High Energy Astrophysics

Solutions to Exercises

10. We can put

$$I_\nu \nu^\alpha = \frac{I_\nu}{\nu^3}\nu^{\alpha+3} = \left(\frac{I_\nu}{\nu^3}\right)\nu^{\frac{a+5}{2}}$$

and when the $\nu^{\frac{a+5}{2}}$ factor is raised to the $2/(a+5)$ power, this just gives a term of the dimensions of frequency. It can easily be shown that

$$\frac{c}{m_e}\frac{I_\nu}{\nu^3}$$

is dimensionless. Since $\frac{eB}{m_e}$ has the dimensions of frequency (it is the cyclotron frequency), then the dimensions are OK.

11. The total energy density and pressure are given by:

$$\varepsilon = (1+c_E)\varepsilon_e + \frac{B^2}{2\mu_0}$$

$$p_{\text{tot}} = (1+c_p)\frac{\varepsilon_e}{3} + \frac{B^2}{2\mu_0}$$

Hence the expression for the minimum pressure follows from the expression for the minimum energy density by the substitution

$$(1+c_E) \rightarrow \frac{(1+c_p)}{3}$$

The corresponding expression for the minimum pressure magnetic field is:

$$B_{\text{p-min}} = \frac{m_e}{e}\left[\frac{a+1}{2}\frac{(1+c_p)}{3}C_2^{-1}(a)\frac{c}{m_e}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a,\gamma_1,\gamma_2)\right]^{\frac{2}{a+5}}$$

Hence,

$$\frac{B_{\text{p-min}}}{B_{\text{e-min}}} = \left[\frac{(1+c_p)}{3(1+c_E)}\right]^{\frac{2}{a+5}}$$

For $c_p = c_E = 0$ and $a = 2$ $B_{\text{p-min}}/B_{\text{e-min}} = \left(\frac{1}{3}\right)^{2/7} = 0.73$.

The subsidiary condition on the particle pressure follows from

$$\frac{\partial p_{\rm tot}}{\partial B} = \frac{(1+c_p)}{3}\frac{\partial \varepsilon_e}{\partial B} + \frac{B}{\mu_0} = \frac{\partial p_p}{\partial B} + \frac{B}{\mu_0}$$

$$\Rightarrow \frac{\partial p_p}{\partial B} = -\frac{B}{\mu_0}$$

Also, analogously to the minimum energy case:

$$\left(\frac{p_p}{mc^2}\right)\Omega_0^{\frac{a+1}{2}} = \frac{(1+c_p)}{3}C_2^{-1}(a)\left(\frac{e^2}{\varepsilon_0 c}\right)^{-1}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a,\gamma_1,\gamma_2)$$

so that

$$\frac{1}{p_p}\frac{\partial p_p}{\partial B} + \frac{a+1}{2}\times\frac{1}{B} = 0 \Rightarrow \frac{1}{p_p}\left(-\frac{B}{\mu_0}\right) + \frac{a+1}{2B} = 0 \Rightarrow p_p = \frac{4}{a+1}\left(\frac{B^2}{2\mu_0}\right)$$

The total minimum pressure:

$$p_{\rm tot,min} = \left[\frac{4}{a+1}+1\right]\frac{B^2_{\rm p\text{-}min}}{2\mu_0} = \frac{a+5}{a+1}\left(\frac{B^2_{\rm p\text{-}min}}{2\mu_0}\right)$$

Hence,

$$\frac{p_{\rm tot,min}}{\varepsilon_{\rm tot,min}} = \frac{B^2_{\rm p\text{-}min}}{B^2_{\rm e\text{-}min}} = \left[\frac{(1+c_p)}{3(1+c_E)}\right]^{\frac{4}{a+5}}$$

For example, for $c_p = c_E = 0$ and $a = 2$,

$$\frac{p_{\rm tot,min}}{\varepsilon_{\rm tot,min}} = \left(\frac{1}{3}\right)^{4/7} = 0.53$$

