In principle the Liénard-Wiechart potentials derived in the previous class contain all the information needed to calculate electromagnetic radiation from an arbitrary charge distribution. After all, since Maxwell's equations are linear, we can again use the Green's function approach: since the Liénard-Wiechart potentials tell us the exact scalar and vector potentials generated by a point charge with an arbitrary history, we can write out the potentials for an arbitrary time-dependent charge distribution as simply the sum of the potentials generated by an ensemble of point charges. In practice, however, this is can be extremely cumbersome, and it is often preferable to work with the potentials in various limits of interest. We will spend this class developing the simplest and lowest-order approximation and its applications.

I. The far field limit

To begin our discussion, let us remind ourselves of the general retarded potentials:

$$\begin{bmatrix} \phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{bmatrix} = \int \begin{bmatrix} \rho_e(\mathbf{x}',t') \\ \mathbf{j}_e(\mathbf{x}',t')/c \end{bmatrix} \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} \tag{1}$$

where

$$t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c} \tag{2}$$

is the retarded time.

In astrophysical applications, we are usually interested in situations where the distance to the emitting region is immense compared to both the wavelength of the emission and the physical size of the emitting region. This is even more true if we remember that the "emitting region" that matters here is not a macroscopic emitting object, since the fields produce by (for example) two points on opposite sides of a star are incoherent and just add; the "emitting region" that matters here is the size of a region over which the charges are moving coherently. In most cases this is of atomic dimensions. To make this formal, let us place the emitting region near the origin, and the observer at position $\mathbf{x}$, and we will assume

$$|\mathbf{x}| \gg \lambda \qquad \text{and} \qquad |\mathbf{x}| \gg |\mathbf{x}'| \text{ for any } \mathbf{x}' \text{ where } \rho_e \neq 0 \text{ or } \mathbf{j}_e \neq 0. \tag{3}$$

Regions that satisfy these assumptions are referred to in electromagnetism as the far field (as distinguished from the near field, regions whose distance from the emitting source is comparable to the source size), or by the equivalent terms the wave zone or the radiation zone. We will see why these terms are appropriate momentarily.

We can use our assumption to Taylor expand the separation term:

$$|\mathbf{x}-\mathbf{x}'| = \sqrt{x^2 - 2\mathbf{x}\cdot\mathbf{x}' + x'^2} = x\sqrt{1 - 2\hat{\mathbf{n}}\cdot\frac{\mathbf{x}'}{x} + \left(\frac{x'}{x}\right)^2} = x\left[1 - \hat{\mathbf{n}}\cdot\frac{\mathbf{x}'}{x} + O\left(\frac{x'^2}{x^2}\right)\right], \tag{4}$$

where $x = |\mathbf{x}|$ and $\hat{\mathbf{n}} = \mathbf{x}/x$ is a unit vector pointing from the emitting region to the point of interest. The corresponding retarded time is

$$t' = t - \frac{x}{c}\left[1 - \hat{\mathbf{n}}\cdot\frac{\mathbf{x}'}{x} + O\left(\frac{x'^2}{x^2}\right)\right]. \tag{5}$$

Keeping only terms of leading order[1], we have

$$\left[\begin{array}{c}\phi(\mathbf{x},t)\\ \mathbf{A}(\mathbf{x},t)\end{array}\right] = \frac{1}{x}\int \left[\begin{array}{c}\rho_e(\mathbf{x}',t')\\ \mathbf{j}_e(\mathbf{x}',t')/c\end{array}\right] d^3x'. \tag{6}$$

This is a considerable simplification, since we have been able to pull the separation-dependent term out of the integral, and we are now left with just an integral over the position distribution of the emitting charges themselves. Moreover, Equation 6 implies that the potentials decrease as $1/x$, i.e., as the inverse of the distance to the observer.

