High Energy Astrophysics

Solutions to Questions 1-4

1. The relationship between energy and wavelength is

$$\varepsilon = \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times 3.0\times10^{8}}{1.60\times10^{-19}\times\left(\frac{\lambda}{\mathrm{m}}\right)} = 1.24\times10^{-6}\left(\frac{\lambda}{\mathrm{m}}\right)^{-1}\ \mathrm{eV}$$

Also, to determine frequency and wavelength given energy, use

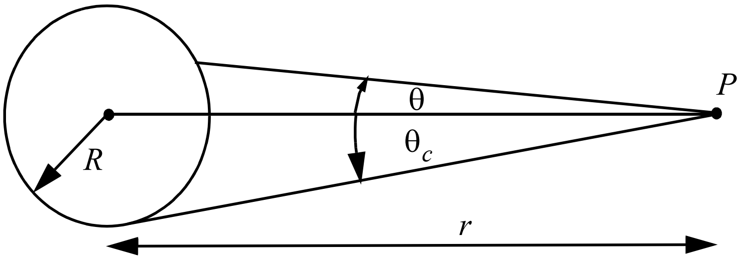
$$\varepsilon = h\nu \Rightarrow \nu = \frac{\varepsilon}{h} = \frac{1.60\times10^{-19}}{6.62\times10^{-34}}\times\left(\frac{\varepsilon}{\mathrm{eV}}\right) = 2.41\times10^{14}\left(\frac{\varepsilon}{\mathrm{eV}}\right)\ \mathrm{Hz}$$

$$\varepsilon = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{\varepsilon} = 1.24\times10^{-6}\left(\frac{\varepsilon}{\mathrm{eV}}\right)^{-1}\ \mathrm{m} = 1.24\times10^{4}\left(\frac{\varepsilon}{\mathrm{eV}}\right)^{-1}\ \mathrm{Angstroms}$$

Hence the following table (the primary quantity is given in bold):

Photon	ν (Hz)	λ (Angstroms)	λ (metres)	ε (eV)
Typical microwave			**0.20**	6.2×10^{-6}
Typical far infrared			**100×10^{-6} = 10^{-4}**	1.24×10^{-2}
Typical near infrared			**2×10^{-6}**	0.62
Typical optical		**5000**	5×10^{-7}	2.48
Typical UV		**1000**	10^{-7}	12.4
Soft X-ray	2.41×10^{17}	12.4	1.24×10^{-9}	**1 keV**
Hard X-ray	2.41×10^{19}	0.124	1.24×10^{-11}	**100 keV**
TeV γ-ray	2.41×10^{26}	1.24×10^{-8}	1.24×10^{-18}	**1 TeV**

2.



The flux density received at point P is:

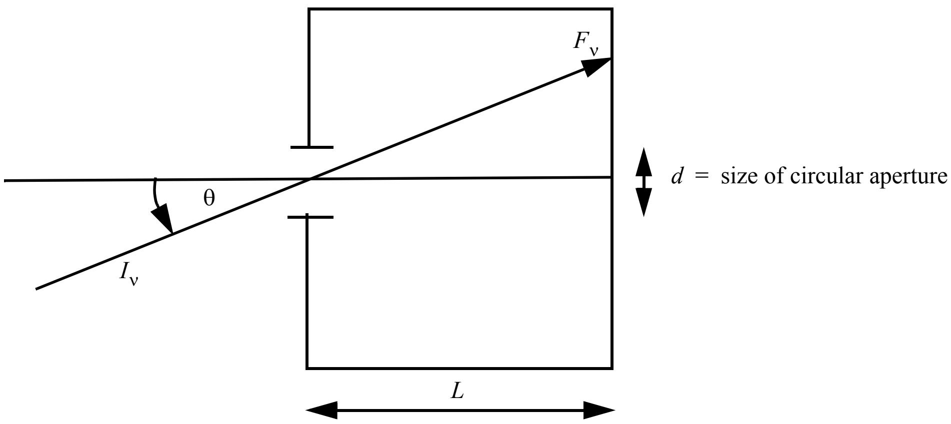
$$F_\nu = \int_\Omega I_\nu \cos\theta \, d\Omega = \int_{\theta,\,\phi} I_\nu \cos\theta \sin\theta \, d\theta \, d\phi$$

