Having completed our (very) whirlwind pass through the classical theory of light, we next begin to build up the machinery necessary to calculate radiation of light from astrophysical sources. The first step in this is to reformulate Maxwell's equations and the wave equation for light in terms of electromagnetic potentials.

I. Maxwell's equations in terms of potentials

A. The inhomogeneous wave equation

From the fact that $\nabla \cdot \mathbf{B} = 0$, it follows that we can write the magnetic field as the curl of some vector field $\mathbf{A}$, known as the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1}$$

The choice of $\mathbf{A}$ is arbitrary up to the addition of any vector field that is the gradient of a scalar, since the curl of a gradient is always zero. That is, we are allowed to replace $\mathbf{A}$ with

$$\mathbf{A}' = \mathbf{A} + \nabla\psi \tag{2}$$

for any arbitrary scalar field ψ. This addition is known as a gauge transformation.

If we replace the magnetic field with the vector potential in Maxwell's equation for the curl of the electric field, we have

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \mathbf{A}, \tag{3}$$

and since the time and spatial derivatives commute, it follows that

$$\nabla \times \left(\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{4}$$

Thus the quantity in brackets has zero curl, and can be expressed as the gradient of a scalar field, so we can write

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \tag{5}$$

where ϕ is a scalar function. Since the electric field depends on $\mathbf{A}$, if we make a gauge transformation by replacing $\mathbf{A}$ with $\mathbf{A}+\nabla\psi$, in order to keep the electric field unchanged we must make a corresponding transformation from ϕ to

$$\phi' = \phi - \frac{1}{c}\frac{\partial \psi}{\partial t}. \tag{6}$$

Direct substitution allows one to immediately verify that if $\mathbf{A} \to \mathbf{A}'$ and $\phi \to \phi'$, then $\mathbf{E}$ and $\mathbf{B}$ are unchanged.

If we now replace the electric field with the potential in Maxwell's equation for the divergence of the electric field, we have

$$4\pi\rho_e = \nabla \cdot \mathbf{E} = -\nabla^2\phi - \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla \cdot \mathbf{A}\right). \tag{7}$$

Finally, if we do replace the electric and magnetic fields with their potential equivalents in Maxwell's equation for the curl of $\mathbf{B}$, we have

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}_e + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{8}$$
$$\nabla \times (\nabla \times \mathbf{A}) = 4\pi \mathbf{j}_e + \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) \tag{9}$$

Making use of the vector calculus identity $\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$, this reduces to

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}\right) = -\frac{4\pi}{c}\mathbf{j}_e. \tag{10}$$

We now do something clever. This equation is irritating in that it has the complicated term $\nabla \cdot \mathbf{A} + (1/c)(\partial\phi/\partial t)$. However, we are free to make a change of gauge by choosing an arbitrary scalar function ψ and transforming $\mathbf{A} \to \mathbf{A}'$ and $\phi \to \phi'$. Let us write out this irritating term after making this transformation. It is

$$\nabla \cdot (\mathbf{A} + \nabla\psi) + \frac{1}{c}\frac{\partial}{\partial t}\left(\phi - \frac{1}{c}\frac{\partial \psi}{\partial t}\right) = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla^2\psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{11}$$

Thus if we choose our function ψ to be the solution to the equation

$$\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} - \nabla^2\psi = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t}, \tag{12}$$

then if follows that $\nabla \cdot \mathbf{A}' + (1/c)(\partial\phi'/\partial t) = 0$. The critical point here is that we do not in fact have to find the solution to this equation. Simply the fact that such a solution exists means that are are free to choose our gauge such that $\mathbf{A}$ and ϕ obey the condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \tag{13}$$

This choice is called the Lorentz gauge. With this choice, Maxwell's equations simplify greatly. Using this in Equation 7 gives

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho_e, \tag{14}$$

and using it in Equation 10 gives

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}_e. \tag{15}$$

These two can be combined into the following compact form

$$\Box^2 \begin{pmatrix} \phi \\ \mathbf{A} \end{pmatrix} = -4\pi \begin{pmatrix} \rho_e \\ \mathbf{j}_e/c \end{pmatrix}, \tag{16}$$

where

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2} \tag{17}$$

is the d'Alembertian operator. Equation 16 is an inhomogeneous wave equation. It is a wave equation because the left hand side, involving the d'Alembertian operator, is the form of a wave equation just like the one we wrote down for the propagation of light. It is inhomogeneous because the right hand side is not zero, but instead contains source terms.