We can simplify even more when we attempt to calculate the electric and magnetic fields. These fields depend on various derivatives of ϕ and $\mathbf{A}$, which simplify even more than ϕ and $\mathbf{A}$ themselves. Let us start by computing the gradient of the scalar potential:

$$\nabla\phi = -\frac{\hat{\mathbf{n}}}{x^2}\int \rho_e(\mathbf{x}',t')\, d^3x' + \frac{1}{x}\int (\nabla t')\,\frac{\partial\rho_e}{\partial t'}\, d^3x' \tag{7}$$

Note that, in evaluating this integral, we have used the fact that $\nabla x = \hat{\mathbf{n}}$. To handle the time term, we can similarly note that

$$\nabla t' = \nabla\left[t - \frac{x}{c}\left(1 - \hat{\mathbf{n}}\cdot\frac{\mathbf{x}'}{x}\right)\right] = -\frac{\hat{\mathbf{n}}}{c} \tag{8}$$

Using Equation 1 to replace the integrals, and noting that differentiation with respect to t and t' are the same to leading order, we therefore have

$$\nabla\phi = -\hat{\mathbf{n}}\left(\frac{1}{x}\phi + \frac{1}{c}\frac{\partial\phi}{\partial t}\right). \tag{9}$$

The natural timescale on which ϕ oscillates is the inverse of the radiation frequency, so

$$\frac{1}{c}\frac{\partial\phi}{\partial t} \sim \frac{\nu}{c}\phi \sim \frac{\phi}{\lambda}. \tag{10}$$

Thus under our assumption that $x \gg \lambda$, we can drop the first term in Equation 9 compared to the second one, and we obtain in the end

$$\nabla\phi = -\frac{\hat{\mathbf{n}}}{c}\frac{\partial\phi}{\partial t}. \tag{11}$$

Using the exact same reasoning, it is straightforward to show that the derivatives of the vector potential are

$$\nabla\cdot\mathbf{A} = -\frac{\hat{\mathbf{n}}}{c}\cdot\frac{\partial\mathbf{A}}{\partial t} \qquad \nabla\times\mathbf{A} = -\frac{\hat{\mathbf{n}}}{c}\times\frac{\partial\mathbf{A}}{\partial t}. \tag{12}$$

Note that, since we are in the Lorentz gauge, we also have $(1/c)(\partial\phi/\partial t) = -\nabla\mathbf{A}$, and thus we also have

$$\nabla\phi = \hat{\mathbf{n}}(\nabla\cdot\mathbf{A}) = -\hat{\mathbf{n}}\left(\frac{\hat{\mathbf{n}}}{c}\cdot\frac{\partial\mathbf{A}}{\partial t}\right). \tag{13}$$

We are now in a position to directly write down the electric and magnetic fields in terms of derivatives of the potentials. The magnetic field is

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{c}\left(\frac{\partial\mathbf{A}}{\partial t}\times\hat{\mathbf{n}}\right) \tag{14}$$

[1] We have to be a little careful with this. While we can drop all non-leading terms in most places, we need to retain the $\hat{\mathbf{n}}\cdot\mathbf{x}'/x$ term in t', for reasons that we will explore in detail below.

Note that this implies that the magnetic field direction is perpendicular to $\hat{\mathbf{n}}$, the vector pointing from the emitting region to the observer. It also implies that the magnetic field decreases as $1/x$.

The analogous calculation for the electric field is

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \hat{\mathbf{n}}\left(\frac{\hat{\mathbf{n}}}{c}\cdot\frac{\partial \mathbf{A}}{\partial t}\right) - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\times\hat{\mathbf{n}}\right)\times\hat{\mathbf{n}}, \tag{15}$$

where in the last step we made use of the triple cross-product identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. Thus we have

$$\mathbf{E} = \mathbf{B}\times\hat{\mathbf{n}}. \tag{16}$$

This should look very familiar: we have a magnetic field that is orthogonal to the direction in which the radiation is travelling and to the electric field. This is exactly what a plane wave looks like. Thus we have shown in very general terms that, no matter what the charges generating the field are doing (which will determine $\partial\mathbf{A}/\partial t$), the electromagnetic fields in the far field just look like plane waves. This is why the far field is also called the wave zone or the radiation zone: it is the region where solutions to Maxwell's equations approach the form of radiation in free space.

