In this class we continue the process of filling in the missing microphysical details that we need to make a stellar model. To recap, in the last two classes we computed the pressure of stellar material and the rate of energy transport through the star. These were two of the missing pieces we needed. The third, which we'll sketch today, is the rate for nuclear reactions, and the energy that they generate.

I. Energetics

A. Energy Release

All nuclear reactions fundamentally work by converting mass into energy. (In some ways the same could be said of chemical reactions, but for those the masses involved are so tiny as to not be worth worrying about.) The masses of the reactants involved therefore determine the energy released by the reaction.

Consider a reaction between two species that produced some other species

$$\mathcal{I}(\mathcal{A}_i, \mathcal{Z}_i) + \mathcal{J}(\mathcal{A}_j, \mathcal{Z}_j) \rightarrow \mathcal{K}(\mathcal{A}_k, \mathcal{Z}_k) + \mathcal{L}(\mathcal{A}_l, \mathcal{Z}_l), \tag{1}$$

where as usual $\mathcal{Z}$ is the atomic number and $\mathcal{A}$ is the atomic mass. At this point we must distinguish between atomic mass and actual mass, so let $\mathcal{M}$ be the mass of a given species. The atomic mass times m_H and the true mass are nearly identical, $\mathcal{M} \approx \mathcal{A} m_\mathrm{H}$, but not quite, and that small difference is the source of energy for the reaction. For the reaction we have written down, the energy released is

$$Q_{ijk} = (\mathcal{M}_i + \mathcal{M}_j - \mathcal{M}_k - \mathcal{M}_l)c^2, \tag{2}$$

i.e., the initial mass minus the final mass, multiplied by c^2.

To remind you, we showed a few classes ago that the rate per unit volume at which the reaction we have written down occurs is

$$\frac{\rho^2}{m_\mathrm{H}^2} \left(\frac{1}{1+\delta_{ij}}\right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk}, \tag{3}$$

where R_{ijk} is the rate coefficient. If each such reaction released an energy Q_{ijk}, then the rate of nuclear energy release per unit volume is simply given by this rate, multiplied by Q_{ijk}, and summed over all possible reactions:

$$\frac{\rho^2}{m_\mathrm{H}^2} \sum_{ijk} \left(\frac{1}{1+\delta_{ij}}\right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk}. \tag{4}$$

The rate of nuclear energy release per unit mass is just this divided by the mass per volume ρ:

$$q_\mathrm{nuc} = \frac{\rho}{m_\mathrm{H}^2} \sum_{ijk} \left(\frac{1}{1+\delta_{ij}}\right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk}. \tag{5}$$

If the reaction produces neutrinos, they will carry away some of the energy and escape the star, and thus the amount by which the star is heated will be reduced. However this loss is small in most stars under most circumstances.

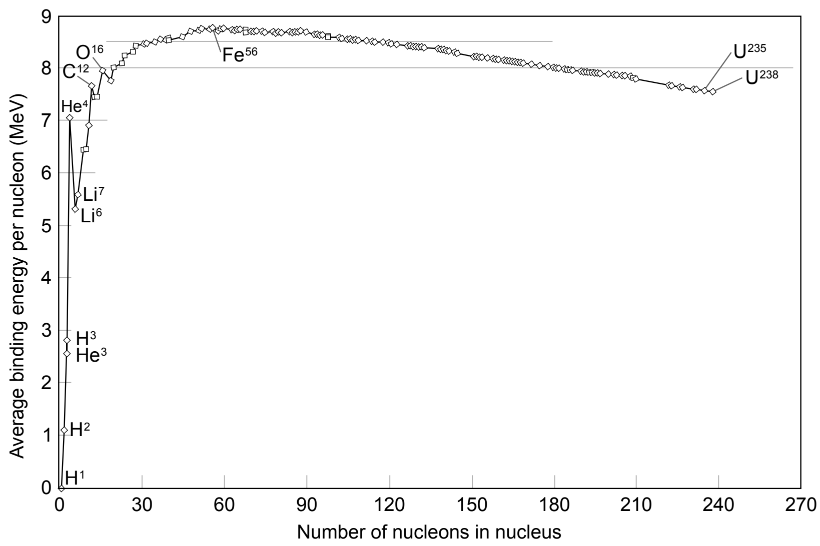

Figure 1: Binding energy per nucleon versus number of nucleons. Source: https://en.wikipedia.org/wiki/Nuclear_binding_energy.

B. Binding energy per nucleon

A very useful way to think about the amount of energy available in nuclear reactions is to compute the binding energy per nucleon. Suppose that we start with hydrogen, which consists of one proton of mass m_{H} (ignoring electrons), and we define that to have zero binding energy. Since binding energy is potential energy, we can do this, since we can choose the zero of potential energy to be anywhere.

Now consider some other element, with atomic number $\mathcal{A}$ and mass $\mathcal{M}$, and consider how much energy is released in the process of making that element from hydrogen. The exact reaction processes used don't matter, just the initial and final masses. Since atomic number is conserved, we must use $\mathcal{A}$ hydrogen atoms to make the new nucleus, so the difference between the final and initial mass is $\mathcal{M} - \mathcal{A} m_{\mathrm{H}}$. We define the mass excess as this quantity multiplied by c^2:

$$\Delta M = (\mathcal{M} - \mathcal{A} m_{\mathrm{H}})c^2. \tag{6}$$

This is just the difference in energy between the bound nucleus and the equal number of free protons. The name is somewhat confusing, since this is really an energy not a mass. The reason for the name is that in relativity one doesn't really need to distinguish between mass and energy. They're the same thing, just measured in different units.

A more useful quantity than this is the binding energy per nucleon, i.e., minus the mass excess divided by the number of nucleons (protons or neutrons) in the nucleus. The minus here is added so that the binding energy is positive if the nucleus is more strongly bound than the corresponding number of free nucleons. Thus we define the binding energy per nucleon as

$$-\frac{\Delta M}{\mathcal{A}} = \left(1 - \frac{\mathcal{M}}{\mathcal{A} m_{\mathrm{H}}}\right) m_{\mathrm{H}} c^2. \tag{7}$$

Since $\mathcal{M}$ and $\mathcal{A}$ can be determined experimentally, this quantity is fairly straightforward to measure. The results are very illuminating, as shown in Figure 1.

This plot contains an enormous amount of information, and looking at it immediately explains a number of facts about stars and nuclear physics. To interpret this plot, recall that number of nucleons is conserved by nuclear reactions. Thus any nuclear reaction just involves taking a fixed number of nucleons and moving them to the left or right on this plot. The energy released or absorbed in the process is just the number of nucleons involved multiplied by the change in binding energy per nucleon.