B. The geometric optics limit

Before embarking on a general solution to the inhomogeneous wave equation, let us consider an important limiting case. Suppose we are in a region containing no sources, and for simplicity let us focus on the scalar part of the inhomogeneous wave equation, for the electric potential ϕ; the argument for the vector potential $\mathbf{A}$ is analogous, but involves more bookkeeping. The equation describing ϕ in free space is

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{18}$$

One valid solution to this is a plane wave, which in general for a wave with vector wavenumber $\mathbf{k}$ we can write out as

$$\phi = \phi_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{19}$$

with $\omega = |\mathbf{k}|c$ and $\phi_0 = \text{const}$. One can verify that this is a valid solution just by substituting. However, this solution is uniform throughout all space, and is clearly not applicable to radiation from arbitrary sources in arbitrary geometries. In mathematical terms, our solution to Equation 18 in free space will need to eventually match on to the boundary conditions imposed by the source terms on the right hand side of Equation 16, and in general a simple plane wave will not.

Despite this, however, we now proceed to show that other solutions to Equation 18, under very general assumptions, will locally resemble plane waves. This is called the geometric optics approximation. For a more general case, consider a solution of the form

$$\phi = A(\mathbf{x},t)e^{iS(\mathbf{x},t)}, \tag{20}$$

where $A(\mathbf{x},t)$ is a real function describing the amplitude of the wave and $S(\mathbf{x},t)$ is a real function describing its phase. The plane wave solution just corresponds to $A(\mathbf{x},t) = \phi_0 = \text{const}$ and $S(\mathbf{x},t) = \mathbf{k}\cdot\mathbf{x} - \omega t$. In the more general case, we can still use the function $S(\mathbf{x},t)$ to define the direction of propagation of the wave and its frequency, even if these are not constant in time or space. Locally, the direction in which a wave is propagating is the direction normal to the surface of constant phase. (Think about a curved surface wave on the ocean: the local direction in which the wave is moving is perpendicular to the line describing the wave crest.) Mathematically, this means that

$$\mathbf{k} = \nabla S. \tag{21}$$

Similarly, the local angular frequency is given by the negative time derivative of the phase, since it measures the inverse of the time required to complete a cycle:

$$\omega = -\frac{\partial S}{\partial t}. \tag{22}$$

One can see immediately that, in the case of a plane wave, these expressions give us $\mathbf{k}$ and ω constant throughout space, as we expect.

Now let us substitute our general solution into the wave equation, Equation 18. This gives

$$\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + iA\left(\nabla^2 S - \frac{1}{c^2}\frac{\partial^2 S}{\partial t^2}\right) + 2i\left(\nabla A\cdot\nabla S - \frac{1}{c^2}\frac{\partial S}{\partial t}\frac{\partial A}{\partial t}\right) - A\left(\nabla S\right)^2 + \frac{A}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 = 0. \tag{23}$$

The real and imaginary terms on the left hand side must both independently sum to zero. Let us focus on the real part, which with some slight re-arrangement is

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S)^2 = \frac{1}{A}\left(\nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2}\right) \tag{24}$$

Now we come to the geometric optics approximation, which is to notice that, if we are far from the source of the radiation and in free space, the length scale and time scales on which the amplitude of the wave vary must be vastly larger than the length and time scales on which the phase of the wave vary. Think about sunlight reaching the Earth: the intensity of the light varies on size scales of AU, while the phase of waves varies on size scales of smaller than a micron. The amplitude varies on time scales of years (or longer), and the phase oscillates with a period of $\sim 10^{-15}$ s. Consequently, the right hand side of this equation, which measures the characteristic length and time scale on which the amplitude varies, is completely negligible in comparison to the left hand side, which describes variation of the phase, and we can therefore set it to approximately zero.

We therefore have to good approximation

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S)^2 = 0 \tag{25}$$

This is known as the eikonal equation. Substituting in the definitions of $\mathbf{k}$ and ω, we immediately see that this implies that

$$|\mathbf{k}|^2 - \frac{\omega^2}{c^2} = 0, \tag{26}$$

which is the same as the dispersion relation for plane waves. Examining the form of the equation, it is clear that if ∇S points in some direction $\mathbf{k}$ at some point in time, that solution will slide along the $\mathbf{k}$ direction at speed c. The solutions to the eikonal equation therefore move in straight lines at constant speed c, just like plane waves. Consequently, we learn that, locally electromagnetic waves all look like plane waves, even if they are produced by arbitrary and complex charge and current distributions.

