In the last class we wrote down a basic equation for the transport of radiation through matter, and showed how it behaves in the limit of thermal equilibrium. In this class we will extend that theory by adding another process to the story: *scattering*. We will then discuss solutions to the transfer equation in various limiting cases, with an emphasis on ones that are relevant to high-energy processes.

I. The transfer equation with scattering

Our starting point is the transfer equation including emission and absorption that we derived in the last class, which can be written in the two equivalent forms

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{n}}\cdot\nabla I_\nu = j_\nu - \alpha_\nu I_\nu \qquad \frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = j_\nu - \alpha_\nu I_\nu, \tag{1}$$

where $\hat{\mathbf{n}}$ is an arbitrary unity vector, and s is a length coordinate along $\hat{\mathbf{n}}$. This equation describes matter emitting radiation, thus decreasing its internal energy, and matter absorbing radiation, thus increasing its internal energy.

However, it is possible for matter to interact with the radiation field without altering its internal energy at all. The matter neither absorbs nor emits the radiation, but merely changes its direction, while keeping the total internal energy of the matter and radiation fields the same.[1] Processes that change the direction of radiation without changing the energy content of the matter or the radiation field are called scattering processes. Terrestrial examples include scattering of optical light off air molecules or water droplets, leading to blue skies and rainbows; in these processes, the scatterers change the direction of the radiation without actually absorbing or emitting it. Astrophysical examples include radiation propagating through a screen of plasma and being scattered by free electrons, as around an active galactic nucleus, compact object, or in the early universe when the cosmic microwave background emerged.

In its most general form, we can describe scattering in terms of an overall scattering coefficient $\kappa_\nu(\hat{\mathbf{n}}')$ (which in complete generality can depend on the direction $\hat{\mathbf{n}}'$ of the incoming radiation field) and a redistribution function $\phi_\nu(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$ that describes, for incoming radiation in direction $\hat{\mathbf{n}}'$, what fraction of that radiation will be redirected into direction $\hat{\mathbf{n}}$; the phase function obeys $\int \phi_\nu(\hat{\mathbf{n}}, \hat{\mathbf{n}}')d\Omega' = 1$, i.e., all the incoming radiation that gets scattered has to go back out in some direction (or else we could call the process absorption, not scattering), so the integral of the redistribution function over all directions is unity.

Scattering appears in the transfer equation in two forms: there is a term that looks just like the absorption term that represents the decrease in radiation intensity along a ray due to radiation being absorbed or scattered, and there is an extra source term that represents

[1] This statement is glossing over some details. Since radiation has momentum, any interaction between matter and radiation that changes the direction of the radiation field necessarily changes the momentum of the radiation and the matter, and therefore can do work on the matter: $dW/dt = \mathbf{v}\cdot d\mathbf{p}/dt$.. However, one can show that the work done due to radiation momentum act on matter is smaller than the energy of the radiation field (which is the amount by which the matter energy would change if it absorbed the radiation) by a factor of order v/c, where v is the matter velocity. Thus for non-relativistic flows there is a fairly clean distinction between processes that just involve matter redirecting the radiation field versus those that involve emission or absorption, and we will mostly ignore this small term. However, correctly including this order v/c effect is crucial to formulating a consistent theory of radiation-hydrodynamics – see the classic textbook Foundations of Radiation Hydrodynamics by Mihalas & Mihalas for details.

the extra radiation being added in a given direction due to scattering of radiation that was originally going in other directions. Formally, we can write the transfer equation with scattering as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{n}}\cdot\nabla I_\nu = j_\nu - \left[\alpha_\nu + \kappa_\nu(\hat{\mathbf{n}}')\right] I_\nu + \int \kappa_\nu(\hat{\mathbf{n}}')\phi_\nu(\hat{\mathbf{n}},\hat{\mathbf{n}}')I_\nu(\hat{\mathbf{n}}',\mathbf{x})\,d\Omega'. \tag{2}$$

Comparing to the transfer equation with just emission and absorption, the extra $\kappa_\nu(\hat{\mathbf{n}}')$ is the extra reduction in intensity due to scattering, and the final integral term is the extra source due to scattering.

