

# High Energy Astrophysics

## Solutions to Exercises

5. (a) The beam is defined by

$$B(x, y) = \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right]$$

The FWHM contour is defined by:

$$B(x, y) = \frac{1}{2} \Rightarrow \frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} = \ln 2$$

The axes of this ellipse are therefore:

$$(\theta_x, \theta_y) = (\sqrt{8\ln 2}\sigma_x, \sqrt{8\ln 2}\sigma_y)$$

(b) The flux per beam is

$$F_v^B(x, y) = \int I_v(x', y') B(x - x', y - y') dx' dy'$$

Now, if  $I_v(x', y')$  varies slowly within the extent of the beam, then we put  $I_v(x', y') \approx I_v(x, y)$  and

$$F_v^B(x, y) \approx I_v(x, y) \int B(x - x', y - y') dx' dy' = I_v(x, y) \times A$$

and

$$I_v(x, y) \approx \frac{F_v^B(x, y)}{A}$$

(c) For a Gaussian beam

$$A = \int \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right] dx dy = \sqrt{2\pi}\sigma_x \times \sqrt{2\pi}\sigma_y = 2\pi\sigma_x\sigma_y$$

(d) Since  $(\sigma_x, \sigma_y) = \frac{(\theta_x, \theta_y)}{\sqrt{8\ln 2}}$  then

$$\begin{aligned} A &= \frac{2\pi}{8\ln 2} \theta_x \theta_y = \left(\frac{\pi}{4\ln 2}\right) \left(\frac{\pi}{180} \times \frac{1}{3600}\right)^2 \left(\frac{\theta_x \theta_y}{\text{square arcsecond}}\right) \text{Str} \\ &= 2.66 \times 10^{-11} \left(\frac{\theta_x}{1''}\right) \left(\frac{\theta_y}{1''}\right) \text{Str} \end{aligned}$$

(e) For an unresolved source represented by  $I_\nu(x, y) = A_\nu \delta(x) \delta(y)$  the flux per beam is

$$F_\nu^B(x, y) = \int A_\nu \delta(x') \delta(y') B(x - x', y - y') dx' dy' = A_\nu B(x, y)$$

The peak flux per beam is therefore

$$F_\nu(0, 0) = A_\nu B(0, 0) = A_\nu$$

The flux density of the source is

$$F_\nu = \int A_\nu \delta(x) \delta(y) \cos \theta dx dy$$

where  $\theta$  is the angle between the ray and the point on the sky. Because of the delta-function,

$$F_\nu = A_\nu$$

Hence, for an unresolved source, the flux per beam  $F_\nu^B$  is equal to the flux density of the source.

6. From the contour image of Cygnus A, we have:

- Point A = Contour 7  $\Rightarrow 0.04 \times 14.5$  Jy/beam = 0.58 Jy/beam
- Point B = Contour 2  $\Rightarrow 0.002 \times 14.5$  Jy/beam =  $2.9 \times 10^{-2}$  Jy/beam

The beam is circular with a FWHM of 1 arcsecond. Hence  $A = 2.66 \times 10^{-11}$  Str

Hence the respective surface brightnesses are given by:

$$I_\nu(A) \approx \frac{0.58 \times 10^{-26}}{2.66 \times 10^{-11}} = 2.18 \times 10^{-16} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ Str}^{-1}$$

$$I_\nu(B) \approx \frac{2.9 \times 10^{-2} \times 10^{-26}}{2.66 \times 10^{-11}} = 1.09 \times 10^{-17} \text{ W m}^{-2} \text{ Hz}^{-1} \text{ Str}^{-1}$$

7. We are required to prove that

$$\frac{d}{dt'} \left[ \frac{\mathbf{n} \times (\mathbf{n} \times \beta(t'))}{1 - \beta(t') \cdot \mathbf{n}} \right] = \frac{\mathbf{n} \times [(\mathbf{n} - \beta(t')) \times \dot{\beta}(t')]}{[1 - \beta(t') \cdot \mathbf{n}]^2}$$

This is fairly straightforward.