II. Formal solution to the potential equations

We now proceed to derive a formal solution to the inhomogeneous wave equation for the scalar and vector potential, which we will use to derive the potential associated with a moving charge.

A. Digression on the method of Green's functions

Inhomogeneous wave equations of the general form

$$\Box^2 \Psi = -4\pi f(\mathbf{x}, t), \tag{27}$$

such as those we have derived, can be solved by the method of Green's functions. While Green's functions are often presented in a dauntingly-mathematical manner, the intuition behind them is in fact super-simple. The basic idea is the following: the d'Alembertian is a linear operator, so we can just add solutions Ψ on the left hand side. Thus we do not need to find the solution for an arbitrary function f on the right hand side. Instead, let us find the solution for the simplest possible non-zero right-hand side: just a δ function, in our example a single, simple point charge. Once we have the solution Ψ that corresponds to the right hand side being just a single point charge, if we want to find the solution for an arbitrary charge distribution, we just build that distribution out of the sum of a bunch of point charges. Each point charge generates a solution Ψ on the left hand side, and, since the differential operator is linear, the solution Ψ for our arbitrary charge distribution is just the sum of the solutions Ψ for each of the point charges of which it is composed.

Putting this mathematically, suppose we find a function $G(\mathbf{x}, t; \mathbf{x}', t')$ that satisfies

$$\Box^2 G(\mathbf{x}, t; \mathbf{x}', t') = -4\pi\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'), \tag{28}$$

which corresponds to placing a unit charge at position $\mathbf{x}'$ at time t'. Then one can verify by direct substitution that the general solution to Equation 27 is

$$\Psi(\mathbf{x}, t) = \int G(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t')\, d^3x'\, dt'. \tag{29}$$

Formally, this follows simply because

$$\begin{aligned}
\Box^2\Psi(\mathbf{x}, t) &= \int \Box^2 G(\mathbf{x}, t; \mathbf{x}', t') f(\mathbf{x}', t')\, d^3x'\, dt' &(30)\\
&= -4\pi \int \delta(\mathbf{x} - \mathbf{x}')\delta(t - t') f(\mathbf{x}', t')\, d^3x'\, dt' &(31)\\
&= -4\pi f(\mathbf{x}, t). &(32)
\end{aligned}$$

Intuitively, the idea is that $G(\mathbf{x}, t; \mathbf{x}', t)$ is the solution generated by a single point charge that exists at a single instant in time, and the way we build up our solution is just by summing / integrating up a bunch of $G(\mathbf{x}, t; \mathbf{x}', t)$ factors corresponding to all the point charges that make up the distribution we're interested in.

B. Green's function solution

So what is the function $G(\mathbf{x}, t; \mathbf{x}', t')$ that solves Equation 28? First note that, by symmetry, the spatial part of the solution must depend only on $r = |\mathbf{r}| = |\mathbf{x} - \mathbf{x}'|$; there is no preferred direction, so G must be symmetric about the location $\mathbf{x}'$ of the point charge. Similarly, the temporal solution must depend only on $\tau = t - t'$, since there is no special time. Thus we can rewrite the problem in terms of r and τ, transforming it into

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} G(r, \tau)\right] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} G(r, \tau) = -4\pi\delta(\mathbf{r})\delta(\tau), \tag{33}$$

where on the left hand side we have written out the ∇^2 operator in spherical coordinates.

The right hand side is non-zero only at the origin, and at all other points the left hand side is just the spherical version of the wave equation we have seen many times before. It has the general solution

$$G(r, \tau) = \frac{1}{r}\left[g_+(\tau - r/c) + g_-(\tau + r/c)\right], \tag{34}$$

as one can readily verify by direct substitution. The two parts of the solution g_+ and g_- are arbitrary functions, and constitute waves moving radially outward and radially inward from the origin. They are completely analogous to the upward- and downward-propagating plane waves dealt with in our discussion of the classical theory of light.

How do we pick the functions $g_{\pm}$ that we want? In order to do that, we need to match our solutions away from the origin with those at the origin. Note that, as we take $r \to 0$, the first term on the left hand side of Equation 33 must become infinitely large compared to the second one. Thus in the vicinity of $r = 0$ we can drop the second term, and we ahve

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}G(r,\tau)\right] = -4\pi\delta(\mathbf{r})\delta(\tau) \tag{35}$$

Since there are no time derivatives on the left, the temporal part is separable, and the solution must lok like $G(r,\tau) = G_r(r)\delta(\tau)$, with

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}G_r(r)\right] = -4\pi\delta(\mathbf{r}). \tag{36}$$