We can obtain one further important result for the far field by calculating the Poynting vector, which describes the energy carried by the field. This is

$$\mathbf{S} = \frac{c}{4\pi}\left(\mathbf{E}\times\mathbf{B}\right) = \frac{c}{4\pi}\left(\mathbf{B}\times\hat{\mathbf{n}}\right)\times\mathbf{B} = \frac{B^2}{4\pi}c\hat{\mathbf{n}}. \tag{17}$$

Recalling that we have shown that $B \propto 1/x$, this implies that the energy flux carried by the field decreases as $1/x^2$. Thus we have just demonstrated the inverse square law for electromagnetic radiation: in the far field, the energy carried falls off as $1/x^2$.

We can compute the power dP carried by the field that passes through an infinitesimal solid angle $d\Omega$ centred on direction $\hat{\mathbf{n}}$ just by noting that the area of such a solid angle is $x^2\,d\Omega$. Thus

$$dP = |\mathbf{S}|x^2\,d\Omega = \frac{(Bx)^2}{4\pi}c\,d\Omega. \tag{18}$$

Since $B \propto 1/x$, the power carried by the field per unit solid angle, as opposed to per unit area, is constant.

II. Applications of the far field limit

Having derived the far field limit, we now proceed to derive a number of important results from it. We will make use of these results through our astrophysical applications.

A. Radiation non-relativistic accelerating charges

We begin with the simplest possible case, one that we have already encountered: a single point charge in non-relativistic motion. Consider a charge q with position $\mathbf{r}(t)$ and velocity $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$ as a function of time. Using the far field limit, the vector potential is

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{x}\int \frac{\mathbf{j}_e(\mathbf{x}', t')}{c}\,d^3x', \tag{19}$$

where the current density is $\mathbf{j}_e(\mathbf{x}, t) = q\mathbf{v}(t)\delta\left[\mathbf{x} - \mathbf{r}(t)\right]$. The retarded time t' at which this is to be evaluated is

$$t' = t - \frac{x}{c}\left[1 - \hat{\mathbf{n}}\cdot\frac{\mathbf{x}'}{x}\right] = t - \frac{x}{c} + \frac{\hat{\mathbf{n}}\cdot\mathbf{x}'}{c} \tag{20}$$

1. The dipole approximation

The second term in t', $\hat{\mathbf{n}}\cdot\mathbf{x}'/c$, has a specific physical meaning: it is the difference in retarded time across the emitting source. In other words, this term accounts for the fact that radiation coming from the parts of the emitting region that are slightly farther from the observer takes longer to get to the observer, and thus the emission properties have to be evaluated at a slightly more retarded time than for the parts of the emitting region that are slightly closer to the observer.

The correction is slight because, by assumption, $|\mathbf{x}'| \ll |\mathbf{x}|$. However, we cannot necessarily neglect this small correction, because it may have significant effects on the phase of the emitted radiation. For example, suppose that the light travel time across the emitting region is 10^{-15} s, as compared to 1000 years to get from the emitting region to Earth. One might think that this would be completely negligible. However, if we are talking about visible light, then 10^{-15} s is enough for the electric field from emission on the far side to have gone through half an oscillation cycle. This can change whether it is in phase or out of phase with the emission coming from the near side, making the difference between the light from the two sides adding coherently or cancelling out. The need to capture these effects is the reason we kept this term when we wrote down Equation 5.

This argument, however, suggests circumstances under which we can drop the second term. If the emitting region is of size L, then the light travel time across it is L/c. In comparison, the time required for the electric field of the radiation to change significantly is $\sim 1/\nu$, where ν is the radiation frequency. Thus we are justified in dropping this second term if

$$\frac{L}{c} \ll \frac{1}{\nu} \quad \Longrightarrow \quad \lambda \gg L, \tag{21}$$

where λ is the wavelength of the radiation. If this condition is satisfied, i.e., the emitting region is small compared to the wavelength of light under consideration, then we can drop the second term in the retarded time and just set

$$t' = t - \frac{x}{c}. \tag{22}$$

This is called the dipole approximation.

