We have now written down the basic equations of stellar structure, and we are therefore in a position to start building actual models for stars. There is one more piece of physics that we will require (convection), which we will discuss in the next class, but for today we will build simple models that do not include this effect. Even with this omission, these models provide a great deal of insight into how stars work, and set the stage for a full explanation of the main sequence.

I. The structure equations

To begin this class, we review the basic equations of stellar structure that our model has to solve. In Lagrangian form, these are

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{2}$$

$$\frac{dL}{dm} = q_{\text{nuc}} \tag{3}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa_R}{T^3}\frac{L}{(4\pi r^2)^2}. \tag{4}$$

These equations are supplemented by an equation of state that gives $P(\rho, T)$, and by functions that specify the nuclear energy generation rate $q_{\text{nuc}}(\rho, T)$ and the opacity $\kappa_R(\rho, T)$. With these functions specified, the system has four unknowns: r, L, T, and P or ρ,which are related by the equation of state.

This is a system of four ordinary differential equations, and requires four boundary conditions. Two of them are immediately obvious: the centre of the star had better be at zero radius, and there had better be zero luminosity entering the star from zero radius. Thus two of the boundary conditions are $r = 0$ and $L = 0$ at $m = 0$.

The other two boundary conditions are specified at the surface of the star. In the simplest form, we can take them to state that, at the stellar surface, the pressure must be go to zero, and the luminosity must be that of a black body. This gives $P = 0$ and $L = 4\pi R^2 \sigma_{\text{SB}} T^4$ as the final two boundary conditions. In reality we usually do something a bit more sophisticated than this, since, as we have seen, stars have atmospheres that are not really black bodies, and that are not at exactly zero pressure. However, we will skip that complication for now.

Full solution to these equations, including the complicated functional forms for P, q_{nuc}, and κ_R, can only be done numerically. However, we can come up with approximate solutions that yield a great deal of insight analytically.

II. Eddington models

The first real model of how stars work was produced by Arthur Eddington in the 1920s. The timing here is significant: Eddington's model dates from the 1920s, while the understanding of exactly how the Sun generates its energy via nuclear reactions was not really understood until the 1940s. Indeed, Eddington's model leaves the nature of the power source of stars completely unspecified. By the 1920s Eddington suspected, correctly, that the power source had to be something like nuclear energy, but the exact mechanism was not understood. Fortunately, it is possible to construct a reasonable equilibrium model for a star without understanding in detail exactly how the energy is generated.

A. Radiation Pressure and the Eddington Limit

The Eddington model begins by considering the temperature structure of stars, and its implication for the importance of radiation pressure. The central insight in the Eddington model is that gas pressure follows $P_{\rm gas} \propto T$, but radiation pressure has a much steeper dependence: $P_{\rm rad} = aT^4/4$. Thus at sufficiently high temperatures radiation pressure always dominates. You computed this crossover in the tutorial.

The strong dependence of $P_{\rm rad}$ on T has important consequences for stellar structure. To see this, it is helpful to rewrite Equation 4 in terms of radiation pressure rather than temperature:

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa_R}{T^3}\frac{L}{(4\pi r^2)^2} \tag{5}$$

$$\frac{4}{3}aT^3\frac{dT}{dm} = -\frac{\kappa_R}{c}\frac{L}{(4\pi r^2)^2} \tag{6}$$

$$\frac{dP_{\rm rad}}{dm} = -\frac{\kappa_R}{c}\frac{L}{(4\pi r^2)^2} \tag{7}$$

Now consider what this implies for hydrostatic balance. Since $P = P_{\rm gas} + P_{\rm rad}$, the equation of hydrostatic balance (Equation 2) can be written as

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{8}$$

$$\frac{dP_{\rm rad}}{dm} = -\frac{Gm}{4\pi r^4} - \frac{dP_{\rm gas}}{dm} \tag{9}$$