However, we can immediately recognise this as just the equation describing the potential around a point charge at the origin, which has the trivial solution $G_r(r) = 1/r$. Thus we conclude that, in the vicinity of the origin, the solution must obey

$$\lim_{r\to 0} G(r,\tau) = \frac{\delta(\tau)}{r}, \tag{37}$$

which can be satisfied by taking $g_+(\tau - r/c) = \delta(\tau - r/c)$ and similarly for g_-. Thus we have shown that the general Green's function can be written

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta(\tau \pm r/c)}{r} = \frac{\delta\left[t' - (t \pm |\mathbf{x}-\mathbf{x}'|/c)\right]}{|\mathbf{x}-\mathbf{x}'|}. \tag{38}$$

The two possible solutions are known as the advanced and the retarded Green's functions, corresponding to the positive and negative signs in the solution. They are given these names because of what they imply about causality. The negative sign, corresponding to the retarded Green's function, implies that the Green's function contains a contribution coming from a time when $t' = t - r/c$, i.e., from something that happened at a time t' that is prior to the present time t. The positive sign, corresponding the the advanced Green's function, means that the Green's function depends on what is going on at time $t' = t + r/c$, i.e, something that will happen in the future. This is clearly not causal, and thus we may throw out this solution, leaving only the retarded one:

$$G(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta(\tau - r/c)}{r} = \frac{\delta\left[t' - (t - |\mathbf{x}-\mathbf{x}'|/c)\right]}{|\mathbf{x}-\mathbf{x}'|}. \tag{39}$$

Applying this to our original problem of Maxwell's equations cast in potential form, we arrive out our general expression for the electric and vector potentials created by an arbitrary distribution of charge and current:

$$\begin{pmatrix}\phi\\ \mathbf{A}\end{pmatrix} = \int \left[\begin{array}{c}\rho_e(\mathbf{x}',t')\\ \mathbf{j}_e(\mathbf{x}',t')/c\end{array}\right]\frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|} \tag{40}$$

where

$$t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}. \tag{41}$$

These expressions are known as the retarded potentials, and they have an easy interpretation: to calculate the retarded electric potential at a given point $\mathbf{x}$, we compute the integral of $\rho_e/|\mathbf{x}-\mathbf{x}'|$ over all points $\mathbf{x}'$, exactly as we would in electrostatics (and similarly for $\mathbf{j}_e/c$ and the magnetic field), with one catch: rather than evaluate ρ_e as it is *now*, we evaluate it as it *was* a time $|\mathbf{x}-\mathbf{x}'|/c$ in the past, properly accounting for the fact that information about electric and magnetic fields propagates at speed c.

III. The Liénard-Wiechert potentials

Now let us apply our formal solution to the most basic problem: the potential of a single moving charge. Suppose we have a charge q whose position as a function of time is given by $\mathbf{r}(t)$. In this case we can write out the charge density and current as

$$\rho_e = q\delta[\mathbf{x}-\mathbf{r}(t)] \tag{42}$$

$$\mathbf{j}_e = q\mathbf{v}(t)\delta[\mathbf{x}-\mathbf{r}(t)], \tag{43}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the particle velocity.

We can evaluate the electric and vector potentials using these values in our formal solution, Equation 40. To make this a bit easier, however, we will write our charge and current distributions in a slightly different form. The formal solution depends on the full history of where the charge has been over time, since as we look at the contribution from points $\mathbf{x}'$ further and further from the point $\mathbf{x}$ in which we are interested, we need to go back further and further in time. We therefore make this dependence on the history explicit by writing

$$\begin{pmatrix} \rho_e(\mathbf{x}',t') \\ \mathbf{j}_e(\mathbf{x}',t')/c \end{pmatrix} = \int \begin{pmatrix} q \\ q\mathbf{v}(\tau)/c \end{pmatrix} \delta[\mathbf{x}'-\mathbf{r}(\tau)]\delta(\tau-t')\,d\tau \tag{44}$$

It is immediately clear that this expression is exactly identical to the charge and current distributions we already wrote out, since the δ-function in $\tau - t'$ just changes very τ into a t'. The reason for making this change will become clear in a moment.

