

Twinkle, twinkle little star,
I don't wonder what you are
for by spectroscopic ken
I know that you're hydrogen...

Anonymous astronomer

In this tutorial we will replicate part of the work of Cecilia Payne-Gaposchkin in computing the abundances of elements in the Sun. In particular, we will show that both calcium and sodium are much less abundant than hydrogen. The full calculation involves the following steps:

1. Measure the absorption of light in for the each line of interest.
2. Use the measurement of light absorption to infer the amount of the species producing the absorption; this species is a particular excitation state of a particular ionisation state of a particular element.
3. Use the Boltzmann distribution to infer the total abundance of each ion from our measured abundance in a particular excitation state.
4. Use the Saha equation to turn this into a measurement of the total abundance of each element.

Step 1.

As a first step, we must measure the amount of light absorbed by a particular transition. **Figure 1** shows the spectrum of the Sun in the vicinity to three prominent spectral lines: the K line produced by calcium, the D2 line produced by sodium, and the $H\alpha$ (Balmer α) line produced by hydrogen. Note that all the absorption dips you see here are real, not noise; there are a *lot* of absorption features in the spectrum of the Sun if one looks closely!

Line absorption is measured by the equivalent width, which measures the amount of light missing due to line absorption. Formally, the equivalent width in wavelength is defined as

$$W_\lambda = \int \left[1 - \frac{F_\lambda}{F_\lambda(0)} \right] d\lambda, \quad (1)$$

where F_λ is the observed flux as a function of wavelength and $F_\lambda(0)$ is the flux that would have been observed were the line absent. The quantity W_λ has the same units as whatever the wavelength is measured in, which traditionally is Angstrom.

Exercise 1. Estimate the equivalent widths of the CaK, NaD₂, and $H\alpha$ lines. You may do this from the plots provided, or, if you are feeling more ambitious, grab the raw data from the course web page at http://www.mso.anu.edu.au/~krumholz/teaching/astr3007_s1_2017/ (under class-by-class topics) and use them to perform your estimate.

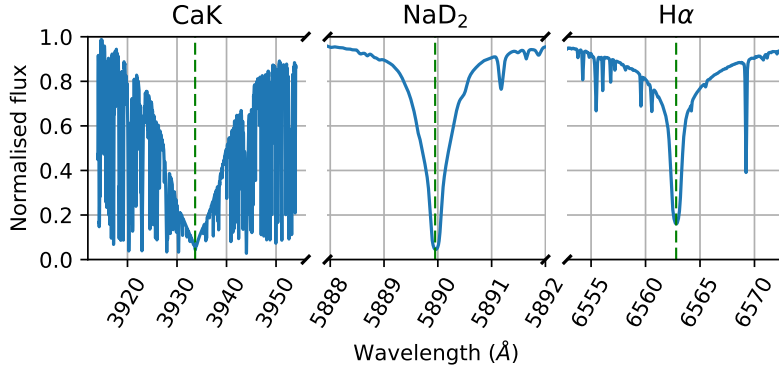


Figure 1: The spectrum of the Sun in the vicinity of three prominent spectral lines. Blue shows the observed spectrum and green dashed lines indicate the central wavelength for each line. The spectra have been normalised so that the continuum level (i.e., the flux for light without any line absorption) is at 1.0. Data taken from http://bass2000.obspm.fr/solar_spect.php.

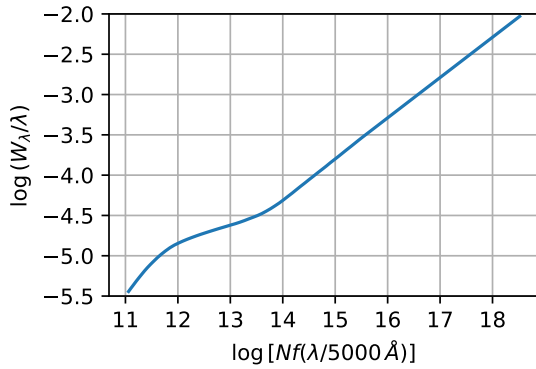


Figure 2: A curve of growth computed for the Sun. The quantity on the x axis is the column of absorbers N (measured in cm^{-2}) multiplied by the dimensionless oscillator strength f for the transition multiplied by the wavelength λ normalised to $5000 \text{ \AA}$. The quantity plotted on the y axis is the equivalent width W_λ divided by the wavelength.

Step 2.

The second step is to turn the observed equivalent width into a column density, i.e., a number of atoms per unit area along the line of sight from us to the stellar photosphere. This is somewhat tricky, because the equivalent width of the line does not linearly proportional to the column density except when it is very small. For larger columns, the relationship between equivalent width and column density is known as the curve of growth. The curve of growth depends on the pressure and temperature of the stellar surface (which affect the velocity distribution of atoms and the frequency with which they collide) as well as intrinsic quantum mechanical properties of the absorbing species.