Since density and temperature always fall as one moves outward within a star, $dP_{\rm gas}/dm$ is always negative, so the term $-dP_{\rm gas}/dm > 0$. Thus we have

$$\frac{dP_{\rm rad}}{dm} > -\frac{Gm}{4\pi r^4} \tag{10}$$

$$-\frac{\kappa_R}{c}\frac{L}{(4\pi r^2)^2} > -\frac{Gm}{4\pi r^4} \tag{11}$$

$$L < \frac{4\pi GcM}{\kappa_R} = 3.2\times 10^4\left(\frac{M}{M_\odot}\right)\left(\frac{0.4\ \mathrm{cm^2\ g^{-1}}}{\kappa_R}\right)L_\odot. \tag{12}$$

Here we have normalised κ_R to the electron scattering opacity because that is usually the smallest possible opacity in a star, producing the maximum possible L. This result is known as the Eddington limit, and the quantity on the right-hand side is called the Eddington luminosity:

$$L_{\rm Edd} = \frac{4\pi Gcm}{\kappa_R}. \tag{13}$$

This result represents a fundamental limit on the luminosity of any object in hydrostatic equilibrium. It applies to stars, but it applies equally well to any other type of astronomical system, and the Eddington limit is important for black holes, entire galaxies, and in many other contexts.

This also lets us derive a useful relation describing how the ratio of radiation pressure to total pressure varies within a star. Taking the ratio of Equation 7 and Equation 2 gives

$$\frac{dP_{\rm rad}}{dP} = \frac{\kappa_R L}{4\pi G c m} = \frac{L}{L_{\rm Edd}}. \tag{14}$$

B. The Eddington standard model

The Eddington standard model stems from the following approximation: suppose that we posit that $L/L_{\rm Edd}$ is constant. This might seem like a crazy assumption, but it turns out to be reasonably good. For low mass stars, recall that we showed that burning on the $p-p$ chain occurs at a rate that depends on temperature as roughly $q_{\rm nuc} \propto \rho T^4$. Since $dL/dm = q_{\rm nuc}$, this means that we expect $L/m \sim T^4$ as long as nuclear burning mostly takes place near the centre of the star where ρ doesn't vary much. On the other hand, free-free opacity varies with $\kappa_R \propto T^{-3.5}$. Thus the product $\kappa_R L/m$ depends relatively weakly on temperature. This is why the Eddington approximation works reasonably well.

This assumption amounts to saying that

$$\beta = \frac{P_{\rm gas}}{P} = 1 - \frac{L}{L_{\rm Edd}} \tag{15}$$

is constant throughout the star. This in turn, lets us model the star as a polytrope, a system for which $P = K_P \rho_P^{\gamma}$ for some constants K_P and γ_P. To see this, note that

$$P = \frac{P_{\rm rad}}{1-\beta} = \frac{aT^4}{3(1-\beta)} \tag{16}$$

$$P = \frac{P_{\rm gas}}{\beta} = \frac{\mathcal{R}}{\beta\mu}\rho T \tag{17}$$

Using the first equation to solve for T gives

$$T = \left[\frac{3}{a}(1-\beta)P\right]^{1/4}, \tag{18}$$

and inserting this into the second equation gives

$$P = \frac{\mathcal{R}}{\beta\mu}\rho\left[\frac{3}{a}(1-\beta)P\right]^{1/4} \tag{19}$$

$$P = \left[\frac{3\mathcal{R}^4(1-\beta)}{a\mu^4\beta^4}\right]^{1/3}\rho^{4/3}. \tag{20}$$

Thus if β is constant throughout the star, then the star is a polytrope with $\gamma_P = 4/3$ (corresponding to an index $n = 1/(\gamma_P - 1) = 3$) and

$$K_P = \left[\frac{3\mathcal{R}^4(1-\beta)}{a\mu^4\beta^4}\right]^{1/3}. \tag{21}$$

