High Energy Astrophysics

Solutions to Exercises

5. (a) The beam is defined by

$$B(x, y) = \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right]$$

The FWHM contour is defined by:

$$B(x, y) = \frac{1}{2} \Rightarrow \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} = \ln 2$$

The axes of this ellipse are therefore:

$$(\theta_x, \theta_y) = (\sqrt{8\ln 2}\sigma_x, \sqrt{8\ln 2}\sigma_y)$$

(b) The flux per beam is

$$F_\nu^B(x, y) = \int I_\nu(x', y')B(x - x', y - y')dx'dy'$$

Now, if $I_\nu(x', y')$ varies slowly within the extent of the beam, then we put $I_\nu(x', y') \approx I_\nu(x, y)$ and

$$F_\nu^B(x, y) \approx I_\nu(x, y)\int B(x - x', y - y')dx'dy' = I_\nu(x, y) \times A$$

and

$$I_\nu(x, y) \approx \frac{F_\nu^B(x, y)}{A}$$

(c) For a Gaussian beam

$$A = \int \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right]dxdy = \sqrt{2\pi}\sigma_x \times \sqrt{2\pi}\sigma_y = 2\pi\sigma_x\sigma_y$$

(d) Since $(\sigma_x, \sigma_y) = \dfrac{(\theta_x, \theta_y)}{\sqrt{8\ln 2}}$ then

$$A = \frac{2\pi}{8\ln 2}\theta_x\theta_y = \left(\frac{\pi}{4\ln 2}\right)\left(\frac{\pi}{180} \times \frac{1}{3600}\right)^2\left(\frac{\theta_x\theta_y}{\text{square arcsecond}}\right) \text{ Str}$$

$$= 2.66 \times 10^{-11}\left(\frac{\theta_x}{1''}\right)\left(\frac{\theta_y}{1''}\right) \text{ Str}$$

(e) For an unresolved source represented by $I_\nu(x, y) = A_\nu\delta(x)\delta(y)$ the flux per beam is

$$F_\nu^B(x, y) = \int A_\nu\delta(x')\delta(y')B(x - x', y - y')dx'dy' = A_\nu B(x, y)$$

The peak flux per beam is therefore

$$F_\nu(0,0) = A_\nu B(0,0) = A_\nu$$

The flux density of the source is

$$F_\nu = \int A_\nu \delta(x)\delta(y)\cos\theta \, dx dy$$

where θ is the angle between the ray and the point on the sky. Because of the delta-function,

$$F_\nu = A_\nu$$

Hence, for an unresolved source, the flux per beam F_ν^B is equal to the flux density of the source.

6. From the contour image of Cygnus A, we have:
   - Point A = Contour 7 => 0.04×14.5 Jy/beam $= 0.58$ Jy/beam
   - Point B = Contour 2 => 0.002×14.5 Jy/beam $= 2.9\times10^{-2}$ Jy/beam

The beam is circular with a FWHM of 1 arcsecond. Hence $A = 2.66\times10^{-11}$ Str
Hence the respective surface brightnesses are given by:

$$I_\nu(A) \approx \frac{0.58\times10^{-26}}{2.66\times10^{-11}} = 2.18\times10^{-16} \text{ W m}^{-2}\text{ Hz}^{-1}\text{ Str}^{-1}$$

$$I_\nu(B) \approx \frac{2.9\times10^{-2}\times10^{-26}}{2.66\times10^{-11}} = 1.09\times10^{-17} \text{ W m}^{-2}\text{ Hz}^{-1}\text{ Str}^{-1}$$

7. We are required to prove that

$$\frac{d}{dt'}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))}{1-\beta(t')\cdot\boldsymbol{n}}\right] = \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\beta(t'))\times\dot{\beta}(t')]}{[1-\beta(t')\cdot\boldsymbol{n}]^2}$$

This is fairly straightforward.

$$\frac{d}{dt'}\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))}{1-\beta(t')\cdot\boldsymbol{n}} = \frac{[1-\beta(t')\cdot\boldsymbol{n}]\times\frac{d}{dt'}[\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))]-[\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))]\frac{d}{dt'}[1-\beta(t')\cdot\boldsymbol{n}]}{[1-\beta(t')\cdot\boldsymbol{n}]^2}$$