Although this is the most general form, in most astrophysical applications we can simplify a lot. First of all, if the scattering is being done by matter that is, at a microphysical scale, isotropic, then we expect that κ_ν will not in fact depend on the direction of the incoming radiation $\hat{\mathbf{n}}'$, but will instead be the same in all directions.[2] Moreover, for matter that is isotropic on small scales, the redistribution function ϕ_ν generally depends not on $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ independently, but only on $\cos\theta = \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}'$ – in words, the strength of scattering depends only on the angle between the direction the incoming radiation and the direction into which it is being scattered, not the overall orientation of these directions in space. In this case, the transfer equation simplifies to

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{n}}\cdot\nabla I_\nu = j_\nu - (\alpha_\nu + \kappa_\nu)\, I_\nu + \kappa_\nu \int \phi_\nu(\cos\theta) I_\nu(\hat{\mathbf{n}}',\mathbf{x})\,d\Omega'. \tag{3}$$

An even further simplification occurs if the scattering is completely isotropic, in which case $\phi_\nu = 1/4\pi$ independent of direction, and we have

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \hat{\mathbf{n}}\cdot\nabla I_\nu = j_\nu - \alpha_\nu I_\nu + \kappa_\nu\left(\frac{c}{4\pi}E_\nu - I_\nu\right) \equiv j_\nu - (\alpha_\nu+\kappa_\nu) I_\nu + \kappa_\nu J_\nu \tag{4}$$

where E_ν is the radiation energy density, and $J_\nu = E_\nu/4\pi$ is the angle-averaged intensity.

II. Solutions to the transfer equation with scattering

Even with scattering, the formal solution to the transfer equation we derived in the last class still applies, as long as we redefine the source function S_ν as

$$S_\nu(\hat{\mathbf{n}}) = \frac{1}{\alpha_\nu + \kappa_\nu(\hat{\mathbf{n}})}\left(j_\nu + \int \kappa_\nu(\hat{\mathbf{n}}')\phi_\nu(\hat{\mathbf{n}},\hat{\mathbf{n}}')I_\nu(\hat{\mathbf{n}}',\mathbf{x})\,d\Omega'\right). \tag{5}$$

With this redefinition, it is easy to show that the functional form of the transfer equation becomes exactly the same as in the pure emission-absorption case, and its solution is therefore exactly the same as the one we wrote down for that case. The physical interpretation is the same as well: the intensity at any given position in any given direction can be found just by integrating back along the ray in that direction, adding up the radiation sources along the ray, attenuated by the optical depth between the point of interest and the location of the source.

However, the character of the solution when scattering is present is very different. In the case of pure emission and absorption, the source function S_ν depends only on the matter emission and absorption coefficients j_ν and α_ν, and, if these are known, the problem is in principle solved. In this case, however, the source function depends on I_ν, so we do not

[2]While this is often the case, there are some counterexamples, mostly involving matter in the presence of strong magnetic fields. Large-scale magnetic fields can produce matter that scatters radiation propagating in different directions by different amounts. However, we will neglect that complication here.

have an explicit solution, only an implicit one, i.e., we don't have a formula for I_ν, we have an equation for I_ν. Nor is it a simple equation: it is an integral equation whereby the value if I_ν at any point and in any direction depends on an integral over all points and directions. Physically, this makes sense: if there is scattering, then the radiation intensity incident on a given point depends on the radiation intensity everywhere else being scattered to that point, so the problem is a global one where one has to find the full radiation field at all points in space. A general solution to this problem can only ever be numerical, and there is an extensive literature on numerical methods for finding solutions. However, we can gain some intuition and develop some useful approximations by considering the transfer equation in various limiting cases.

A. Pure isotropic scattering

As one simple case, suppose we take a homogenous medium where there is no emission or absorption, only scattering, and we assume that the scattering is isotropic, with phase function $\phi_\nu = 1/4\pi$. We will also assume steady-state, so we can drop the time derivative term. In this case the transfer equation reads

$$\hat{\mathbf{n}} \cdot \nabla I_\nu = \kappa_\nu \left(J_\nu - I_\nu \right). \tag{6}$$

What do solutions to this equation look like? In general the solutions have to depend on the boundary conditions, since the equation as written does not contain any term capable of injecting or removing photons. However, we can sharpen the question a bit by asking what solutions look like if we take as a boundary condition that there is a point source of radiation somewhere inside the homogenous medium. What does the solution look like then?