Polytropes with $n = 3$ have a special, useful property. To remind you from the tutorial, if a star is a polytrope, then we can write rewrite the equation of hydrostatic balance (Equation 2) for the star as the Lane-Emden equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Theta}{d\xi}\right) = -\Theta^n. \tag{22}$$

Here Θ is the dimensionless density, related to the true density by

$$\rho = \rho_c \Theta^n, \tag{23}$$

where ρ_c is the central density. The quantity ξ is the dimensionless radius, which is related to the true radius by

$$r = \alpha \xi, \tag{24}$$

where you showed on the tutorial that

$$\alpha = \left[\frac{(n+1)\, K_P}{4\pi G \rho_c^{(n-1)/n}}\right]^{1/2}. \tag{25}$$

For any given n, the Lane-Emden equation has a solution $\Theta(\xi)$ that runs from $\xi = 0$ to the outer edge of the star at $\xi = \xi_1$, where $\Theta(\xi_1) = 0$. For any given n, one can integrate the Lane-Emden equation to obtain this outer radius ξ_1, and thus the true physical radius of the star,

$$R = \alpha \xi_1. \tag{26}$$

We can now prove our useful result for $n = 3$ polytropes. We can obtain the physical mass of the star just by integrating over the density:

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho \, dr && (27) \\ &= 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \Theta^n \, d\xi, && (28) \end{aligned}$$

where the second step just amounts to inserting Equation 23 and Equation 24 for ρ and r. We next substitute for $\xi^2 \Theta^n$ using the Lane-Emden equation (Equation 22):

$$\begin{aligned} M &= -4\pi \alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{\xi} \left(\xi^2 \frac{d\Theta}{d\xi}\right) d\xi && (29) \\ &= -4\pi \alpha^3 \rho_c \xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1}. && (30) \end{aligned}$$

Now we just have to do a little bit of algebraic manipulation. First, we re-arrange Equation 25 to get ρ_c, giving

$$\rho_c = \left[\frac{(n+1) K_P}{4\pi G \alpha^2}\right]^{n/(n-1)}. \tag{31}$$

Next we insert this into our formula for the mass, giving

$$M = -4\pi \alpha^3 \left[\frac{(n+1) K_P}{4\pi G \alpha^2}\right]^{n/(n-1)} \xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1}. \tag{32}$$

Finally, we eliminate α using $\alpha = R/\xi_1$. This gives

$$\left[\frac{GM}{-\xi_1^2 (d\Theta/d\xi)_{\xi_1}}\right]^{n-1} \left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[(n+1) K_P]^n}{4\pi G}. \tag{33}$$

The equation we have just derived is the mass-radius relation for polytropes, meaning that, if you know the polytropic index n and polytropic constant K_P, this equation lets you solve for the radius R corresponding to any total mass M. However, we note that something special happens for $n = 3$: the radius term vanishes. Thus for $n = 3$, we do not gas a mass-radius relation. Instead, we simply get a relationship between the mass M and the polytropic constant. Specifically, for $n = 3$, we have

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1} \left(\frac{K_P}{G}\right)^{3/2} = 4.56 \left(\frac{K_P}{G}\right)^{3/2}, \tag{34}$$

where in the final step we inserted the numerical solution for $\xi_1^2(d\Theta/d\xi)_{\xi_1}$ for $n = 3$.

For the Eddington model, K_P is simply set by the value of β. This in turn means that the value of β uniquely determines the value of M. Inserting the value of K_P we just obtained (Equation 21) into the relationship between M and K_P gives

$$\begin{aligned} M &= -\frac{4}{\sqrt{\pi}}\xi_1^2 \left(\frac{d\Theta}{d\xi}\right)_{\xi_1} \left[\frac{3\mathcal{R}^4(1-\beta)}{aG^3\mu^4\beta^4}\right]^{1/2} && (35) \\ &= \frac{18.1\, M_\odot}{\mu^2} \left(\frac{1-\beta}{\beta^4}\right)^{1/2}. && (36) \end{aligned}$$