$$\frac{d}{dt'} \frac{\mathbf{n} \times (\mathbf{n} \times \beta(t'))}{1 - \beta(t') \cdot \mathbf{n}} = \frac{[1 - \beta(t') \cdot \mathbf{n}] \times \frac{d}{dt'} [\mathbf{n} \times (\mathbf{n} \times \beta(t'))] - [\mathbf{n} \times (\mathbf{n} \times \beta(t'))] \frac{d}{dt'} [1 - \beta(t') \cdot \mathbf{n}]}{[1 - \beta(t') \cdot \mathbf{n}]^2}$$

Look at the various parts of this:

$$\frac{d}{dt'} [\mathbf{n} \times (\mathbf{n} \times \beta(t'))] = \frac{d}{dt'} [\mathbf{n} \cdot \beta(t') \mathbf{n} - \beta(t')] = \mathbf{n} \cdot \dot{\beta}(t') \mathbf{n} - \dot{\beta}(t')$$

$$\frac{d}{dt'} [1 - \beta(t') \cdot \mathbf{n}] = -\dot{\beta}(t') \cdot \mathbf{n}$$

Combining terms

$$\begin{aligned}\frac{d}{dt'} \frac{\mathbf{n} \times (\mathbf{n} \times \beta(t'))}{1 - \beta(t') \cdot \mathbf{n}} &= \frac{[\mathbf{n} - \beta(t')] \beta(t') \cdot \mathbf{n} - \dot{\beta}(t') [1 - \beta(t') \cdot \mathbf{n}]}{(1 - \beta(t') \cdot \mathbf{n})^2} \\ &= \frac{\mathbf{n} \times [(\mathbf{n} - \beta(t')) \times \beta(t')]}{(1 - \beta(t') \cdot \mathbf{n})^2}\end{aligned}$$

8. In order to determine the transformation of the Stokes parameters we can utilise the transformations of the electric field in a wave to determine the quantities  $E_\alpha E_\beta^*$ . For a rotation by  $\psi$  in the plane of the wave:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

i.e.

$$\begin{aligned}E'_x &= \cos\psi E_x + \sin\psi E_y \\ E'_y &= -\sin\psi E_x + \cos\psi E_y\end{aligned}$$

We then evaluate all of the combinations  $E'_x E_x^*$ ,  $E'_x E_y^*$  etc. This just involves a bit of algebra with the results:

$$\begin{aligned}E'_x E_x^* &= \cos^2\psi E_x E_x^* + \sin\psi \cos\psi E_x E_y^* + \sin\psi \cos\psi E_y E_x^* + \sin^2\psi E_y E_y^* \\ E'_x E_y^* &= -\sin\psi \cos\psi E_x E_x^* + \cos^2\psi E_x E_y^* - \sin^2\psi E_y E_x^* + \sin\psi \cos\psi E_y E_y^* \\ E'_y E_x^* &= -\sin\psi \cos\psi E_x E_x^* - \sin^2\psi E_x E_y^* + \cos^2\psi E_y E_x^* + \sin\psi \cos\psi E_y E_y^* \\ E'_y E_y^* &= \sin^2\psi E_x E_x^* - \sin\psi \cos\psi E_x E_y^* - \sin\psi \cos\psi E_y E_x^* + \cos^2\psi E_y E_y^*\end{aligned}$$

One then has to form the combinations that give the Stokes parameters:

$$\begin{aligned}E'_x E_x^* + E'_y E_y^* &= E_x E_x^* + E_y E_y^* \\ E'_x E_x^* - E'_y E_y^* &= \cos 2\psi (E_x E_x^* - E_y E_y^*) + \sin 2\psi (E_x E_y^* + E_y E_x^*) \\ E'_x E_y^* + E'_y E_x^* &= -\sin 2\psi (E_x E_x^* - E_y E_y^*) + \cos 2\psi (E_x E_y^* + E_y E_x^*) \\ E'_x E_y^* - E'_y E_x^* &= E_x E_y^* - E_y E_x^*\end{aligned}$$

We can then express the transformations for the Stokes parameters in matrix form:

$$\begin{bmatrix} I' \\ Q' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}$$