The first thing to notice about this plot is that there is a maximum at ^{56}Fe – iron-56. This is the most bound nucleus. At smaller atomic masses the binding energy per nucleon generally increases with atomic number, while at larger atomic masses it decreases. This marks the divide between fusion and fission reactions. At atomic masses below 56, energy is released by increasing the atomic number, so fusion is exothermic and fission is endothermic. At atomic number above 56, energy is released by decreasing the atomic number, so fission is exothermic and fission is endothermic.

Second, notice that the rise is very sharp at small atomic number. This means that fusing hydrogen into heavier things is generally the most exothermic reaction available, and that it releases far more energy per nucleon than later stages of fusion, say helium into carbon. This has important implications for the fate of stars that exhaust their supply of hydrogen.

Third, notice that there are several local maxima and minima at small atomic number. ^{4}He is a maximum, as are ^{12}C and ^{16}O. There is a good reason that helium, carbon, and oxygen are the most common elements in the universe after hydrogen: they are local maxima of the binding energy, which means that they are the most strongly bound, stable elements in their neighbourhood of atomic number. Conversely, lithium is a minimum. For this reason nuclear reactions in stars destroy lithium, and they do not produce it.

Finally, notice that these are big numbers as far as the energy yield. The scale on this plot is MeV per nucleon. In terms of more familiar units, 1 MeV per H mass corresponds to 10^{18} erg g^{-1}, or roughly 22 tons of TNT per gram of hydrogen fuel.

II. Reaction Rates

A. The Coulomb Barrier

The binding energy curve tells us the amount of energy available from nuclear reactions, but not the rates at which they occur. Given that the reaction for fusing hydrogen to helium is highly exothermic, why doesn't the reaction happen spontaneously at room temperature? The answer is the same as the reason that coal doesn't spontaneously combust at room temperature: the reaction has an activation energy, and that energy is quite high.

To understand why, consider the potential energy associated with two nuclei of charge $\mathcal{Z}_i$ and $\mathcal{Z}_j$ separated by a distance r. The Coulomb (electric) potential energy is

$$U_C = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{r} = \mathcal{Z}_i \mathcal{Z}_j \frac{1.4 \text{ MeV}}{r/\text{fm}}, \tag{8}$$

where 1 fm $= 10^{-13}$ cm $= 10^{-15}$ m. Since this is positive, the force between the protons is repulsive, as it should be.

In addition to that positive energy, there is a negative energy associated with nuclear forces. The full form of the proton-proton force is complicated, but we can get an idea of its behavior by noting that, at larger ranges, it is mediated by the exchange of virtual mesons such as pions. Because these particles have mass, their range is limited by the Heisenberg uncertainty principle: they can only exist for a short time, and they only exert significant force at distances they can reach within that time. Specifically, the uncertainty principle tells us that

$$\Delta E \, \Delta t \geq \frac{\hbar}{2} \tag{9}$$

If the particle travels at the maximum possible speed of c, its range is roughly

$$r \sim c\Delta t \sim \frac{c\hbar}{E}, \tag{10}$$

where E is the rest energy of the particle being exchanged. For pions, which mediate the proton-proton force, $\Delta E = 135$ MeV or 140 MeV, depending on whether they are neutral or charged. Plugging this in for ΔE gives $r \sim 1$ fm. Thus the nuclear force is negligible at distances greater than ~ 1 fm. Within that range, however, the nuclear force is dominant. Potentials arising from exchanges of massive particles like this are called Yukawa potentials, and they have the form

$$U_{\mathrm{Y}} = -g^2 \frac{e^{-r/\lambda}}{r}, \tag{11}$$

where g is a constant and $\lambda = c\hbar/E$ is the range of the force. This is only an approximation to the true potential energy, but it is reasonably good one at large ranges. The total potential is the sum of the Yukawa and Coulomb potentials. The functional form of this potential is something like a $1/r$ rise that is cut off by a sharp decrease at small radii.

For the reaction to proceed, the two particles must get close enough to one another to reach the region where the potential drops, and the force becomes attractive. If they do not, they will simply bounce off one another without reacting. This is called the Coulomb barrier, and it applies to chemical as well as nuclear reactions. The existence of the Coulomb barrier means that there is a minimum relative velocity the particles must have in order for the reaction to go, which we can calculate from the height of the Coulomb barrier. This is much like rolling a ball up a steep hill with a peak – there is a minimum velocity with which you must roll the ball if you want it to reach the top of the hill.

Suppose that the potential follows the Coulomb form until some minimum radius $r_0 \sim 1$ fm, then suddenly drops at smaller radii. The maximum potential energy is

$$U_C = \frac{\mathcal{Z}_i \mathcal{Z}_j e^2}{r_0} = \mathcal{Z}_i \mathcal{Z}_j \frac{1.4 \text{ MeV}}{r_0/\text{fm}}. \tag{12}$$

The minimum relative velocity of the particles is given by the condition that the kinetic energy in the centre of mass frame exceed this value:

$$\frac{1}{2}\mu_{\mathrm{red}} m_{\mathrm{H}} v^2 \geq U_c, \tag{13}$$

where $\mu_{\mathrm{red}} m_{\mathrm{H}}$ is the reduced mass.

A more useful calculation than this is to ask what temperature the gas must have such that the typical collision is at sufficient velocity for the reaction to occur. The typical collision energy is

$$\frac{1}{2}\mu_{\mathrm{red}} m_{\mathrm{H}} v^2 = \frac{3}{2} k_B T, \tag{14}$$

so setting this equal to U_C and solving gives

$$T \geq \frac{2\mathcal{Z}_i\mathcal{Z}_j e^2}{3k_B r_0} = 1.1 \times 10^{10}\ \mathrm{K}\ \frac{\mathcal{Z}_i\mathcal{Z}_j}{r_0/\mathrm{fm}}. \tag{15}$$

Thus the typical particle does not have enough energy to penetrate the Coulomb barrier until the temperature is $\sim 10^{10}$ K for proton-proton reactions, and even higher temperatures for higher atomic numbers. This is much higher than the temperatures for stars' centres than we estimated earlier in the class. You might think that it's not a problem because some particles move faster than the average, and thus are going fast enough to penetrate the Coulomb barrier. You will show on your homework that this solution doesn't work. At the temperature of $\sim 10^7$ K in the centre of the Sun, if this calculation is correct then fusion should not be possible.