Look at the various parts of this:

$$\frac{d}{dt'}[\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))] = \frac{d}{dt'}[\boldsymbol{n}\cdot\beta(t')\boldsymbol{n}-\beta(t')] = \boldsymbol{n}\cdot\dot{\beta}(t')\boldsymbol{n}-\dot{\beta}(t')$$

$$\frac{d}{dt'}[1-\beta(t')\cdot\boldsymbol{n}] = -\dot{\beta}(t')\cdot\boldsymbol{n}$$

Combining terms:

$$\frac{d}{dt'}\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))}{1-\beta(t')\cdot\boldsymbol{n}} = \frac{[\boldsymbol{n}-\beta(t')]\dot{\beta}(t')\cdot\boldsymbol{n}-\dot{\beta}(t')[1-\beta(t')\cdot\boldsymbol{n}]}{(1-\beta(t')\cdot\boldsymbol{n})^2}$$

$$= \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\beta(t'))\times\dot{\beta}(t')]}{(1-\beta(t')\cdot\boldsymbol{n})^2}$$

8. (a) In order to determine the transformation of the Stokes parameters we can utilise the transformations of the electric field in a wave to determine the quantities $E_\alpha E_\beta^*$. For a rotation by ψ in the plane of the wave:

$$\begin{bmatrix} E_x' \\ E_y' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

i.e.

$$\begin{aligned} E_x' &= \cos\psi E_x + \sin\psi E_y \\ E_y' &= -\sin\psi E_x + \cos\psi E_y \end{aligned}$$

We then evaluate all of the combinations $E_x' E_x'^*$, $E_x' E_y'^*$ etc.This just involves a bit of algebra with the results:

$$\begin{aligned} E_x' E_x'^* &= \cos^2\psi E_x E_x^* + \sin\psi\cos\psi E_x E_y^* + \sin\psi\cos\psi E_x^* E_y + \sin^2\psi E_y E_y^* \\ E_x' E_y'^* &= -\sin\psi\cos\psi E_x E_x^* + \cos^2\psi E_x E_y^* - \sin^2\psi E_x^* E_y + \sin\psi\cos\psi E_y E_y^* \\ E_y' E_x'^* &= -\sin\psi\cos\psi E_x E_x^* - \sin^2\psi E_x E_y^* + \cos^2\psi E_x^* E_y + \sin\psi\cos\psi E_y E_y^* \\ E_y' E_y'^* &= \sin^2\psi E_x E_x^* - \sin\psi\cos\psi E_x E_y^* - \sin\psi\cos\psi E_x^* E_y + \cos^2\psi E_y E_y^* \end{aligned}$$

One then has to form the combinations that give the Stokes parameters:

$$\begin{aligned} E_x' E_x'^* + E_y' E_y'^* &= E_x E_x^* + E_y E_y^* \\ E_x' E_x'^* - E_y' E_y'^* &= \cos 2\psi (E_x E_x^* - E_y' E_y'^*) + \sin 2\psi (E_x E_y^* + E_y E_x^*) \\ E_x' E_y'^* + E_x'^* E_y' &= -\sin 2\psi (E_x E_x^* - E_y' E_y'^*) + \cos 2\psi (E_x E_y^* + E_y E_x^*) \\ E_x' E_y'^* - E_x'^* E_y' &= E_x E_y^* - E_y E_x^* \end{aligned}$$

We can then express the transformations for the Stokes parameters in matrix form:

$$\begin{bmatrix} I' \\ Q' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}$$

We can write this as

$$I' = I$$

$$\begin{bmatrix} Q' \\ U' \end{bmatrix} = \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ -\sin 2\psi & \cos 2\psi \end{bmatrix} \begin{bmatrix} Q \\ U \end{bmatrix}$$

$$V' = V$$

(b) The frame in which $U' = 0$ is given by

$$U' = -Q\sin 2\psi + U\cos 2\psi = 0$$

$$\Rightarrow \tan 2\psi = \frac{U}{Q}$$