The quantitative answer here is less interesting than the qualitative one. To obtain the qualitative one, note that we can interpret κ_ν as just the inverse of the photon mean free path. That is, in the absence of sources, the radiation intensity along a given ray diminishes exponentially, as $I_\nu \propto e^{-\kappa_\nu s}$. Thus $1/\kappa_\nu$ is the e-folding length for the radiation intensity to diminish due to scattering, and thus is the mean free path.

Thus we can think of the trajectory of a photon emitted by a source embedded in this medium as a random walk. The photon goes a distance $\sim 1/\kappa_\nu$ before being scattered into a random direction, goes another distance of this order, etc. The total displacement after N steps of this process is the sum of N randomly-oriented vectors for which the expectation value for their length is $1/\kappa_\nu$.

The expected value for a vector that is the sum of N independent, randomly-oriented vectors of expected length $1/\kappa_\nu$ is $\sqrt{N}/\kappa_\nu$. To demonstrate this, let

$$\mathbf{R} = \sum_{i=1}^{N} \mathbf{r}_i \tag{7}$$

be the displacement after N steps, so

$$|\mathbf{R}|^2 = \sum_{i=1}^{N} \mathbf{r}_i \cdot \mathbf{r}_i + 2 \sum_{i,j}^{i \neq j} \mathbf{r}_i \cdot \mathbf{r}_j \tag{8}$$

where the first term is the dot product of each $\mathbf{r}_i$ with itself, and the second is the cross-term of every $\mathbf{r}_i$ with every other $\mathbf{r}_j$. We are interested in the expectation values of both sides, which are

$$\left\langle |\mathbf{R}|^2 \right\rangle = \sum_{i=1}^{N} \langle \mathbf{r}_i \cdot \mathbf{r}_i \rangle + 2 \sum_{i,j}^{i\neq j} \langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle \tag{9}$$

$$= N/\kappa_\nu^2. \tag{10}$$

In the second step, we noted that, if the $\mathbf{r}_i$, $\mathbf{r}_j$ pairs are all randomly oriented with respect to one another, then the expectation value of their dot product is zero, and the final term vanishes. Taking the square root of the final line gives the desired result.

We therefore learn an important fact about scattering: because photons in a scattering medium random-walk, in order to travel a *net* distance L, a photon requires $(\kappa_\nu L)^2$ scatterings. Since before each scattering the photon travels a distance $\approx 1/\kappa_\nu$, the photon must therefor travel a *total* distance $\kappa_\nu L^2$, larger by a factor of $\kappa_\nu L$. If the medium is optically thick, $\kappa_\nu L \gg 1$, this is a large factor.

B. Isotropic scattering plus emission and absorption

Now let us consider a more general case where we still have steady state in a homogenous medium with isotropic scattering, but the medium can absorb and emit radiation as well. We further assume that the medium is in thermal equilibrium at temperature T. The transfer equation for this situation reads

$$\hat{\mathbf{n}} \cdot \nabla I_\nu = \alpha_\nu (B_\nu - I_\nu) + \kappa_\nu (J_\nu - I_\nu), \tag{11}$$

where we have used the fact that $j_\nu = \alpha_\nu B_\nu(T)$ for a medium in thermal equilibrium. The first term represents emission and absorption, while the second represents scattering. If is convenient to rewrite this in terms of the source function

$$S_\nu = \frac{\alpha_\nu B_\nu + \kappa_\nu J_\nu}{\alpha_\nu + \kappa_\nu}, \tag{12}$$

so that the transfer equation becomes

$$\hat{\mathbf{n}} \cdot \nabla I_\nu = (\alpha_\nu + \kappa_\nu)(S_\nu - I_\nu) \tag{13}$$

This is very similar in functional form to the pure scattering case (Equation 6), but with the important difference that S_ν contains a contribution from both thermal emission and scattering.

To guide us toward a solution in this case, let us analyse this situation in terms of our random walk treatment of the pure scattering case. The quantity $\alpha_\nu + \kappa_\nu$ is called the extinction coefficient, and its inverse is the photon mean free path. However, because we have both scattering and absorption in this medium, each time the photon takes one its random steps there is some probability that it will be absorbed and stop moving. This probability is

$$\epsilon_\nu = \frac{\alpha_\nu}{\alpha_\nu + \kappa_\nu}. \tag{14}$$

We refer to the quantity $1 - \epsilon_\nu$, the probability that the end of each step of the random walk is a scatter rather than an absorption, as the single-scattering albedo. Written in terms of this quantity, the source function is

$$S_\nu = (1 - \epsilon_\nu) J_\nu + \epsilon_\nu B_\nu. \tag{15}$$

Thus ϵ_ν serves the role of interpolating for us between a medium that is purely scattering ($\epsilon_\nu = 0$, single-scattering albedo = 1) and a medium that has pure emission and absorption ($\epsilon_\nu = 1$, single-scattering albedo = 0).