Calculation of the curve of growth for a star is a non-trivial exercise that we will not attempt in this tutorial. **Figure 2** shows a curve of growth computed for the Sun. The curve of growth expresses a relationship between the equivalent width normalised by the wavelength, W_λ/λ , and the following combination of parameters: $Nf\lambda$. Here N is the column density of the absorbers, i.e., how many of them there are per unit area. The quantity f is called the oscillator strength, and it is a dimensionless quantum mechanical measurement of the transition strength.

Exercise 2. Use the curve of growth plus your measured equivalent width to infer the column of sodium atoms in the ionisation and excitation states responsible for the three lines. The required data are given below.

Line	$\lambda [\text{\AA}]$	f
CaK	3934.77	0.682
NaD ₂	5891.58	0.641
H α	6562.79	0.641

Step 3.

We have now determined the columns of ions in the states responsible for producing the CaK, NaD₂, and H α transitions. Our next step will be to correct from atoms in a particular state to atoms in all quantum states using the Boltzmann distribution. As a reminder, the Boltzmann distribution is that the fraction of atoms in some state i with energy E_i and degeneracy g_i is given by

$$f_i = \frac{g_i e^{-E_i/k_B T}}{Z(T)}, \quad (2)$$

where T is the temperature and $Z(T)$ is the partition function, given by

$$Z(T) = \sum_i g_i e^{-E_i/k_B T}. \quad (3)$$

We need to determine f_i for each of our states in order to correct from the column in a particular state to the total column in all states. For all cases we will use the estimated Solar surface temperature $T = 5780$ K.

The H α lines comes from neutral hydrogen, and for hydrogen we can calculate f_i exactly. Hydrogen has states denoted by quantum number n , and if we take the ground state to have zero energy, then the degeneracy and binding energy of state n is

$$g_n = 2n^2 \quad E_n = 13.6 \text{ eV} \left(1 - \frac{1}{n^2}\right) \quad (4)$$

for $n = 1 \dots \infty$. The H α line comes from the $n = 2$ state.

For NaD₂, the state that produces the line is the ground state ($E_i = 0$), which has a degeneracy of $g_i = 2$. The partition function must be determined by a brute force sum over all the possible levels and states, of which there are a significant number. At $T = 5780$ K, partition function is approximately $Z_{\text{Na}} \approx 2.1$. Similarly, the CaK line is produced by atoms of Ca⁺ in the ground state, which has degeneracy $g_i = 2$ and partition function $Z_{\text{Ca}^+} \approx 2.3$.

Exercise 3. Compute f_i for the states responsible for producing the H α , CaK, and NaD₂ lines. Do this even if you didn't find the columns for all three in the previous part. Use this value to correct the columns you have obtained from the columns of the state producing the transition to the columns of all ions.

Step 4.

The final step is to use the Saha equation to correct from total numbers of ions to total numbers of atoms in all ionisation states. That is, we have determined the columns of H⁰, Na⁰, and Ca⁺, and we want to know the columns of all H atoms, all Na atoms, and all Ca atoms, regardless of ionisation state.

The ratios of atoms in two different ionisation states is given by the Saha equation, first derived by **Meghnad Saha** in 1920. The equation states that, in equilibrium, the number densities n_{i+1} and n_i of an atomic species that has been ionised $i + 1$ and i times, respectively, have a ratio

$$\frac{n_{i+1}}{n_i} = \frac{2}{n_e} \frac{Z_{i+1}}{Z_i} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T}, \quad (5)$$

where Z_{i+1} and Z_i are the partition functions of the two ionisation states, n_e is the number density of electrons, and χ is the ionisation potential. Since the pressure is usually easier to

measure than the electron abundances, this equation is often also rewritten using the ideal gas law, which gives $P_e = n_e k_B T$, where P_e is the partial pressure of electrons. This gives

$$\frac{n_{i+1}}{n_i} = \frac{2k_B T}{P_e} \frac{Z_{i+1}}{Z_i} \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T}. \quad (6)$$

To apply this equation to figure out the corrections from H^0 to H , from Na^0 to Na , and from Ca^+ to Ca , we need to know the partition functions of both the neutral and once-ionised states. (The population of even more ionised states is negligibly small, as one can readily confirm by using the Saha equation for them.) The partition function for H^0 we computed in the previous part, and the partition function for H^+ is trivial: $Z_{H^+} = 1$, since H^+ is just a proton, which has only a single state. For the other species one must again resort to brute force. The data on ionisation potential and partition function that you will need are:

Ion	χ (eV)	$Z(T)$ at $T = 5780$ K
H^0	-	For you to compute
H^+	13.6	1
Na^0	-	1.0
Na^+	5.1	2.4
Ca^0	-	1.3
Ca^+	6.1	2.4

Exercise 4. Use the Saha equation to correct your columns from one ionisation state to the total abundance of each atom. Use an electron pressure $P_e = 15 \text{ dyne cm}^{-2}$, and use the partition functions and ionisation potentials given above. What do you conclude about the relative abundances of Na and Ca compared to H ?