B. Quantum Tunneling

The resolution to this problem lies in the phenomenon of quantum tunneling. The calculation we just did is based on classical physics, and predicts that no nuclei will get within r_0 of one another unless they reach a high enough velocity to overcome the Coulomb barrier. However, in quantum mechanics there is a non-zero probability of finding the particle inside r_0 even if it does not have enough energy to break the Coulomb barrier. This phenomenon is known as tunneling, because it is like the particle takes a tunnel through the peak rather than going over it.

We can make a crude estimate of when tunneling will occur using wave-particle duality. Recall that each proton can be thought of as a wave whose wavelength is dictated by the uncertainty principle. The wavelength associated with a particle of momentum p is

$$\lambda = \frac{h}{p}. \tag{16}$$

This is known as the particle's de Broglie wavelength.

As a rough estimate of when quantum tunneling might allow barrier penetration, we can estimate that the two particles must be able to get within one de Broglie wavelength of one another. This in turn requires that the kinetic energy of the particles be equal to their Coulomb potential energy at a separation of one de Broglie wavelength:

$$\frac{\mathcal{Z}_i\mathcal{Z}_j e^2}{\lambda} = \frac{1}{2}\mu_{\mathrm{red}} m_{\mathrm{H}} v^2 = \frac{p^2}{2\mu_{\mathrm{red}} m_{\mathrm{H}}} = \frac{h^2}{2\mu_{\mathrm{red}} m_{\mathrm{H}} \lambda^2} \tag{17}$$

Solving this for λ, we find that barrier penetration should occur is the particles are able to get within a distance

$$\lambda = \frac{h^2}{2\mu_{\mathrm{red}} m_{\mathrm{H}} \mathcal{Z}_i\mathcal{Z}_j e^2}. \tag{18}$$

of one another.

To figure out the corresponding temperature, we can just evaluate our result from the classical problem using λ in place of r_0:

$$T \geq \frac{2\mathcal{Z}_i\mathcal{Z}_j e^2}{3k_B \lambda} = \frac{4\mathcal{Z}_i^2\mathcal{Z}_j^2 e^4 \mu_{\mathrm{red}} m_{\mathrm{H}}}{3h^2 k_B} = 9.6 \times 10^6\ \mathrm{K}\ \mathcal{Z}_i^2\mathcal{Z}_j^2 \left(\frac{\mu_{\mathrm{red}}}{1/2}\right). \tag{19}$$

Thus proton-proton reactions, which have $\mathcal{Z}_i = \mathcal{Z}_j = 1$ and $\mu_{\mathrm{red}} = 1/2$, should begin to occur via quantum tunneling at a temperature of $\sim 10^7$ K, much closer to the temperatures we infer in the centre of the Sun.

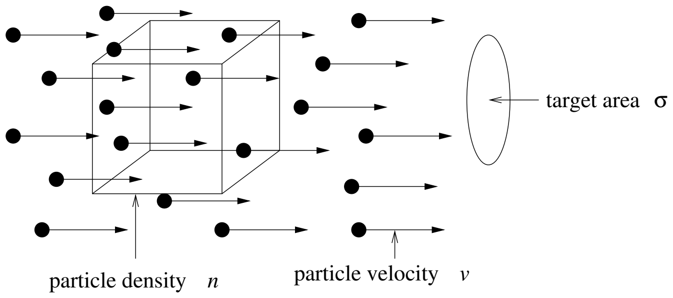

Figure 2: Schematic of the calculation of a reaction rate from a density, velocity, and cross-section.

C. The Gamow Peak

Having seen that quantum effects are important, we will now try to perform a more rigorous calculation of the reaction rate. Consider reactions between two nuclei with number densities n_i and n_j in a gas at temperature T. In order to compute the reaction rate, we need to know the rate at which these nuclei collide with enough energy to tunnel through the Coulomb barrier. That's what we'll calculate now.

The first step is to compute the rate at which particles strike one another closely enough to interact. This is very much like calculating the pressure. We consider a particle, and we want to know how often other particles run into it. If we had a beam of particles of density n and velocity v, and the target particle had a cross-sectional area σ, the impact rate would be $n\sigma v$. Note that this formula is almost exactly like the one describing the rate at which particles strike the wall of a vessel, which we used to compute pressures. Figure 2 illustrates the situation.

In reality the particle in question isn't a solid target with a fixed area. We're interested in interactions that lead to reactions, which require that the collision be close enough to allow the nuclei to tunnel through the Coulumb barrier, but also require that the interaction have enough energy to make such tunneling possible. A direct bullseye at a very low energy won't lead to a reaction, so the cross-section at very low energies is basically zero. However, we can still extend the analogy of shooting a beam of particles at a target by defining the cross-section at energy E. Let $dN_{\text{reac}}(E)/dt$ be the number of reactions per time interval dt produced by shooting a beam of particles of density n at velocity v at a target nucleus. We define the cross-section $\sigma(E)$ via the relation

$$\frac{dN_{\text{reac}}(E)}{dt} = n\sigma(E)v(E). \tag{20}$$

Next we want to generalise from a the case of a beam to the case of a thermal gas where not all particles have the same energy. We proved a few classes ago that the momentum distribution of particles of mass m at temperature T is

$$\frac{dn}{dp} = \frac{4n}{\pi^{1/2}(2mk_BT)^{3/2}} p^2 e^{-p^2/(2mk_BT)}. \tag{21}$$

Since we're interested in particle energies, we'll change this to a distribution over energy instead of momentum. Since $E = p^2/(2m)$, or $p = \sqrt{2mE}$, we have

$$\frac{dn}{dE} = \frac{dn}{dp}\frac{dp}{dE} = \frac{4n}{\pi^{1/2}(2mk_BT)^{3/2}} p^2 e^{-p^2/(2mk_BT)} \cdot \sqrt{\frac{m}{2E}} = \frac{2n}{\pi^{1/2}(k_BT)^{3/2}} E^{1/2} e^{-E/k_BT}. \tag{22}$$

Note that this only applies to non-relativistic particles, since we used $E = p^2/(2m)$ instead of $E = pc$. However, nuclei are generally always non-relativistic, except in neutron stars.