Photons emitted in this medium still undergo random walks, but they will survive only a finite number of steps in the walk before being absorbed. Since for each step of the walk the probability that the photon survives to make another step is $1 - \epsilon_\nu$, The probability that the photon dies after one step is ϵ_ν, while the probability that the photon makes it to step 2 is $1 - \epsilon_\nu$. Thus the probability that it dies after two steps is $\epsilon_\nu(1 - \epsilon_\nu)$, the probability that it does after three steps is $\epsilon_\nu(1 - \epsilon_\nu)^2$, and so forth. The probability that the photon survives exactly N steps is

$$p(N) = \epsilon_\nu(1 - \epsilon_\nu)^{N-1} \tag{16}$$

The expected number of steps is

$$\langle N \rangle = \sum_{n=1}^{\infty} n\epsilon_\nu(1 - \epsilon_\nu)^{N-1} = \frac{1}{\epsilon_\nu} \tag{17}$$

Since the expected distance travelled per step is $\ell = 1/(\alpha_\nu + \kappa_\nu)$, the expected total displacement from its original position at which a photon is absorbed is therefore

$$\ell_* = \frac{\ell}{\sqrt{\langle N \rangle}} = \frac{1}{\sqrt{\alpha_\nu(\alpha_\nu + \kappa_\nu)}}. \tag{18}$$

The quantity ℓ_* is known as the thermalisation length for the medium. It is the effective length scale over which photons carry energy across the medium, homogenising its temperature.

Media whose physical size L is $\ll \ell_*$ behave very differently from those whose size is $\gg \ell_*$. If $L \ll \ell_*$, most photons emitted inside the medium will scatter out of it before being absorbed. In this case we call the medium optically thin, and the emergent luminosity per unit frequency approaches

$$L_\nu = 4\pi\alpha_\nu B_\nu V, \tag{19}$$

where V is the volume of the medium. This simply says that each unit volume in the medium emits with an emissivity $\alpha_\nu B_\nu$, which is isotropic, so the total energy emission rate is this time 4π. Since no photons are absorbed and all escape, the total emission rate is just this emission per unit volume times the volume.

If $L \gg \ell_*$, on the other hand, the medium is optically thick. In this case most photons do not escape before being absorbed. In this case the fact that there is scattering internal to the medium doesn't really matter, and the medium acts like an opaque body with just emission and absorption. The radiation field inside it approaches the thermal value $I_\nu \to B_\nu(T)$. Photons escape only if they are emitted within $\sim \ell_*$ of the surface, and so the medium looks like a black body shining from its surface. The luminosity is

$$L_\nu \approx B_\nu A, \tag{20}$$

where A is the surface area.

III. The Eddington approximation

We have seen that the qualitative solution to the transfer equation in a medium with both scattering and absorption depends on the optical depth, and that a full quantitative solution necessarily depends on the boundary conditions and must be obtained numerically. However, there is an intermediate approach that allows us to obtain a reasonably-accurate approximate solution to a reasonably broad class of problems where we can approximate an optically thick medium as plane-parallel, such that the properties of the medium and the radiation field with which it interacts are only functions of depth into the medium. (No approximation is needed if the medium is optically thin, since in that case we can just compute the emission from every volume element independently.) The surfaces of accretion disks around compact objects and the atmospheres are stars are examples of structures that satisfy this assumption reasonably well. The method we introduce to deal with these situations is called the Eddington approximation.

A. The diffusion equation

To set up the problem, consider a medium that is homogenous in the x and y directions. We will assume for now that the medium scatters radiation isotropically, though we will see below that we can actually drop this assumption. We set up our coordinate system so that the surface of the medium is at $z = 0$, and the medium lies as $z < 0$; the space at $z > 0$ is empty. We let θ be the angle between $\hat{\mathbf{z}}$ and any direction of radiation $\hat{\mathbf{n}}$, so that $\mu \equiv \cos\theta = \hat{\mathbf{z}} \cdot \hat{\mathbf{n}}$. In our geometry, angles $\mu > 0$ correspond to rays going upward out of the atmosphere, while angles $\mu < 0$ correspond to rays going downward into the atmosphere.