An important reminder here (without which it would seem that the dipole condition is never satisfied in real life) is that L is not the size of the macroscopic emitting object, it is the size of the region within which the electric field is behaving coherently, all in phase. This may be a macroscopic size if we're talking about a radio antenna, for example, but if we're talking about visible light or higher energy radiation, it is more likely to be a region of atomic dimensions.

One situation in which the dipole approximation turns out to apply is for emission by a non-relativistic accelerating particle. To see why, note that the timescale on which the electric field created by the particle oscillates must be of order the timescale over which the particle's position changes significantly, which we will denote τ, and the corresponding radiation frequency must be of order $\nu \sim 1/\tau$. However, in this case the condition that $\lambda \ll L$ amounts to a requirement that

$$L \gg \frac{c}{\nu} \sim c\tau \quad \Longrightarrow \quad \frac{L}{\tau} \sim v \ll c, \tag{23}$$

where $v \sim L/\tau$ is the particle's velocity. Thus a non-relativistic particle, $v \ll c$, automatically satisfies the dipole condition.

2. The Larmor formula

   Armed with the dipole approximation, we are now ready to derive the properties of radiation emitted by an accelerating charge. The vector potential now reduces to

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{x} \int \frac{\mathbf{j}_e(\mathbf{x}', t - x/c)}{c} \, d^3x' = \frac{q}{cx} \mathbf{v}(t - x/c). \tag{24}$$

   The resulting magnetic and electric fields are

$$\mathbf{B} = \frac{1}{c} \left(\frac{\partial \mathbf{A}}{\partial t} \times \hat{\mathbf{n}} \right) = \frac{q}{c^2 x} \dot{\mathbf{v}} \times \hat{\mathbf{n}} \tag{25}$$

$$\mathbf{E} = \mathbf{B} \times \hat{\mathbf{n}} = \frac{q}{c^2 x} \left(\dot{\mathbf{v}} \times \hat{\mathbf{n}} \right) \times \hat{\mathbf{n}}, \tag{26}$$

   where we have omitted the argument of $\dot{\mathbf{v}}$ for brevity, but recall that it is to be evaluated at the retarded time $t - x/c$.

   The Poynting vector for this field is

$$\mathbf{S} = \frac{B^2}{4\pi} \hat{\mathbf{n}} = \frac{q^2}{4\pi x^2 c^3} \left| \dot{\mathbf{v}} \times \hat{\mathbf{n}} \right|^2 \hat{\mathbf{n}} = \frac{q^2 \dot{v}^2}{4\pi x^2 c^3} \sin^2\theta \, \hat{\mathbf{n}}, \tag{27}$$

   where θ is the angle between $\hat{\mathbf{n}}$ and $\dot{\mathbf{v}}$ and $\dot{v} = |\dot{\mathbf{v}}|$. The corresponding power radiated per solid angle is

$$\frac{dP}{d\Omega} = \frac{(Bx)^2}{4\pi} c = \frac{q^2 \dot{v}^2}{4\pi c^3} \sin^2\theta \tag{28}$$

   Integrating over all angles gives the total power radiated by the charge,

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{2q^2 \dot{v}^2}{3c^3}, \tag{29}$$

   a famous result known as Larmor's formula.