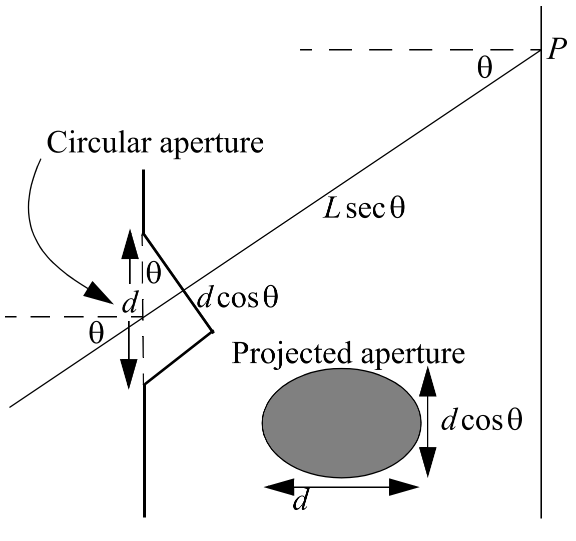
$$= 2\pi I_\nu \int_0^{\theta_c} \cos\theta \sin\theta \, d\theta = \pi I_\nu \sin^2\theta_c$$

$$= \pi I_\nu \frac{R^2}{r^2}$$

That is, we recover the inverse square law.

3.





First calculate the solid angle of rays hitting the back of the camera. Viewed from the point P the aperture appears as an ellipse with major and minor axes d and $d\cos\theta$. The area of the aperture is therefore

$$\Delta A \approx \frac{\pi}{4} d\cos\theta \times d = \frac{\pi}{4} d^2 \cos\theta$$

Hence the (small) solid angle subtended by the aperture at P is

$$\Delta\Omega = \frac{\frac{\pi}{4} d^2 \cos\theta}{(L\sec\theta)^2} = \frac{\pi}{4}\frac{d^2}{L^2}\cos^3\theta$$

Next, the flux density incident on the back of the camera at P is

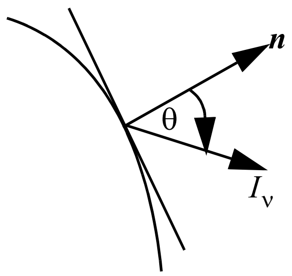
$$F_\nu = \int_\Omega I_\nu \cos\theta d\Omega \approx I_\nu \cos\theta \Delta\Omega = I_\nu \frac{\pi d^2}{4L^2}\cos^4\theta = I_\nu \frac{\frac{\pi}{4}\cos^4\theta}{f^2}$$

where the focal ratio $f = L/d$.

4. (a) The mean intensity at a particular point, a radius r from the centre of the sphere is:

$$J_\nu = \frac{1}{4\pi}\int_\Omega I_\nu d\Omega = \frac{1}{2}\int_0^{\theta_c} I_\nu \sin\theta d\theta = \frac{1}{2}I_\nu[1-\cos\theta_c] = \frac{1}{2}I_\nu\left[1-\left(1-\frac{R^2}{r^2}\right)^{1/2}\right]$$

(b) Flux through the surface of the sphere is:



$$F_\nu = \int_0^{2\pi}\left\{\int_0^{\pi} I_\nu \cos\theta \sin\theta d\theta\right\}d\phi = \pi I_\nu$$

Hence, the total energy per unit time through the surface of the sphere (i.e. the Luminosity) is:

$$L_\nu = F_\nu \times 4\pi R^2 = 4\pi^2 I_\nu R^2$$

Therefore,

$$I_\nu = \frac{L_\nu}{4\pi^2 R^2}$$

(c) The energy density per unit frequency is

$$u_\nu = \frac{4\pi}{c}J_\nu = \frac{4\pi}{c}\times\frac{1}{2}I_\nu\left[1-\left(1-\frac{R^2}{r^2}\right)^{1/2}\right]$$

$$= \frac{L_\nu}{2\pi c R^2}\left[1-\left(1-\frac{R^2}{r^2}\right)^{1/2}\right]$$

$$\approx \frac{L_\nu}{4\pi c R^2}$$

Hence, the total (i.e. frequency integrated) energy density is

$$u \approx \frac{L}{4\pi c R^2}$$

(d) This expression can also be derived as follows. The total energy passing through a sphere of radius r is

$$u \times c \times 4\pi r^2$$

and this is equal to the total luminosity, L . Hence

$$u \times c \times 4\pi r^2 = L \Rightarrow u = \frac{L}{4\pi c r^2}$$

(e) For $L = 3.83\times10^{26}$ W and $r = 1.50\times10^{11}$ m

$$u = 4.5\times10^{-6} \text{ J m}^{-3} = 2.8\times10^{13} \text{ eV m}^{-3}$$