By symmetry the intensity I_ν must depend only on z and μ. The transfer equation therefore reads

$$\hat{\mathbf{n}} \cdot \nabla I_\nu(z, \mu) = (\alpha_\nu + \kappa_\nu)(S_\nu - I_\nu). \tag{21}$$

By assumption the problem is homogenous in the x and y directions, so the x and y components of the ∇ operator are zero; only the z component is non-zero. Thus the transfer equation reduces to

$$\mu \frac{\partial I_\nu(z, \mu)}{\partial z} = (\alpha_\nu + \kappa_\nu)(S_\nu - I_\nu), \tag{22}$$

where we have used the definition $\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = \mu$. This question holds for every μ, but if there is scattering in the problem then the source term S_ν includes and integral over all μ, which couples all the equations for different μ values together.

We start by re-casting this in terms of the optical depth instead of z. Let $\tau_\nu = -\int_0^z (\alpha_\nu + \kappa_\nu)\, dz$; the $-$ sign is because the medium is at $z < 0$, so this choice ensures that the optical depth increases into the medium, and is zero at its surface. With this definition, the transfer equation becomes

$$\mu \frac{\partial I_\nu(z, \mu)}{\partial \tau_\nu} = I_\nu - S_\nu. \tag{23}$$

This equation has to hold at every μ, and the source term couples all the μ's together. Our next step is to reduce this complexity by taking moments of this equation, and to recasting the infinite number of transfer equations (one for each μ) as a series of moment equations. Let us define moments of the radiation field:

$$J_\nu(z) \equiv \frac{1}{2} \int_{-1}^{1} I_\nu(z, \mu)\, d\mu \tag{24}$$

$$H_\nu(z) \equiv \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\mu\,d\mu \tag{25}$$

$$K_\nu(z) \equiv \frac{1}{2}\int_{-1}^{1} I_\nu(z,\mu)\mu^2\,d\mu \tag{26}$$

Note that these are the same as the moments we defined in the previous class, except for omitting factors of c, and taking advantage of the symmetry that I_ν depends only on μ and z. In this case the "moment" of the radiation field just describes how many powers of μ we multiply by, and the objects we get by taking the moments are all one-element scalars, rather than vectors, tensors, etc.

Using these definitions, we can take moments of the transfer equation by integrating over μ. The zeroth moment is:

$$\frac{1}{2}\int_{-1}^{1} \mu\frac{\partial I_\nu(z,\mu)}{\partial\tau_\nu}\,d\mu = \frac{1}{2}\int_{-1}^{1}(I_\nu - S_\nu)\,d\mu \qquad \Longrightarrow \qquad \frac{\partial H_\nu}{\partial\tau_\nu} = J_\nu - S_\nu. \tag{27}$$

Note that S_ν is isotropic under our assumption that scattering is isotropic, and thus it does not depend on μ. For the next moment we must multiply by μ before integrating:

$$\frac{1}{2}\int_{-1}^{1} \mu^2\frac{\partial I_\nu(z,\mu)}{\partial\tau_\nu}\,d\mu = \frac{1}{2}\int_{-1}^{1}\mu(I_\nu - S_\nu)\,d\mu \qquad \Longrightarrow \qquad \frac{\partial K_\nu}{\partial\tau_\nu} = H_\nu. \tag{28}$$

Note that the integral of μS_ν vanishes by symmetry, since S_ν is independent of μ. We could go on, and in general since there are an infinite number of different μ values in the transfer equation we would need infinitely many moment equations to fully represent all possible solutions.