   Note that, if we have a collection of non-relativistic accelerating charges that all obey $x' \ll x$, then we can sum the currents they provide; as long as they satisfy the dipole condition, we need not worry about the fact that the different charges may have different retarded times due to their different positions, because we are approximating all the charges' retarded times as $t - x/c$. In this case all that happens is that the vector potential becomes

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{cx} \sum_i q_i \mathbf{v}_i(t - x/c) = \frac{\dot{\mathbf{d}}}{cx}, \tag{30}$$

   where the subscript i runs over all the particles contributing to the current, and we have defined

$$\mathbf{d} = \sum_i q_i \mathbf{r}_i \tag{31}$$

as the dipole moment of the charge distribution. Carrying out the same steps then gives

$$\mathbf{B} = \frac{1}{c^2 x}\ddot{\mathbf{d}} \times \hat{\mathbf{n}} \tag{32}$$

$$\mathbf{E} = \frac{1}{c^2 x}\left(\ddot{\mathbf{d}} \times \hat{\mathbf{n}}\right) \times \hat{\mathbf{n}} \tag{33}$$

$$\frac{dP}{d\Omega} = \frac{|\ddot{\mathbf{d}}|^2}{4\pi c^3}\sin^2\theta \tag{34}$$

$$P = \frac{2|\ddot{\mathbf{d}}|^2}{3c^3}. \tag{35}$$

This gives the generalisation of the Larmor formula to an arbitrary dipolar charge distribution; however, recall that is valid only for wavelengths $\lambda \gg L$.

3. Spectrum of emitted radiation

We have calculated the total power emitted by the accelerating charge(s), and we have calculated its angular distribution, but at times we may also be interested in knowing the spectrum of the emitted radiation. Indeed, this will be a very important concern when we come to astrophysical applications, since we will want to be able to predict and interpret the spectra of astrophysical sources. We will limit ourselves for now to the simplified problem where **d** does not change direction over time, just amplitude.

In this case the electric field amplitude as a function of time is given by

$$E(t) = \ddot{d}(t)\frac{\sin\theta}{c^2 x}. \tag{36}$$

To figure out the corresponding frequency, let us take the Fourier transform of the dipole moment and the electric field:

$$d(t) = \int e^{-i\omega t}\tilde{d}(\omega)\, d\omega \qquad E(t) = \int e^{-i\omega t}\tilde{E}(\omega)\, d\omega \tag{37}$$

Then we have

$$\ddot{d}(t) = -\int \omega^2 \tilde{d}(\omega) e^{-i\omega t}\, d\omega, \tag{38}$$

and the Fourier transform of the electric field is therefore

$$\tilde{E}(\omega) = -\frac{1}{c^2 x}\omega^2 \tilde{d}(\omega)\sin\theta. \tag{39}$$

Using our earlier result that the energy per unit area per unit time in frequency space is just $c|\tilde{E}(\omega)^2|$, the differential power radiated per solid angle per angular frequency is therefore

$$\frac{d^2 W}{d\Omega\, d\omega} = \frac{1}{c^3}\omega^4 |\tilde{d}(\omega)|^2 \sin^2\theta \tag{40}$$

and the total power per unit angular frequency is

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2. \tag{41}$$

Thus the frequency spectrum of the emitted radiation is the same as the frequency of the dipole oscillation, but weighted so that higher frequency oscillations emit more power by a factor of ω^4.

B. Thomson scattering

A second very important application of radiation the far-field limit is Thomson scattering: the scattering of radiation by a free charge, usually a free electron.

1. Scattering of a linearly polarised wave

Consider a free point mass m of charge q upon which a linearly-polarised plane wave of angular frequency ω is incident. Without loss of generality we will choose our coordinate system so that the plane wave is propagating in the $\hat{\mathbf{z}}$ direction, and the electric field is in the $\hat{\mathbf{x}}$ plane. As long as the free charge moves non-relativistically in response, the magnetic force is small compared to the electric one (as you will show on your homework), so the force that the wave exerts on the electron is

$$\mathbf{F} = q\mathbf{E}(t) = qE\sin\omega t\,\hat{\mathbf{x}} \tag{42}$$

where $\mathbf{E}$ is the electric field, E is its magnitude, we have chosen to define $t = 0$ as a time when the electric field is zero, and by assumption the force and electric field both lie in the x direction.