This gives us M in terms of β and μ. Alternately, we can re-arrange to get an equation for β in terms of M:

$$\begin{aligned} 0 &= -1 + \beta + \left(\frac{aG^3}{3\mathcal{R}^4}\right) \frac{\pi}{16\xi_1^4 (d\Theta/d\xi)_{\xi_1}^2} \mu^4 M^2 \beta^4 && (37) \\ &= -1 + \beta + 0.003 \left(\frac{M}{M_\odot}\right)^2 \mu^4 \beta^4 && (38) \\ &= -1 + \beta + 0.004 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\mu}{0.61}\right)^4 \beta^4 && (39) \end{aligned}$$

This is known as the Eddington quartic.

C. Mass-luminosity relations in the Eddington standard model

The Eddington standard model makes strong predictions for the way the properties of stars depend on mass, which, when compared to detailed numerical calculations (and to reality) turn out to be roughly correct. In particular, we can use the Eddington standard model to compute how the luminosity of a star will vary with its mass.

In the Eddington standard model, we assume that $\beta = 1 - L/L_{\mathrm{Edd}}$ is constant throughout the star. If we apply this at the star's surface, and use Equation 37 to evaluate $1 - \beta$, we have

$$\begin{aligned} L &= (1-\beta) L_{\mathrm{Edd}} && (40) \\ &= \left(\frac{aG^3}{3\mathcal{R}^4}\right) \frac{\pi}{16\xi_1^4 (d\Theta/d\xi)_{\xi_1}^2} \mu^4 M^2 \left(\frac{4\pi c G}{\kappa_s} M\right) && (41) \\ &= \frac{\pi^2}{12\xi_1^4 (d\Theta/d\xi)_{\xi_1}^2} \frac{acG^4}{\mathcal{R}^4 \kappa_s} \mu^4 \beta^4 M^3 && (42) \\ &= 5.5\, \beta^4 \left(\frac{\mu}{0.61}\right)^4 \left(\frac{1 \text{ cm}^2 \text{ g}^{-1}}{\kappa_s}\right) \left(\frac{M}{M_\odot}\right)^3 L_\odot, && (43) \end{aligned}$$

where κ_s is the value of κ_R at the stellar surface. We have thus derived, for the first time, a theoretical mass-luminosity relation.

We can get some idea of how this mass-luminosity relation behaves by solving the Eddington quartic in the limits of high and low masses. First consider stars with masses $\sim M_\odot$ or less. For these stars, the term

$$0.004 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\mu}{0.61}\right)^4 \beta^4 \tag{44}$$

in Equation 39 is very small. Thus the solution is near $\beta = 1$. If κ_s doesn't vary much between stars, then at these masses we therefore have $L \propto M^3$.

On the other hand, consider very massive stars, those with $M \sim 100\ M_\odot$ or more. In this case the coefficient of β^4 in Equation 39 is large. We can very roughly approximate the solution in that case by dropping the β term, which gives

$$0.004 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\mu}{0.61}\right)^4 \beta^4 \approx 1 \qquad \Longrightarrow \qquad \beta^4 \propto M^{-2}. \tag{45}$$

The approximation is rough because, even for $M = 100\ M_\odot$, the coefficient of the β^4 term is only 4.

Nonetheless, plugging this into the mass-luminosity relation gives $L \propto \beta^4 M^3 \propto M$. Thus we expect that for very massive stars the mass-luminosity relation should flatten and approach $L \propto M$. Again, this expectation is assuming constant surface opacity κ_s, which is an ok approximation, but not a great one, since the surface temperature varies significantly between low and high mass stars, and the opacity therefore varies as well.

This rough trend that at low masses $L \propto M^3$ (it's actually a bit closer to 3.5 in reality), flattening to $L \propto M$ at high masses, is actually seen in the observations. Thus this model at least roughly reproduces reality.