In this case, the number of reactions dN per time interval dt that a given target nucleus undergoes is given by integrating over the possible energies of the impacting particles. In particular, the number of reactions per unit time for a particle of species i due collisions with particles of species j is

$$\frac{dN_i}{dt} = \int_0^\infty \sigma(E)v(E)\frac{dn_j}{dE}\,dE. \tag{23}$$

Since the velocity that matters here is the relative velocity, we have to compute it in terms of the reduced mass: $v(E) = \sqrt{2E/\mu_{\text{red}}}$, where $\mu_{\text{red}} = m_i m_j/(m_i + m_j)$. Finally, if we want to know the number of reactions per unit time in a given volume of gas, we just have to multiply this by the number of target nuclei per unit volume, n_i. This gives

$$\frac{dn_{\text{reac}}}{dt} = n_i \int_0^\infty \sigma(E)v(E)\frac{dn_j}{dE}\,dE. \tag{24}$$

If the reaction is of a species with itself, we have to multiply by an additional factor of 1/2 to avoid double-counting.

Recall that we defined the rate coefficient R_{ijk} so that the reaction rate is $R_{ijk}n_i n_j$ for different species, or $R_{ijk}n_i^2/2$ for two of the same species. Thus the rate coefficient is

$$\begin{aligned} R_{ijk} &= \frac{1}{n_i n_j}\frac{dn_{\text{reac}}}{dt} && (25)\\ &= \frac{2}{\pi^{1/2}}\frac{1}{(k_B T)^{3/2}}\int_0^\infty \sigma(E)v(E)E^{1/2}e^{-E/k_B T}\,dE && (26)\\ &= \frac{1}{(\pi\mu_{\text{red}}m_{\text{H}})^{1/2}}\left(\frac{2}{k_B T}\right)^{3/2}\int_0^\infty \sigma(E)Ee^{-E/k_B T}\,dE && (27) \end{aligned}$$

The final remaining step is to figure out the cross-section $\sigma(E)$ at energy E. Computing this in general is quite difficult, and often laboratory measurements are required to be sure of exact values. However, we can get a rough idea of how $\sigma(E)$ varies with energy based on general quantum-mechanical principles. The first such principle is that particles should interact when they come within distances that are comparable to their de Broglie wavelengths – a higher energy particles has a smaller wavelength, and thus represents a smaller target. Thus we expect that

$$\sigma(E) \propto \lambda^2 \propto \frac{h^2}{p^2} \propto \frac{1}{E}. \tag{28}$$

The second principle is that nuclear reactions like the ones we are interested in require tunneling through the Coulomb barrier. A quantum mechanical calculation of the probability that tunneling will occur shows that it is proportional to

$$e^{-2\pi^2 U_C/E}, \tag{29}$$

where U_C is the height of the Coulomb barrier at a distance of one de Broglie wavelength. In terms of the energy, the Coulomb barrier U_C is

$$U_C = \frac{\mathcal{Z}_i\mathcal{Z}_j e^2}{\lambda} = \frac{\mathcal{Z}_i\mathcal{Z}_j e^2 p}{h} = \frac{\mathcal{Z}_i\mathcal{Z}_j e^2}{h}\sqrt{2\mu_{\text{red}}m_{\text{H}}E}, \tag{30}$$

so the exponential factor is

$$2\pi^2 \frac{U_C}{E} = 2^{3/2}\pi^2 \frac{\mu_{\rm red}^{1/2} m_{\rm H}^{1/2} \mathcal{Z}_i \mathcal{Z}_j e^2}{h} E^{-1/2} \equiv b E^{-1/2}, \tag{31}$$

where

$$b = 2^{3/2}\pi^2 \frac{\mu_{\rm red}^{1/2} m_{\rm H}^{1/2} \mathcal{Z}_i \mathcal{Z}_j e^2}{h} = 0.0013 \mu_{\rm red}^{1/2} \mathcal{Z}_i \mathcal{Z}_j \ ({\rm erg})^{1/2}. \tag{32}$$

Thus we also expect to have $\sigma \propto e^{-bE^{-1/2}}$. Note that the factor b depends only on the charges and masses of the nuclei involved in the reaction. It is therefore constant for any given reaction.

Combining the two factors our analysis reveals, we define

$$\sigma(E) \equiv \frac{S(E)}{E} e^{-bE^{-1/2}}, \tag{33}$$

where $S(E)$ is, ideally, either a constant or a function that varies only very, very weakly with E. Plugging all this in, the reaction rate coefficient is

$$R_{ijk} = \frac{1}{(\pi \mu_{\rm red} m_{\rm H})^{1/2}} \left(\frac{2}{k_B T}\right)^{3/2} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/k_B T}\, dE. \tag{34}$$

It is instructive to look at the behaviour of the two exponential factors, $e^{-bE^{-1/2}}$ and e^{-E/k_BT}. Clearly the first function increases as E increases, while the second one decreases as E increases. We therefore expect their product to reach a maximum at some intermediate energy. In fact, we can compute the maximum analytically, by taking the derivative and setting in equal to zero:

$$\begin{aligned}
0 &= \frac{d}{dE}\left(e^{-bE^{-1/2}} e^{-E/k_BT}\right) && (35)\\
&= \frac{d}{dE} e^{-(E/k_BT + bE^{-1/2})} && (36)\\
&= -\left(\frac{E}{k_BT} + bE^{-1/2}\right)\left(\frac{1}{k_BT} - \frac{b}{2E^{3/2}}\right) e^{-(E/k_BT + bE^{-1/2})} && (37)\\
E_0 &= \left(\frac{b k_B T}{2}\right)^{2/3} && (38)\\
&= 1.22 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\rm red} \left(\frac{T}{10^6\ {\rm K}}\right)^2\right]^{1/3} \ {\rm keV}, && (39)
\end{aligned}$$

where E_0 is the energy at the maximum. This maximum is known as the Gamow peak, after George Gamow, who discovered it in 1928.

If we let $x = E/E_0$, then we can rewrite the reaction rate coefficient as

$$\begin{aligned}
R_{ijk} &= \frac{E_0}{(\pi \mu_{\rm red} m_{\rm H})^{1/2}} \left(\frac{2}{k_BT}\right)^{3/2} \int_0^\infty S(x) \exp\left[-\left(\frac{b^2}{4k_BT}\right)^{1/3}\left(x + \frac{2}{x^{1/2}}\right)\right] dx && (40)\\
&= \left[2^{11}\pi^5 \frac{\mathcal{Z}_i^4 \mathcal{Z}_j^4 e^8}{\mu_{\rm red} m_{\rm H} (k_BT)^5}\right]^{1/6} \int_0^\infty S(x) \exp\left[-\left(\frac{b^2}{4k_BT}\right)^{1/3}\left(x + \frac{2}{x^{1/2}}\right)\right] dx && (41)
\end{aligned}$$

To get a sense of how narrowly peaked this function is, it is helpful to evaluate the factor $[b^2/(4k_BT)]^{1/3}$ for some typical examples. If we consider proton-proton interactions (so $\mathcal{Z}_i = \mathcal{Z}_j = 1$ and $\mu_{\rm red} = m_{\rm H}/2$) at the Sun's central temperature of 1.57×10^7 K, then we have

$$b = 8.8 \times 10^{-4}\ ({\rm erg})^{1/2} \text{ and } \left(\frac{b^2}{4k_BT}\right)^{1/3} = 4.5. \tag{42}$$

Evaluating the function $e^{-4.5(x+2/x^{1/2})}$ shows that for $x = 3$ (i.e. at energies three times the peak), it is a factor of 180 lower than it is at peak. For $x = 1/3$ (i.e. at energies three times below the peak), it is 35 times smaller than it is at peak. Thus the reaction rate is strongly dominated by energies near the peak, with energies different from the peak by even as little as a factor of 3 contributing negligibly.

