This tutorial introduces *polytropes*, which are a simple model for stars that we can make without any of the detailed physics for the equation of state, energy transport, and nuclear reactions that we are going to introduce later on. The idea behind polytropes is that, if we just posit a simple relationship between pressure and density, we can dispense with the rest of the model for a star.

To treat polytropes, we begin from hydrostatic balance:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho. \tag{1}$$

We bring the r^2/ρ to the other side, then differentiate both sides:

$$\begin{aligned}
\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -G\frac{dm}{dr} && (2)\\
&= -4\pi G\rho r^2 && (3)\\
\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) &= -4\pi G\rho. && (4)
\end{aligned}$$

We now make our approximation for the pressure. We posit that the pressure is related to the density by a simple power-law relationship,

$$P = K_P \rho^{(n+1)/n}, \tag{5}$$

where K_P and n are both constants. The motivation for this scaling is that there are a number of physical situations where a relationship like this would be expected to hold, which we will cover later in the course. Substituting this relationship into Equation 4 and doing a bit of re-arranging gives

$$\frac{(n+1)K_P}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(n-1)/n}}\frac{d\rho}{dr}\right) = -\rho. \tag{6}$$

The next step you will do. To render this equation more useful, we make some substitutions. Let ρ_c be the density at the centre of the star, and define the dimensionless density Θ by

$$\Theta^n = \frac{\rho}{\rho_c}. \tag{7}$$

Exercise 1. *In Equation 6, make a change of variable from ρ to Θ, and define a new position coordinate $\xi = r/\alpha$, where α is a constant you will determine. Show that, with this change of variables, Equation 6 can be rewritten*

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n. \tag{8}$$

Determine the value of α required to put the equation into this form.

Equation 8 is known as the Lane-Emden Equation. It is a second order ordinary differential equation, and therefore requires two boundary conditions. Since we defined ρ_c as the central density, one of these conditions must be that $\Theta = 1$ at $\xi = 0$. For our second boundary condition, we will also require that Θ approaches this value smoothly as $\xi \to 0$, which requires that $d\Theta/d\xi = 0$ at $\xi = 0$. This is the second boundary condition. Equation 8 is to be integrated from the inner boundary to some outer radius ξ_1 where Θ reaches 0. This will define the outer radius of the star.

We are going to solve the Lane-Emden equation numerically, but before doing so we must handle some preliminaries. First, to test that the code we write to solve it is correct, we would like to have an exact solution against which we can test. Fortunately, it is possible to obtain such a solution for the case $n = 1$.

Exercise 2. *For the case $n = 1$, show that $\Theta = \sin\xi/\xi$ is an exact solution of Equation 8, and that in this case the outer radius of the star is $\xi_1 = \pi$.*

The second preliminary we must handle before solving Equation 8 numerically is that most ordinary differential equation (ODE) solvers want to work on systems of first order ODEs rather than on second order ones. We must recast the Lane-Emden equation in this form.

Exercise 3. *Rewrite Equation 8 as a pair of coupled first order ODEs in standard form, i.e.,*

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2) \tag{9}$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2). \tag{10}$$

The final step is that would like to integrate starting at $\xi = 0$, since this is where our boundary conditions are specified, but Equation 8 has a *singular point* at $\xi = 0$. That is, the leading order derivative term in Equation 8 (or equivalently the first order ODEs you derived in Exercise 3) vanishes at $\xi = 0$. Fortunately, there is a standard way to handle this: by series expansion in the vicinity of the singular point. That is, we consider a trial solution that is written out as a Taylor series, substitute into the original equation, and find the solution in the limit $\xi \to 0$. This tell us how the solution behaves in the vicinity of the singular point.

Exercise 4. *Consider a trial solution $\Theta = a_0 + a_1\xi + a_2\xi^2 + \cdots$, where we are interested in the behaviour for $\xi \ll 1$, and the coefficients a_n are to be determined. Substitute this trial solution into Equation 8, and, making use of the the boundary conditions, derive the values of a_0, a_1, and a_2.*

We are now ready to solve the problem numerically. Our goal will be to integrate Equation 8 numerically from a small value of ξ (say $\xi = \xi_0 = 10^-6$), out to the point where $\Theta = 0$, which we will denote ξ_1. For the boundary condition, use the series expansion to obtain values of Θ and $d\Theta/d\xi$ at $\xi = \xi_0$. You can do this numerical integration in whatever program / language you prefer. Some reasonable choices are python, matlab, and mathematica.

Exercise 5. *Numerically integrate Equation 8 for the case $n = 1$, and verify that your numerical result matches the analytic value you derived in step 2.*

Exercise 6. *Once you have verified that your code is working correctly, obtain numerical solutions for $n = 1, 1.5, 2, 2.5, 3$, and make a table of ξ_1 versus n.*