However, we now make an approximation that will let us stop. Deep in an optically thick medium, the radiation field should be nearly isotropic, since the emission and absorption processes are isotropic, and far into the medium most of the photons that are around are ultimately coming from thermal emission and absorption. If the radiation field were truly isotropic, then I_ν would depend on z alone, with no dependence of μ at all. Since the field is nearly isotropic, we approximate this by Taylor expanding I_ν about $\mu = 0$ and dropping the higher-order terms:

$$I_\nu(z,\mu) \approx a_\nu(z) + b_\nu(z)\mu. \tag{29}$$

Plugging this approximation into the definitions of the moments, Equation 24 to Equation 26, we have

$$J_\nu(z) = a_\nu(z) \tag{30}$$

$$H_\nu(z) = \frac{1}{3}b_\nu(z) \tag{31}$$

$$K_\nu(z) = \frac{1}{3}a_\nu(z) = \frac{1}{3}J_\nu(z) \tag{32}$$

This last step constitutes a *closure relation*, because it allows us to express one of the moments of the radiation field directly in terms of other moments, without the need for a series of non-local differential equations linking them all. This is the thing we need to terminate the infinite series of moment equations.[3]

[3]For those with an interest in fluid dynamics: there is a completely analogous process in the way one derives

Note that, in this approximation, we can in fact drop the assumption that the scattering is isotropic. As long as the radiation field is isotropic, or close to it, we don't actually care about the angular dependence of the scattering, just its angle-average. Thus we can drop the assumption that scattering is isotropic, and re-interpret our scattering coefficient that we have implicitly included in the optical depth as just the angle-averaged value.

Using the Eddington approximation, we can rewrite our first moment equation, Equation 28, as

$$\frac{\partial K_\nu}{\partial \tau_\nu} = H_\nu = \frac{1}{3}\frac{\partial J_\nu}{\partial \tau} \tag{33}$$

Taking the derivative of both sides with respect to τ_ν gives

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \frac{\partial H_\nu}{\partial \tau_\nu} = J_\nu - S_\nu, \tag{34}$$

where in the last step we used Equation 27. Rewriting the source function in terms of the single-scattering albedo, Equation 15, we have

$$\frac{1}{3}\frac{\partial^2 J_\nu}{\partial \tau_\nu^2} = \epsilon_\nu (J_\nu - B_\nu). \tag{35}$$

This is known as the radiative diffusion equation, because it has the form of a diffusion equation: on the left hand side we have the second derivative of the thing being diffused (integrated intensity) with respect to a spatial coordinate, on the right hand side we have the intensity itself, plus a source term. This equation can be solved using the standard methods used to solve diffusion equations.

If the albedo ϵ_ν and the photon mean free path $\ell = 1/(\alpha_\nu + \kappa_\nu)$ are constant and independent of depth, the analogy between this and a diffusion equation is even more obvious. In this case the equation can be written

$$\frac{\ell^2}{3\epsilon_\nu}\frac{\partial^2 J_\nu}{\partial z^2} = J_\nu - B_\nu, \tag{36}$$

and it is clear that we can identify $\ell^2/3\epsilon_\nu$ as the diffusion coefficient.

Solution for $J_\nu(\tau)$ of either this simplified form, or the more general one, Equation 35, is relatively straightforward for a specified temperature distribution $T(z)$, which in turn determines $B_\nu(\tau)$. Once we have a solution for J_ν, we can use it to evaluate the source function everywhere, and thereby determine the intensity everywhere, giving a complete solution.

B. Boundary conditions

The last thing we need before proceeding to a solution boundary conditions; this is a second-order equation, so we need two of them. The boundary conditions are slightly tricky because any choice will necessarily be an approximation, since the

the equations of fluid dynamics from the kinetic theory of gasses. One takes moments of the kinetic transport equation. The first moment is mass conservation, which depends on the fluid velocity / momentum. The second is momentum conservation, which depends on the fluid pressure / energy. The next is energy conservation. However, rather than having this depend on an even higher moment of the particle velocity distribution, one invokes what we know about the thermodynamics of gasses to introduce a closure relation that expresses the fluid pressure and viscosity as a function of the gas temperature and similar microscopic properties. This is the closure relation that yields the Navier-Stokes equations for fluid dynamics.

assumption we made in order to derive the radiative diffusion equation – that the radiation field is nearly isotropic – must begin to fail within a distance ℓ of the surface. One common approximation is the so-called two-stream approximation, in which we approximate the radiation field at the surface with only two angles, $\mu = \pm 1/\sqrt{3}$. The motivation for choosing these two angles is that they satisfy the closure Eddington closure relation, as we will see in a moment.