When we are near the peak, i.e. $x \approx 1$, the reaction rate varies exponential with the quantity $[b^2/(k_BT)]^{1/3}$. This means that the reaction rate is extremely sensitive to temperature. For this reason, we often think of nuclear reactions as having a threshold temperature at which they turn on. This threshold temperature clearly increases with nuclear charge: since $b \propto \mathcal{Z}_i\mathcal{Z}_j$, and the reaction rate depends on b^2/T, we expect the temperature needed to ignite a particular reaction to vary as $\mathcal{Z}_i^2\mathcal{Z}_j^2$. Thus higher $\mathcal{Z}$ nuclei require progressively higher temperatures to fuse.

Of course we still have not assigned a value of $S(E)$ near the Gamow peak. We have only said that we expect it to be nearly constant. Its actual value depends on the reaction in question and the type of physics it involves, and must be obtained either by laboratory measurement, theoretical quantum calculation, or a combination of both. Unfortunately these values sometimes have significant uncertainties. In a star, reactions can occur at an appreciable rate at relatively low temperatures because the density is high – recall that the reaction rate per unit volume varies as n_in_j. In a laboratory, we have to work with much lower densities, and as a result the reaction rates at the temperatures found in stars are often unobservably small. Instead, we are forced to make measurements at higher temperatures and extrapolate.

D. Temperature Dependence of Reaction Rates

It is often helpful to know roughly how the reaction rate varies with temperature when one is near the ignition temperature. To find that out, we can approximately evaluate the integral in the formula for the rate coefficient. As a first step in this approximation, we neglect any variation in the $S(E)$ factor across the Gamow peak, and simply set it equal to a constant value $S(E_0)$. Thus the reaction rate coefficient is approximately

$$R_{ijk} = \frac{1}{(\pi\mu_{\rm red}m_{\rm H})^{1/2}}\left(\frac{2}{k_BT}\right)^{3/2} S(E_0)\int_0^\infty \exp\left(\frac{E}{k_BT} - \frac{b}{E^{1/2}}\right)dE. \tag{43}$$

The maximum value of the integrand occurs when $E = E_0$, and is given by

$$I_{\rm max} \equiv \exp\left(-\frac{3E_0}{k_BT}\right) \equiv e^{-\tau}, \tag{44}$$

where we define

$$\tau = \frac{3E_0}{k_BT} = 42.46\left[\mathcal{Z}_i^2\mathcal{Z}_j^2\mu_{\rm red}\left(\frac{T}{10^6\ {\rm K}}\right)^{-1}\right]^{1/3} \tag{45}$$

The second step in the approximation is to approximate the exponential factor in the integral by a Gaussian of width Δ:

$$\exp\left(\frac{E}{k_B T} - \frac{b}{E^{1/2}}\right) \approx I_{\max} \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right]. \tag{46}$$

The width Δ is generally chosen by picking the value such that the second derivatives of the exact and approximate forms for the integrand are equal at $E = E_0$. A little algebra shows that this gives

$$\Delta = \frac{4}{\sqrt{3}} \left(E_0 k_B T\right)^{1/2}. \tag{47}$$

The approximation is reasonably good.

The final step in the approximation is to change the limits of integration from 0 to ∞ to $-\infty$ to ∞. This is not a bad approximation because the vast majority of the power in the Gaussian occurs at positive energies, and if the limits are $-\infty$ to ∞, the integral can be done exactly:

$$\int_{-\infty}^{\infty} \exp\left[-\left(\frac{E - E_0}{\Delta/2}\right)^2\right] dE = \frac{\sqrt{\pi}}{2}\Delta. \tag{48}$$

With this approximation complete, we can write the reaction rate coefficient as

$$R_{ijk} = \frac{1}{(\pi \mu_{\mathrm{red}} m_{\mathrm{H}})^{1/2}} \left(\frac{2}{k_B T}\right)^{3/2} S(E_0) I_{\max} \frac{\sqrt{\pi}}{2}\Delta \tag{49}$$

$$= I_{\max} \left(\frac{2}{\mu_{\mathrm{red}} m_{\mathrm{H}}}\right)^{1/2} \frac{\Delta}{(k_B T)^{3/2}} S(E_0). \tag{50}$$

We can rewrite this in terms of τ by substituting in for Δ and $k_B T$ in terms of τ. Doing so and simplifying a great deal produces

$$R_{ijk} = \frac{4}{3^{5/2}\pi^2} \frac{h}{\mu_{\mathrm{red}} m_{\mathrm{H}} \mathcal{Z}_i \mathcal{Z}_j e^2} S(E_0) \tau^2 e^{-\tau}. \tag{51}$$

All the temperature-dependence is encapsulated in the $\tau^2 e^{-\tau}$ term. The factor τ itself varies as

$$\tau \propto \frac{E_0}{T} \propto T^{-1/3}. \tag{52}$$

It is often useful to approximate the reaction rate as a powerlaw in T, i.e. to set $R_{ijk} \propto T^{\nu}$ for some power ν. Obviously the relationship is not a powerlaw in general, since there is an exponential in τ. However, we can approximate the behavior as a powerlaw if we are in the vicinity of a particular temperature T_0, near which $\tau = \tau_0 (T/T_0)^{-1/3}$. To understand what this entails, recall that a powerlaw is just a straight line in a log-log plot. In effect, fitting to a powerlaw is just the same as computing the slope at some point in the log-log plot. Thus we have

$$\nu = \frac{d \ln R_{ijk}}{d \ln T} \tag{53}$$

Since $R_{ijk} \propto \tau^2 e^{-\tau}$,

$$\ln R_{ijk} = 2 \ln \tau - \tau + \mathrm{const} = -\frac{2}{3} \ln T - \tau_0 \left(\frac{T}{T_0}\right)^{-1/3} + \mathrm{const} \tag{54}$$