D. Implications of the Eddington standard model for stellar evolution

The Eddington standard model, although it does not in itself know anything about nuclear physics and thus about stellar evolution, nonetheless makes useful predictions for stellar evolution. From the standpoint of the model, the main effect of stellar evolution is that, as stars burn hydrogen into helium, the mean molecular mass μ must increase.

Recall that we showed that the mean molecular mass obeys

$$\frac{1}{\mu} = \frac{1}{\mu_I} + \frac{1}{\mu_e}, \tag{46}$$

where μ_I is the mean mass per ion and μ_e is the mean mass per electron. These in turn can be written in terms of the hydrogen mass fraction X, helium mass fraction Y, and metal mass fraction Z as

$$\frac{1}{\mu_I} = X + \frac{Y}{4} + \frac{Z}{\langle \mathcal{A} \rangle_{\mathrm{metals}}} \tag{47}$$

$$\frac{1}{\mu_e} \approx X + \frac{Y}{2} + \frac{Z}{2}, \tag{48}$$

where $\langle \mathcal{A} \rangle_{\mathrm{metals}} \approx 20$ for the Sun, and the approximation for μ_e follows from assuming that metals have equal numbers of protons and neutrons.

The present-day Sun has $X = 0.707$, $Y = 0.274$, and $Z = 0.019$, so and plugging this in gives $\mu_I = 1.29$, $\mu_e = 1.17$, and $\mu = 0.61$. This is for a composition where the hydrogen mass is roughly three times the helium mass. Now consider a star that has burned much of its hydrogen into helium, so the ratio is reversed, say $X = 0.281$, $Y = 0.7$, $Z = 0.019$. This star has $\mu_I = 2.19$, $\mu_e = 1.56$, and $\mu = 0.91$. What does this do to its luminosity?

First consider a low-mass star, one for which $M \sim 1\ M_\odot$, and thus $\beta \approx 1$. In this case, Equation 43 tells us that $L \propto \mu^4$, so the luminosity will rise by a factor of $(0.91/0.61)^4 = 5.0$. Thus the star's luminosity increases by a factor of 5.

For a high-mass star, one where we approximate that

$$0.004 \left(\frac{M}{M_\odot}\right)^2 \left(\frac{\mu}{0.61}\right)^4 \beta^4 \approx 1, \tag{49}$$

we have $\beta \propto 1/\mu$. Since Equation 43 tells us that $L \propto \beta^4 \mu^4$, it follows that for this star the luminosity will remain roughly unchanged.

Thus the prediction of the Eddington standard model is that low mass stars will brighten a significant amount over the course of their lives as they burn hydrogen to helium, but that the effect will be less for more massive stars. Detailed numerical calculations again bear out this prediction, and the increase in luminosity over time is a large part of the reason that the main sequence for stars near the Sun, which have a wide range of ages, is significantly wider than the main sequence for stars in a star cluster, which are all nearly the same age.

This prediction also works the other way in time. The Sun is currently richer in helium than it was when it formed, due to the fact that it has been fusing hydrogen to helium for some time. Its birth composition was probably closer to $X = 0.741$, $Y = 0.240$, $Z = 0.019$, corresponding to $\mu = 0.598$. With the scaling $L \propto \mu^4$, this predicts that at birth the Sun was only about 90% as luminous as it is now. In fact, this somewhat underestimates the effect; detailed models give about 70% instead of about 90%. This gives rise to a problem in geology known as the faint young Sun paradox: if the Sun was really this much less luminous 4.5 billion years ago, then naively one would expect that the Earth's surface would have been too cool to allow water to exist as a liquid. It should all have frozen. This contradicts the geologic record, which suggests that surface water has been present in liquid form over almost all of the Earth's history.

The resolution to the paradox is not entirely clear, but leading hypotheses include that the Earth was kept warmer due to the greenhouse effect and that the Earth was warmed by the release of energy due to the radioactive decay of long-lived isotopes present in interstellar space that were incorporated into the Earth when it formed, and/or the operation of natural nuclear reactors such as the Oklo natural reactor.