV. The associated temperature is $T = 1.0 \times 10^8$ K. Thus the X-ray spectrum allows direct determination of the gas temperature.

One can translate this temperature directly into the mass of the galaxy cluster, because the gas is close to hydrostatic equilibrium. In hydrostatic equilibrium, we have

$$\frac{GM(<r)}{r^2} = \frac{dP}{dr} = \frac{k_B}{\mu m_{\rm H}}\left(\rho\frac{dT}{dr} + T\frac{d\rho}{dr}\right), \tag{27}$$

where $M(< r)$ is the mass interior to radius r, μ is the mean mass in AMU per particle (≈ 0.61 for fully ionised gas that is 75% H, 25% He by mass), and ρ is the gas density. The observations suggest that the gas is close to isothermal, since a single-temperature spectrum fits the data quite well, so $dT/dr \approx 0$. Thus by measuring the temperature T, and the way the density drops off with radius, $d\rho/dr$, we get a direct measurement of the enclosed mass. Measuring cluster masses in turn has important applications for cosmology.