Substituting Equation 44 into Equation 40 and integrating over d^3x', we have

$$\begin{pmatrix} \phi(\mathbf{x},t) \\ \mathbf{A}(\mathbf{x},t) \end{pmatrix} = \int \begin{pmatrix} q \\ q\mathbf{v}(\tau)/c \end{pmatrix} \frac{\delta(\tau-t')}{R(\tau)}\,d\tau, \tag{45}$$

where

$$\mathbf{R}(\tau) \equiv \mathbf{x}-\mathbf{r}(\tau) \tag{46}$$

$$t' = t - \frac{R(\tau)}{c}, \tag{47}$$

and $R(\tau) = |\mathbf{R}(\tau)|$. Thus we have now written the retarded potentials as an explicit integral over the history of the particle's position $\mathbf{r}(\tau)$. We have replaced the integral over volume with one over time. The physical meaning of the δ function in this equation is simple: for each time τ in the past, the particle contributes to the potential at time t and position $\mathbf{x}$ if and only if the distance $R(\tau)$ from the particle to $\mathbf{x}$ is such that a signal traveling at c would reach position $\mathbf{x}$ at time t. In other words, for each point in space-time $(\mathbf{x}, t)$ where the want to know the potential, the δ function is responsible for picking out the exact coordinates in space-time along the particle's past space-time trajectory $(\mathbf{r}(\tau), \tau)$ such that information from $(\mathbf{r}(\tau), \tau)$ travelling at c reaches position $\mathbf{x}$ and time t, since only these points contribute to the potential at $(\mathbf{x}, t)$.

To evaluate the integrals, let us make a change of variable from τ to

$$\tau' = \tau - t' = \tau - t + R(\tau)/c. \tag{48}$$

The integration variable then changes as

$$d\tau' = \left[1 + \frac{\dot{R}(\tau)}{c}\right] d\tau. \tag{49}$$

To differentiate $R(\tau)$, the separation between the position of interest and the particle position at time τ, note that $R^2(\tau) = \mathbf{R}(\tau) \cdot \mathbf{R}(\tau)$. Thus $2\mathbf{R}(\tau) \cdot \dot{\mathbf{R}}(\tau) = -2\mathbf{R}(\tau)\mathbf{v}(\tau)$, where $\mathbf{v}(\tau) = \dot{\mathbf{r}}(\tau)$. Thus we have

$$\frac{d\tau}{R(\tau)} = \frac{d\tau'}{R(\tau) - \mathbf{R}(\tau) \cdot \mathbf{v}(\tau)/c} \tag{50}$$

We can simplify the notation a bit by defining $\hat{\mathbf{n}} = \mathbf{R}/R$ as the unit vector pointing from the position of interest $\mathbf{x}$ to wherever the particle is, and defining

$$\kappa = 1 - \frac{1}{c}\hat{\mathbf{n}} \cdot \mathbf{v}. \tag{51}$$

The quantity κ is just the usual length contraction factor in special relativity, i.e., it is the factor by which a moving object appears to be contracted compared to the same object at rest with respect to an observer. With these definitions, we have

$$\frac{d\tau}{R(\tau)} = \frac{d\tau'}{\kappa(\tau)R(\tau)}. \tag{52}$$

Thus our change of variables turns Equation 45 into

$$\begin{pmatrix} \phi(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{pmatrix} = \int \begin{pmatrix} q \\ q\mathbf{v}(\tau)/c \end{pmatrix} \frac{\delta(\tau')}{\kappa(\tau)R(\tau)}\, d\tau, \tag{53}$$

where the time τ that contributes at any time t is the one that makes $\tau' = 0$, i.e., where $\tau + R(\tau)/c = t$. Again, the δ function is doing the job for us of picking out the unique point in the particle's trajectory through space time (if one exists) that contributes to the potential at any position $\mathbf{x}$ and time t.

This can be written in a slightly more compact form as simply

$$\begin{pmatrix} \phi(\mathbf{x}, t) \\ \mathbf{A}(\mathbf{x}, t) \end{pmatrix} = \begin{bmatrix} q/\kappa R \\ q\mathbf{v}/c\kappa R \end{bmatrix}_{\mathrm{ret}}, \tag{54}$$

where the notation $[\cdot]_{\mathrm{ret}}$ indicates that the quantity in brackets is to be evaluated at the retarded time τ, i.e., at the time τ that satisfies

$$\tau + \frac{R(\tau)}{c} = t. \tag{55}$$

These are called the Liénard-Wiechart potentials, after their discovers. They are the dynamical analogs of the usual potentials for electrostatics, and if $\mathbf{v} = 0$, i.e., the charge in question is static, they reduce to the electrostatic case. They tell us that the proper way to compute the potential for a moving charge is to use its position at the retarded time, but that there is also a relativistic correction factor κ that we must account for which corrects for the compression of distances in relativity. This correction factor will become critically important when we discuss emission from relativistic particles later on.