The equation of motion for the point charge is then

$$\ddot{x} = \frac{qE}{m}\sin\omega t, \tag{43}$$

where x is the charge's position. Integrating, the charge's position as a function of time is just

$$x = -\frac{qE}{m\omega^2}\sin\omega t. \tag{44}$$

The dipole moment is therefore

$$d = qx = -\frac{q^2E}{m\omega^2}\sin\omega t, \tag{45}$$

and its second derivative is

$$\ddot{d} = \frac{q^2E}{m}\sin\omega t. \tag{46}$$

Again, as a reminder these are all x components.

Since the point charge is an oscillating dipole, it will radiate, and we can use our previous results for radiation from an accelerating charge to calculate the nature of the radiation. In particular, the time-averaged angular distribution of the radiated power (Equation 34) and the total power (Equation 35) are

$$\frac{dP}{d\Omega} = \frac{q^4E^2}{4\pi m^2c^3}\sin^2\theta\left\langle\sin^2\omega t\right\rangle = \frac{q^4E^4}{8\pi m^2c^3}\sin^2\Theta \tag{47}$$

$$P = \frac{2q^4E^2}{3m^2c^3}\left\langle\sin^2\omega t\right\rangle = \frac{q^4E^2}{3m^2c^3}. \tag{48}$$

Here $\langle\cdot\rangle$ indicates averaging over an oscillation period, and we have used the fact that the average of $\sin^2$ over an oscillation period is $1/2$. The angle Θ is the angle between the $\hat{\mathbf{x}}$ direction (the direction in which the incoming electric field and the charge are oscillating) and the direction of the direction of propagation of the outgoing radiation. Note that this is not the same as the angle between the incident wave and the outgoing radiation, which is $\theta = \pi/2 - \Theta$.

Also note that, since the charge is oscillating in the x direction and the resulting electric field is as well. Thus for radiation emitted at an angle $\hat{\mathbf{n}}$, the electric field must lie in the plane defined by $\hat{\mathbf{n}}$ and $\hat{\mathbf{x}}$. (If $\hat{\mathbf{n}}$ and $\hat{\mathbf{x}}$ are parallel then they do not define a plane, but in this case $\Theta = 0$ and no radiation is emitted in this direction.) Since the electric field lies in a plane the outgoing wave is linearly polarised as well.

This radiated power can be compared to the flux of the incoming plane wave, characterised by the Poynting vector, which has a time-averaged magnitude

$$\langle S \rangle = \frac{c}{8\pi} E^2. \tag{49}$$

The process of taking an incoming plane wave and redirecting its power in different directions is called scattering. Its defining characteristic is that the matter interacting with the incoming radiation does not absorb any energy (as much be the case here, since on average the charge has constant energy), so all that happens is that some of the arriving electromagnetic energy is sent off in a different direction. We can define the differential scattering cross section as the ratio of output power to input power as a function of direction, i.e.,

$$\frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\langle S \rangle} = \frac{q^4}{m^2 c^4} \sin^2 \Theta. \tag{50}$$

The total scattering cross section is just the integral of this over angle,

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{8\pi}{3} \frac{q^4}{m^2 c^4}. \tag{51}$$

Since the cross section scales as m^{-2}, electrons are much more effective scatters than protons or ions (by a factor of $\approx 10^6$, since $m_e/m_p \approx 10^{-3}$), and thus we generally only bother thinking about scattering by electrons, for which $q = e$ and $m = m_e$, and we define the quantity

$$r_0 \equiv \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13} \text{ cm} \tag{52}$$

as the classical electron radius. The corresponding total cross section is the Thomson cross section

$$\sigma \equiv \sigma_T = \frac{8\pi}{3} r_0^2 = 6.65 \times 10^{-25} \text{ cm}^2. \tag{53}$$

Three more remarks on Thomson scattering are important to make at this point. First, notice that ω dropped out of the transmitted power and scattering cross section. Thomson scattering is therefore a frequency-independent, or "grey", process.