Taking the derivative:

$$\nu = \frac{d\ln R_{ijk}}{d\ln T} = -\frac{2}{3} - \tau_0 T_0^{1/3} \frac{d}{d\ln T} T^{-1/3} \tag{55}$$

$$= -\frac{2}{3} - \tau_0 T_0^{1/3} T \frac{d}{dT} T^{-1/3} \tag{56}$$

$$= -\frac{2}{3} + \frac{\tau_0 T_0^{1/3}}{3T^{1/3}} \tag{57}$$

$$= \frac{\tau}{3} - \frac{2}{3} \tag{58}$$

This lets us approximate the behavior of R_{ijk} as a powerlaw:

$$R_{ijk} = R_{0,ijk} T^{(\tau-2)/3}. \tag{59}$$

We will use this in the next class to evaluated several of the important reactions inside stars. Given such a powerlaw fit, we can come up with an equivalent one for the rate of nuclear energy generation per unit mass when the gas temperature is near the ignition temperature for a given reaction:

$$q_{\text{nuc}} = \frac{\rho}{m_{\text{H}}^2} \sum_{ijk} \left(\frac{1}{1+\delta_{ij}}\right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R_{ijk} Q_{ijk} = \rho \sum_{ijk} \left(\frac{1}{1+\delta_{ij}}\right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} q_{0,ijk} T^{p_{ijk}}, \tag{60}$$

where $q_{0,ijk}$ and p_{ijk} are constants for a given reaction, i.e. they do not depend on gas density, element abundances, or gas temperature, as long as the temperature is near the ignition temperature.

E. Resonances and Screening

The simple model we have just worked out is reasonably good for many reactions of importance in stars, but it omits a number of complications, two of which we will discuss briefly.

First, the assumption that $S(E)$ varies weakly with E over the Gamow peak is not always valid. The most common way for the assumption to fail is if there is a resonance, which means that the energy of the collision corresponds closely to the energy of an excited state of the final product nucleus. If this happens, the cross section increases dramatically in a narrow range of energies, and $S(E)$ becomes sharply peaked. While none of the reactions involved in hydrogen burning in main sequence stars are resonant, some of the important reactions that occur in more evolved stars are. Resonances can enhance the reaction rate by orders of magnitude compared to what our our simple model would suggest.

A second complication is screening. Our calculation of the Coulomb barrier was based on the potential of two nuclei of charge $\mathcal{Z}_i$ and $\mathcal{Z}_j$ interacting with one another. However, this ignores the presence of electrons. For neutral atoms, the electric potential drops to zero for distances greater than a few angstroms, because the nucleus is surrounded by a cloud of electrons of equal and opposite charge. From a point outside the cloud, the net charge seen is zero, because the electronic and nuclear charges cancel – the electrons screen the nucleus. This is why neutral atoms do not violently repel one another.

In the fully ionized plasma inside a star electrons are not bound to atoms, and they float about freely. However, they are still attracted to the positively charged nuclei, and thus they tend to cluster around them, partly screening them. This effect reduces the Coulomb barrier. Screening is strongest at lower temperatures, since when $k_B T$ is smaller compared to the electric potential energy, electrons tend to to cluster more tightly around nuclei. This effect can enhance reaction rates for turning H into He by $\sim 10 - 50\%$ compared to the results of our naive calculation.

III. Nuclear chemistry in hydrogen-burning stars

Now that we have a general theory of nuclear reactions, we are in a position to characterise the nuclear reactions that are of importance in most stars.

A. The $p - p$ chain

The most important mechanism for generating power in the Sun is known as the $p - p$ chain, for proton proton chain. It is not surprising that the reaction involves protons, i.e., hydrogen nuclei. These are by far the most abundant nuclei in main sequence stars, and, since the strength of the Coulomb barrier scales as $\mathcal{Z}_i \mathcal{Z}_j$, it is also the reaction with the lowest Coulomb barrier. Thus it begins at the lowest temperature.

Before going into the details of the reaction, it is useful to re-examine the chart of binding energy per nucleon (Figure 1). Clearly the first big peak is at helium-4, so that is where we expect the reaction to go. However, getting there is not so easy, because all the stable nuclei shown in the chart except ${}^1_1$H contain neutrons. The reason is that neutrons are required to provide enough nuclear force to hold a nucleus together against the Coulomb repulsion of the protons. Thus the most obvious first step for a reaction involving two hydrogen nuclei doesn't work. We can't easily do

$$ {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_2\mathrm{He} \tag{61} $$

because ${}^2_2$He is not a stable nucleus. Any ${}^2_2$He made in such a manner almost immediately disintegrates into two protons, producing no net energy release.

Thus for a reaction to generate energy, one must find a way to bypass ${}^2_2$He and jump to a stable state. One possible solution to this problem was discovered by Hans Bethe in 1939: during the very brief period that ${}^2_2$He lives, a weak nuclear reaction can occur that converts one of the protons into a neutron plus a positron plus a neutrino. The positron and neutrino, which do not feel the strong nuclear force, immediately escape from the nucleus, leaving behind a proton plus a neutron. The proton plus neutron do constitute a stable nucleus: deuterium. The net reaction is exothermic, and the excess energy mostly goes into the recoil of the deuterium and positron. This excess energy is then turned into heat when the nuclei collide with other particle. The final reaction is

$$ {}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + e^+ + \nu_e \tag{62} $$

The electron neutrino, ν_e, escapes the star immediately, while the positron very quickly encounters an electron and annihilates, producing gamma rays which are then absorbed and converted into heat:

$$ e^+ + e^- \rightarrow 2\gamma, \tag{63} $$

where γ is the symbol for photon. As we'll discuss further in a moment, the proton-neutron conversion is very unlikely because it relies on the weak force, so the reaction coefficient for this reaction is very small compared to others in the chain. In terms of our earlier notation, $S(E_0)$ is very small for this reaction.

The next step in the chain is an encounter between the deuterium nucleus and another proton, producing helium:

$$ {}^{2}_{1}\mathrm{D} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{3}_{2}\mathrm{He} + \gamma. \tag{64}$$

This reaction goes very quickly compared to the first step, because the Coulomb barrier is the same (deuterium and ordinary hydrogen both have $\mathcal{Z} = 1$), but no weak nuclear forces are required.