We will denote $I_\nu^+(z) = I_\nu(z, \mu = 1/\sqrt{3})$ and $I_\nu^-(z) = I_\nu(z, \mu = -1/\sqrt{3})$ as the intensities along these two rays; the positive sign corresponds to a ray going out of the slab, the negative sign to a ray going into the slab. The total intensity in the two-stream approximation is

$$I_\nu(z,\mu) = \delta(\mu - 1/\sqrt{3}) I_\nu^+(z) + \delta(\mu + 1/\sqrt{3}) I_\nu^-(z). \tag{37}$$

Inserting this into our definitions of the moments, Equation 24 to Equation 26, we have

$$J_\nu(z) = \frac{1}{2}\left[I_\nu^+(z) + I_\nu^-(z)\right] \tag{38}$$

$$H_\nu(z) = \frac{1}{2\sqrt{3}}\left[I_\nu^+(z) - +I_\nu^-(z)\right] \tag{39}$$

$$K_\nu(z) = \frac{1}{6}\left[I_\nu^+(z) + I_\nu^-(z)\right] = \frac{1}{3} J_\nu(z). \tag{40}$$

The last step shows that this choice is indeed consistent with the Eddington approximation that $K_\nu = (1/3) J_\nu$.

If we now insert H_ν and J_ν into our first moment equation, Equation 33, we can solve for the relationship between theJ_ν and the upward and downward intensities:

$$I_\nu^+ = J_\nu + \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu} \tag{41}$$

$$I_\nu^- = J_\nu - \frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu}. \tag{42}$$

If we specify the ingoing and outgoing intensities at the surface $z = 0$, or we specify the incoming intensity at the surface and the intensity at some other point, these equations then provide the two required boundary conditions.

C. Sample solution

To illustrate how this works in practice, let us consider a simple example. Suppose we have a semi-infinite slab of material at constant temperature T, within which the absorption and scattering coefficients α_ν and κ_ν are constant. There is no external radiation field incident on the slab. What does the radiation field look like?

In this case the radiative diffusion equation, Equation 35, can be integrated directly because ϵ_ν and B_ν are constants. The solution is

$$J_\nu(\tau_\nu) = B_\nu(T) + C_1 e^{\sqrt{3\epsilon_\nu}\tau_\nu} + C_2 e^{-\sqrt{3\epsilon_\nu}\tau_\nu}, \tag{43}$$

where C_1 and C_2 are constants of integration to be determined by the boundary conditions. Deep within the slab, we know that we must reach radiative equilibrium, so $J_\nu \to B_\nu(T)$ for $\tau_\nu \to \infty$. This immediately tells us that $C_1 = 0$, so the solution is of the form

$$J_\nu(\tau_\nu) = B_\nu(T) + C_2 e^{-\sqrt{3\epsilon_\nu}\tau_\nu}, \tag{44}$$

To find the other constant of integration, we use our two-stream approximation. There is zero radiation incoming to the surface at $z = 0$, so $I_\nu^- = 0$ at $z = \tau_\nu = 0$. Thus

$$\frac{1}{\sqrt{3}} \frac{\partial J_\nu}{\partial \tau_\nu} = J_\nu \tag{45}$$

at the surface. This is one of our boundary conditions. Inserting our generic solution at $\tau_\nu = 0$, we have

$$-\sqrt{\epsilon_\nu} C_2 = B_\nu(T) + C_2 \quad \Longrightarrow \quad C_2 = -\frac{B_\nu(T)}{1 + \sqrt{\epsilon_\nu}} \tag{46}$$

The solution for the average intensity is therefore

$$J_\nu = B_\nu(T) \left(1 - \frac{e^{-\sqrt{3\epsilon_\nu}\tau_\nu}}{1 + \sqrt{\epsilon_\nu}} \right). \tag{47}$$

Note that this becomes undefined for $\epsilon_\nu = 0$, but this is exactly what we should expect: $\epsilon_\nu = 0$ corresponds to a purely scattering atmosphere, in which case the problem is ill-posed, because there is no source of photons, since nothing every emits. If $\epsilon_\nu = 1$, then at $\tau_\nu = 0$ we have $J_\nu = B_\nu(T)/2$. This too makes sense: for a slab with no scattering, only emission and absorption, if you sit at the surface and look down, every line of sight will reveal a pure blackbody with intensity $B_\nu(T)$. If you look up, away from the surface, you see intensity zero. Thus the average intensity over all directions is $B_\nu(T)/2$.