12. (i) The minimum energy density solution can be calculated using a simple modification of the spreadsheet on the course web-site. The only difference is that in the template, the surface brightness in physical units is calculated from the flux per beam and the beam dimensions, whereas one needs to use the plotted values of mJy arcsec^{-2} to estimate I_ν in physical units. That is,

$$\frac{I_\nu}{(\mathrm{W\ m^{-2}\ Hz^{-1}\ Sr^{-1}})} = \left(\frac{I_\nu}{\mathrm{mJy\ arcsec^{-2}}}\right)\times\frac{10^{-29}}{\left(\frac{\pi}{180}\times\frac{1}{3600}\right)^2} = 4.25\times10^{-19}\left(\frac{I_\nu}{\mathrm{mJy\ arcsec^{-2}}}\right)$$

I used the plotted values in figures 14 and 16 of the paper to estimate the FWHM, Φ, and the surface brightness in mJy arcsec^{-2} to calculate the minimum energy magnetic field, the corresponding particle energy density and the total minimum energy density, although I only asked for the minimum pressure. I have used $c_E = c_P = 0$ appropriate for a electron-positron plasma. However, the numbers don't vary all that much if you put $c_E = c_p = 1$, for example.

θ (arcsec)	Φ (arcsec)	I_ν (mJy arcsec^{-2})	$B_{\text{e-min}}$(nT) $B_{\text{p-min}}$(nT)	ε_p(10^{-12} J m^{-3}) p_p(10^{-12}N m^{-2})	ε_{tot}(10^{-12} J m^{-3}) p_{tot}(10^{-12}N m^{-2})
25 W	7.4	1.09	1.9 1.4	1.8 0.96	3.2 1.7
50 W	11.8	1.20	1.7 1.3	1.4 0.63	2.6 0.78
100 W	15.6	0.35	1.1 0.82	0.50 0.27	1.1 0.61
25 E	7.4	0.69	1.7 1.2	1.4 0.75	2.5 1.3
50 E	13.2	0.88	1.5 1.1	0.91 0.50	2.5 1.3
100 E	16.9	0.41	1.1 0.84	0.65 0.35	1.2 0.64

(ii) The pressure of the thermal plasma is nkT where n is the total (ion plus electron) number density and T is the temperature. Hence the minimum pressure required to confine the jet is

$$n_{\min} = \frac{p_{\text{tot,min}}}{kT} = \frac{10^{-12}}{1.38\times10^{-23}\times 10^{7}}\left(\frac{p_{\text{tot,min}}}{10^{-12}\ \text{N m}^{-2}}\right)\left(\frac{T}{10^{7}}\right)^{-1}$$

$$= 7.2\times10^{3}\left(\frac{p_{\text{tot,min}}}{10^{-12}\ \text{N m}^{-2}}\right)\left(\frac{T}{10^{7}}\right)^{-1}\ \text{m}^{-3} = 7.2\times10^{-3}\left(\frac{p_{\text{tot,min}}}{10^{-12}\ \text{N m}^{-2}}\right)\left(\frac{T}{10^{7}}\right)^{-1}\ \text{cm}^{-3}$$

Given the entries in the above table the minimum no. densities at various points along the jet are:

θ (arcsec)	$p_{\text{tot,min}}$ 10^{-12} N m^{-2}	$n_{\min}$ cm^{-3}
25 W	1.7	1.2×10^{-2}
50 W	0.78	5.6×10^{-3}
100 W	0.61	4.4×10^{-3}
25 E	1.3	9.4×10^{-3}
50 E	1.3	9.4×10^{-3}

θ (arcsec)	$p_{\text{tot,min}}$ 10^{-12} N m^{-2}	$n_{\min}$ cm^{-3}
100 E	0.64	4.6×10^{-3}

(iii) The answers that one obtains here will differ depending upon the exact assumptions. Here are the parameters that I used:

	West lobe	**East lobe**
Contour representing "average" plateau of surface brightness (i.e. ignore local hot spots or warm spots)	11th contour => 272 mJy	7th contour => 77.8 mJy
Diameter, D (gives distance through lobe as well as volume)	10.4′	11.7′
Redshift	0.01174	
Beam	60″ × 60″	
γ_1, γ_2	10, 10^5	
c_E, c_p	0	
Minimum total energy density	5.7×10^{-14} J m^{-3}	2.7×10^{-14} J m^{-3}
Minimum energy	3.0×10^{51} J	2.0×10^{51} J