For the last part of the chain, there are three possibilities, known as the pp I, pp II, and pp III branches. Branch I involves an encounter between two ${}^{3}_{2}$He nuclei produced in the previous step:

$$ {}^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \rightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H}. \tag{65}$$

This reaction has a Coulomb barrier four times higher than the first one, but, since it does not require a weak nuclear interaction, it actually proceeds faster than the first step. At this point the reaction stops, because ${}^{4}_{2}$He is stable, and there is no route from there to a more massive nucleus that is accessible at the temperatures of $\sim 10^7$ K where hydrogen burning occurs.

Branch II involves an encounter between the ${}^{3}_{2}$He and a pre-existing ${}^{4}_{2}$He nucleus to make beryllium, followed by capture of an electron to convert the beryllium to lithium, followed by capture of one more proton and fission of the resulting nucleus:

$$\begin{aligned} {}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{7}_{4}\mathrm{Be} + \gamma & (66)\\ {}^{7}_{4}\mathrm{Be} + e^{-} &\rightarrow {}^{7}_{3}\mathrm{Li} + \nu_e & (67)\\ {}^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} &\rightarrow 2\,{}^{4}_{2}\mathrm{He}. & (68) \end{aligned}$$

Finally, branch III involves production of beryllium-7 just like the first step of the pp II branch, but then an encounter between that and another proton to produce boron. The boron then decays to beryllium via positron emission, and finally ends at beryllium-8, which spontaneously splits:

$$\begin{aligned} {}^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} &\rightarrow {}^{7}_{4}\mathrm{Be} + \gamma & (69)\\ {}^{7}_{4}\mathrm{Be} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{8}_{5}\mathrm{B} + \gamma & (70)\\ {}^{8}_{5}\mathrm{B} &\rightarrow {}^{8}_{4}\mathrm{Be} + e^{+} + \nu_e & (71)\\ {}^{8}_{4}\mathrm{Be} &\rightarrow 2\,{}^{4}_{2}\mathrm{He}. & (72) \end{aligned}$$

As before, the positron produced in the third step immediately encounters an electron and annihilates into gamma rays.

Which of these chains is most important depends on the local density, temperature, and chemical composition. Obviously pp II and pp III are more likely when there is more ${}^{4}_{2}$He around, since they require it. In Sun, pp I is 69% of all reactions, pp II is 31%, and pp III is 0.1%.

The net reactions associated with these chains can be written:

$$ 4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \text{photons, neutrinos, and light particles,} \tag{73}$$

where the exact number of photons, neutrinos, and light particles depends on which branch is taken. The total energy release is given by subtracting the mass of He-4 from the mass of 4 protons, and is given by

$$\Delta E = (4m_{\rm H} - m_{\rm He})c^2 = 26.73 \text{ MeV}. \tag{74}$$

The actual amount of energy that goes into heating up the gas depends on the amount of energy carried away by neutrinos, which escape the star. This is different for each branch, because each branch involves production of a different number of neutrinos with different energies. The neutrino loss is 2.0% for pp I, 4.0% for pp II, and 28.3% for pp III.

In any of the pp branch, the first step, which requires spontaneous conversion of a proton into a neutron, is by far the slowest, and thus the rate at which it occurs controls the rate for the entire chain. For this reason, we can calculate the rate coefficient simply by knowing the properties of this reaction. The reaction begins to occur at an ignition temperature that is roughly equal $T_i = 4 \times 10^6$ K. The Sun's central temperature $T_0 \approx 1.57 \times 10^7$ K, which gives

$$\tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\rm red} \left(\frac{T}{10^6 \text{ K}} \right)^{-1} \right]^{1/3} = 13.5. \tag{75}$$

The reaction rate varies as temperature to roughly the 4th power. Measuring the value of $S(E_0)$ for this reaction lets us compute the rate coefficient. If we do not make the powerlaw approximation and just plug into

$$R \approx \frac{3}{3^{5/2}\pi^2} \frac{h}{\mu_{\rm red} m_{\rm H} \mathcal{Z}_i \mathcal{Z}_j e^2} S(E_0) \tau^2 e^{-\tau}, \tag{76}$$

we get

$$R \approx 6.34 \times 10^{-37} \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp\left[-\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right] \text{ cm}^{-3} \text{ s}^{-1}. \tag{77}$$

If we multiply this by the number density of protons, we get an estimate for the rate of reactions that a single proton undergoes. The inverse of this is the lifetime of a proton – the amount of time it takes for it to react with another proton and begin the reaction chain that will turn it into helium. Thus the proton lifetime is

$$t = \frac{1}{n_p R} = \frac{m_{\rm H}}{\rho X R} = \frac{8.3 \times 10^4 \text{ yr}}{X} \left(\frac{1 \text{ g cm}^{-3}}{\rho} \right) \left(\frac{T}{10^6 \text{ K}} \right)^{2/3} \exp\left[\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right]. \tag{78}$$

Plugging in a density of 100 g cm^{-3} and a temperature of 1.5×10^7 K, the result is a bit over 10^9 yr. Thus the typical proton in the centre of the Sun requires $> 10^9$ yr to undergo fusion. Averaging over a larger volume of the Sun, which has a lower density and temperature, makes the timescale even longer.

Finally, combining the reaction rate coefficient R with an energy release of $Q = 13.4$ MeV per reaction (since 26.73 MeV is what we get when we use 4 protons, and each pp reaction only uses 2), the corresponding energy generation rate is

$$q \approx \frac{\rho}{m_{\rm H}^2} \left(\frac{1}{1+\delta_{ij}} \right) \frac{X_i X_j}{\mathcal{A}_i \mathcal{A}_j} R Q \tag{79}$$

$$= 2.4 \times 10^6 X^2 \left(\frac{\rho}{1 \text{ g cm}^{-3}} \right) \left(\frac{T}{10^6 \text{ K}} \right)^{-2/3} \exp\left[-\frac{33.8}{(T/10^6 \text{ K})^{1/3}} \right] \text{ erg g}^{-1} \text{ s}^{-1} \tag{80}$$

If we do want to make a powerlaw fit, the index is

$$\nu = \frac{\tau - 2}{3} \approx 4. \tag{81}$$

B. The CNO cycle

The $p-p$ chain faces a relatively small Coulomb barrier, since the rate-limiting step has $\mathcal{Z} = 1$ for both reactants. However, it is slow because it requires spontaneous proton-neutron conversion within the short time that two protons are close to one another in a violently unstable configuration. There is another possible route to turning hydrogen into helium-4 which has a different tradeoff: a larger Coulomb barrier, but no need for a weak reaction in a short period.