Second, notice that, since the scattering charge produces a dipole that oscillates at exactly angular frequency ω, the Fourier transform of the dipole is a delta function at ω. Consequently, following our discussion above, the scattered wave is also monochromatic at frequency ω. The scattering process therefore does not change the frequency of the light.

Third, this calculation is only valid for photons whose energies are much less than the electron rest mass, 511 keV. At higher energies there are quantum mechanical and relativistic corrections to this cross section that are non-negligible. We will return to these when we discuss Compton scattering, which is the generalisation of Thomson scattering to arbitrary photon energy.

2. Scattering of unpolarised light

Our calculation of Thomson scattering thus far is for linearly polarised waves. We now generalise this to the more common case of unpolarised light. To figure out how scattering works in this case, let us again consider light propagating in the $\hat{\mathbf{z}}$ direction, and without loss of generality let us set up our coordinate system so that the observer who sees the scattered light is somewhere in the xz plane. Thus the unit vector $\hat{\mathbf{n}}$ that points from the scattering electron to the observer has zero y component.

We showed earlier that we can describe unpolarised light as simply the sum of two linearly polarised waves of equal electric field strength, with polarisation vectors 90° apart, and where the phases of the two waves are uncorrelated. Since we can orient these two uncorrelated, linearly-polarised waves however we want, we will take one of them to have its electric field in the $\hat{\mathbf{x}}$ direction, and other so it is in the $\hat{\mathbf{y}}$ direction.

Since Maxwell's equations are linear, we can consider scattering of the two incoming plane waves independently. The incoming plane wave that has its electric field in the $\hat{\mathbf{x}}$ direction will make an angle Θ between the electric field vector and the direction of scattering, exactly as in our calculation of scattering of a linearly-polarised wave. The incoming plane wave that has its electric field in the $\hat{\mathbf{y}}$ direction always makes an angle of $\pi/2$ between its electric field and the scattering direction, since the angle between $\hat{\mathbf{y}}$ and any vector $\hat{\mathbf{n}}$ that lies in the xz plane is $\pi/2$.

Since each of the incoming waves carries equal power, the total cross section is simply the average of the cross sections for each of the two waves as computed by Equation 50, i.e.,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left[\left(\frac{d\sigma}{d\Omega}\right)_x + \left(\frac{d\sigma}{d\Omega}\right)_y\right] = \frac{1}{2}r_0^2\left(\sin^2\Theta + 1\right) = \frac{1}{2}r_0^2\left(\cos^2\theta + 1\right), \tag{54}$$

where $(\cdot)_{x,y}$ refer to the incoming plane waves whose electric fields are in the x and y directions, respectively, and where $\theta = \pi/2 - \Theta$ is the direction between the direction of the original incoming plane wave and the outgoing scattered radiation. Note that the total cross section is still $\sigma_T = (8\pi/3)r_0^2$.

An important implication of this calculation is that, even if the incoming radiation is unpolarised, the scattered radiation will be at least partly polarised. This is because the two linearly-polarised components of the incoming radiation do not get scattered equally: the component whose electric field is in the $\hat{\mathbf{x}}$ direction has a smaller scattering cross section than the component whose electric field is in the $\hat{\mathbf{y}}$ direction by a factor of $\cos^2\theta$. Recall our definition of the polarisation fraction Π as the ratio of the energy in the polarised component to the total energy of the wave. In this case, if we have 1 unit of energy in the scattered $\hat{\mathbf{y}}$ component, and $\cos^2\theta$ units in the scattered $\hat{\mathbf{x}}$ component, then the total intensity is $1+\cos^2\theta$, and the intensity of the purely-polarised component is $1-\cos^2\theta$. Thus

$$\Pi = \frac{1-\cos^2\theta}{1+\cos^2\theta}. \tag{55}$$

Thus radiation scattered at a 90° angle is purely polarised, $\Pi = 1$, while radiation that is scattered at a shallower or steeper angle is less and less polarised as we move away from 90°.