(iv) The total minimum energy is therefore approximately 5×10^{51} J – about an order of magnitude less than Cygnus A. The accreted mass that can be responsible for this energy is given by

$$E_{\min} = \alpha M_{\text{acc,min}} c^2$$

where we take $\alpha \sim 0.1$
Thus,

$$M_{\text{acc,min}} = \frac{5\times10^{51}}{0.1c^2} = 2.8\times10^5 \text{ solar masses}$$

This is the minimum mass of the black hole that is implicated in the radio emission from IC 4296 and is less than that implicated in Cygnus A, because of the lower minimum energy. This is probably indicative of a lower accretion rate, rather than a lower black hole mass, i.e. the mass accreted is a significant underestimate of the mass of the black hole.

13. (i) For a uniform (in solid angle) distribution of pitch angles,

$$\langle \sin^2\alpha \rangle = \frac{1}{4\pi}\int_0^{2\pi}\left[\int_0^{\pi} \sin^2\alpha \sin\alpha d\alpha\right]d\phi = \frac{2}{3}$$

(ii) Hence the mean energy loss rate (averaged over man scatterings) is given by:

$$\left\langle \frac{d\gamma}{dt} \right\rangle = -c\sigma_T \frac{B^2\langle \sin^2\alpha \rangle}{\mu_0}\gamma^2 = -\frac{2}{3}\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_0}\right)\gamma^2$$

(iii) The energy of an electron is given by:

$$\frac{d\gamma}{dt} = -\frac{2}{3}\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_0}\right)\gamma^2$$

This integrates to

$$\frac{1}{\gamma} - \frac{1}{\gamma_0} = \frac{2}{3}\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_0}\right)t$$

where γ_0 is the Lorentz factor at $t = 0$. Write this equation as:

$$\gamma = \frac{\gamma_0}{\left[1 + \frac{2}{3}\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_0}\right)\gamma_0 t\right]}$$

The time at which the Lorentz factor is reduced to half of its initial value is:

$$t_{\text{syn}} = \frac{3}{2}\left(\frac{\mu_0 m_e c}{\sigma_T}\right)B^{-2}\gamma_0^{-1}$$

Using

$$c^2 = \frac{1}{\varepsilon_0\mu_0} \Rightarrow \mu_0 c = \frac{1}{\varepsilon_0 c}$$

gives

$$t_{\text{syn}} = \frac{3}{2}\left(\frac{m_e}{\varepsilon_0 c\sigma_T}\right)B^{-2}\gamma_0^{-1}$$

(iv) The critical frequency is

$$\omega_c = \frac{3}{2}\frac{eB}{m_e}\gamma^2\sin\alpha \Rightarrow \nu_c = \frac{3}{4\pi}\frac{eB}{m_e}\gamma^2\sin\alpha$$

To evaluate the mean critical frequency we calculate

$$\langle \sin\alpha \rangle = \frac{1}{2}\int_0^{\pi} \sin\alpha \sin\alpha d\alpha = \frac{\pi}{4}$$

Hence, the critical frequency corresponding to γ_0 is:

$$\nu_0 = \frac{3}{16}\frac{eB}{m_e}\gamma_0^2$$

(v) From the above:

$$\gamma_0^{-1} = \left(\frac{3}{16\pi}\frac{eB}{m_e}\right)^{1/2}\nu_0^{-1/2}$$

and

$$t_{\text{syn}} = \frac{3}{2}\left(\frac{m_e}{\varepsilon_0 c\sigma_T}\right)B^{-2}\left(\frac{3}{16}\frac{eB}{m_e}\right)^{1/2}\nu_0^{-1/2} = \frac{3^{3/2}}{8}\frac{(em_e)^{1/2}}{\varepsilon_0 c\sigma_T}B^{-3/2}\nu_0^{-1/2}$$

(vi) Evaluating the leading constant gives

$$t_{\text{syn}} = 1.40\times10^{6}B^{-3/2}\nu_0^{-1/2}\ \text{secs}$$

leading to the following table:

B	ν_0	t_{syn}
1 nT	1 GHz	4.4×10^{7} yrs
10 nT	6×10^{14} Hz	1800 yrs
1 μT	10^{17} Hz	51 days