This second route is called the CNO cycle, and was discovered independently by Hans Bethe and Carl-Friedrich von Weizsäcker in 1938. It relies on the fact that cabron, nitrogen, and oxygen are fairly abundant in the universe, and are present in a star even before it starts nuclear burning. They can act as catalysts in a proton fusion reaction. The reaction chain is

$$
\begin{aligned}
{}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{13}_{7}\mathrm{N} + \gamma & (82)\\
{}^{13}_{7}\mathrm{N} &\rightarrow {}^{13}_{6}\mathrm{C} + e^{+} + \nu_e & (83)\\
{}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} + \gamma & (84)\\
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma & (85)\\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^{+} + \nu_e & (86)\\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He} & (87)
\end{aligned}
$$

Alternately, the chain can be:

$$
\begin{aligned}
{}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{15}_{8}\mathrm{O} + \gamma & (88)\\
{}^{15}_{8}\mathrm{O} &\rightarrow {}^{15}_{7}\mathrm{N} + e^{+} + \nu_e & (89)\\
{}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{16}_{8}\mathrm{O} + \gamma & (90)\\
{}^{16}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{17}_{9}\mathrm{F} + \gamma & (91)\\
{}^{17}_{9}\mathrm{F} &\rightarrow {}^{17}_{8}\mathrm{O} + e^{+} + \nu_e & (92)\\
{}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{14}_{7}\mathrm{N} +{}^{4}_{2}\mathrm{He} & (93)
\end{aligned}
$$

The first route is generally the more important one, by a large factor.

Note that both of these chains have the property that it neither creates nor destroys any carbon or nitrogen nuclei. One starts with ${}^{12}_{6}\mathrm{C}$ and ends with it, or starts with ${}^{14}_{7}\mathrm{N}$ and ends with it. Thus the net reaction is exactly the same as for the $p-p$ chain:

$$4\,{}^{1}_{1}\mathrm{H} \rightarrow {}^{4}_{2}\mathrm{He} + \text{photons, neutrinos, and light particles,} \tag{94}$$

In this sense, the carbon or nitrogen acts as a catalyst. They enable the reaction to take place, but are not themselves consumed or created by it. Since the net reaction is the same as for $p-p$, the net energy release is also the same, except for slightly different neutrino losses. For the CNO cycle, $Q \approx 25$ MeV once the neutrino losses are factored in, as opposed to 27 MeV for $p-p$.

In each of these reaction chains, it makes sense to distinguish between reactions that involve creation of a positron e^{+} and reactions that do not. The former are called

β decays, and they rely on the weak nuclear force. However, they are much faster than the first step of the $p-p$ chain, because the take place in nuclei that are stable except for the weak reaction they undergo. Thus there is no need for precisely timing the reaction with the period when two protons are in close proximity.

At the temperature found in the Sun, however, the rate-limiting step is not the β decays, but the need to overcome strong Coulomb barriers. The ignition temperature is about 1.5×10^7 K, about the Sun's central temperature. Analysis of the full reaction rate is tricky because which step is the rate-limiting one depends on the relative abundances of C, N, O, and the other catalysts, which are in turn determined by the reaction cycle itself. Once things reach equilibrium, however, it turns out that the step

$$ {}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{15}_{8}\mathrm{O} + \gamma \tag{95} $$

is the rate-limiting one. This step appears in both cycles.

Plugging $\mathcal{Z}_i = 7$, $\mathcal{Z}_j = 1$, and $\mu_{\mathrm{red}} = (14)(1)/(14+1) = 0.93$ into our equation for the temperature-dependence gives

$$ \tau = \frac{3E_0}{k_B T} = 42.46 \left[\mathcal{Z}_i^2 \mathcal{Z}_j^2 \mu_{\mathrm{red}} \left(\frac{T}{10^6\ \mathrm{K}} \right)^{-1} \right]^{1/3} = 152 \left(\frac{T}{10^6\ \mathrm{K}} \right)^{-1/3} \tag{96} $$

Using the laboratory measurement for $S(E_0)$ for this reaction, the rate coefficient is

$$ R = 8.6 \times 10^{-19} \left(\frac{T}{10^6\ \mathrm{K}} \right)^{-2/3} \exp\left[-\frac{152}{(T/10^6\ \mathrm{K})^{1/3}} \right]\ \mathrm{cm}^{-3}\ \mathrm{s}^{-1}, \tag{97} $$

and the corresponding energy generation rate is

$$ q = 8.7 \times 10^{27} X X_{\mathrm{CNO}} \left(\frac{\rho}{1\ \mathrm{g\ cm}^{-3}} \right) \left(\frac{T}{10^6\ \mathrm{K}} \right)^{-2/3} \exp\left[-\frac{152}{(T/10^6\ \mathrm{K})^{1/3}} \right]\ \mathrm{erg\ g}^{-1}\ \mathrm{s}^{-1}, \tag{98} $$

where X_{CNO} is the total mass fraction of carbon, nitrogen, and oxygen. This is roughly $Z/2$, where Z is the total mass fraction of metals.

It is informative to evaluate q for the $p-p$ chain and for the CNO cycle using values appropriate to the centre of the Sun: $\rho \approx 10$ g cm^{-3}, $T \approx 1.5 \times 10^7$ K, $X = 0.71$, $Z = 0.02$. This gives

$$ q_{p-p} = 82\ \mathrm{erg\ g}^{-1}\ \mathrm{s}^{-1} \tag{99} $$

$$ q_{\mathrm{CNO}} = 6.4\ \mathrm{erg\ g}^{-1}\ \mathrm{s}^{-1} \tag{100} $$

Thus the $p-p$ chain dominates in the Sun by about a factor of 10. However, it is important to notice that, because it has 152 instead of 33.8 in the exponential, the CNO cycle is much more temperature-sensitive than the $p-p$ chain. If we assign a powerlaw approximation, the index is

$$ \nu = \frac{\tau - 2}{3} = 20. \tag{101} $$

Thus stars a bit more massive than the Sun, which we will see have higher central temperatures, the CNO cycle dominates. In stars smaller than the Sun, the CNO cycle is completely irrelevant.

This also brings out a general feature of all the nuclear reactions we will consider: the temperature-sensitivity is determined by τ, and τ in turn depends on the charges of the nuclei involved, $\mathcal{Z}$, because it is determined by the Coulomb barrier. The stronger the nuclear charge, the stronger the Coulomb barrier, and thus the higher the ignition temperature and the more temperature-sensitive the reaction becomes. We have already seen that the CNO cycle produces energy as a rate that varies as T^{20}, and the temperature-sensitivity only gets stronger as we march up the periodic table.