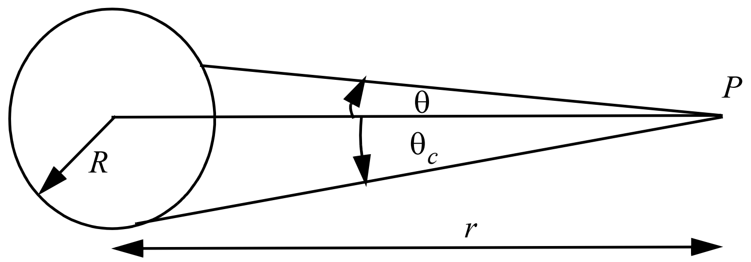
High Energy Astrophysics
Problem Set

1. (5 marks) <u>*Photon energies.*</u>
   Calculate the typical energies in eV of a microwave photon, a far infrared photon, a near-infrared photon, an optical photon, and a UV photon. Calculate the frequencies and wavelengths of 1 keV, 100keV and 1 TeV photons.

2. *(*10 marks*)* <u>*Inverse square law for a uniformly bright sphere.*</u>
   The aim of this question is to demonstrate that there is no conflict between the constancy of specific intensity along a ray with the notion of the inverse square law.
   Consider the flux from a uniformly bright sphere as shown in the following diagram:



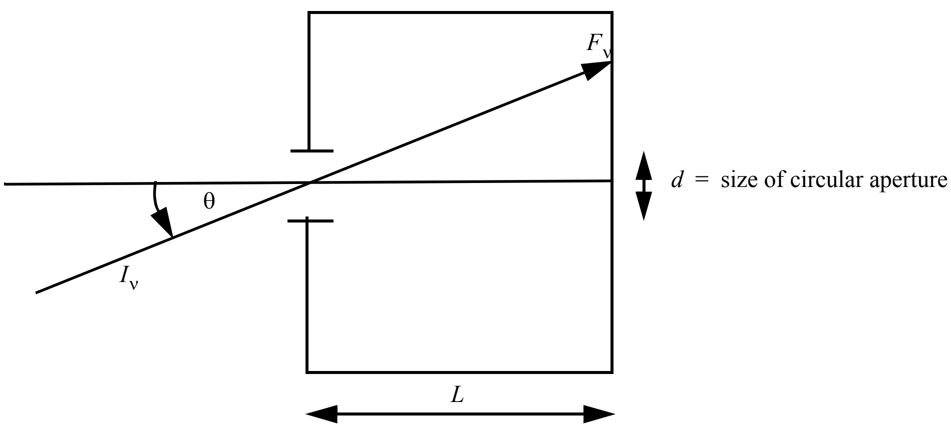
   The angle θ_c is the angle at which a ray from the point P is tangent to the sphere.
   Assume that all rays leaving the sphere have the same specific intensity (surface brightness or just brightness) I_ν , i.e. the sphere is uniformly bright.

   Show that the flux density of the radiation from the sphere is given by:

$$F_\nu = 2\pi I_\nu \int_0^{\theta_c} \sin\theta \cos\theta d\theta = \pi I_\nu \left(\frac{R}{r}\right)^2$$

   i.e. the flux density decreases with the inverse square of the distance from the sphere.

3. (10 marks) <u>*Relationship between flux density and specific intensity.*</u>
   A pinhole camera is as depicted in the following diagram



   Let θ be the polar angle, as shown and let ϕ be the corresponding azimuthal angle. The specific intensities of rays entering the pinhole will, in general, be functions of θ and ϕ. Show that the flux density at the film plane depends upon the brightness field $I_\nu(\theta, \phi)$ by:

$$F_\nu = \frac{\pi \cos^4\theta}{4f^2} I_\nu(\theta, \phi)$$

where the focal ratio $f = \dfrac{L}{d}$.

4. (15 marks) _Energy density of radiation from a central source._
(a) Show that the mean intensity at a distance r from a uniformly bright sphere of radius R (as in Q2) is given by:

$$J_\nu = \frac{1}{2} I_\nu \left[1 - \sqrt{1 - \frac{R^2}{r^2}}\right]$$

(b) Show that the surface brightness I_ν of the sphere is given in terms of its monochromatic luminosity, L_ν by:

$$I_\nu = \frac{L_\nu}{4\pi^2 R^2}$$

(c) Hence show that the total (i.e. frequency integrated) energy density of radiation at a distance r from a uniformly emitting sphere that is large compared to its radius, is given at large distances from the sphere by

$$u = \frac{L}{4\pi c r^2}$$

where L is the total luminosity. Note that u is independent of the radius of the sphere.

(d) Verify this relationship using another argument.

(e) Estimate the energy density of sunlight at the Earth's orbit but outside the Earth's atmosphere. You will need the following parameters:

- Luminosity of the Sun $\approx 3.83 \times 10^{26}$ W
- Sun - Earth distance $\approx 1.5 \times 10^{11}$ m

Express your answer both in J m^{-3} and eV m^{-3}.

5. (15 marks) _Flux densities and surface brightness in radio astronomy._
The surface brightnesses of images in radio astronomy are always represented in terms of the units of "flux density per beam". (See the image of Cygnus A in question 5). One can think of the "beam" as a smoothing function which convolves the real surface brightness. For an arbitrary surface brightness, the flux per beam is therefore given by

$$F_\nu(x, y) = \int_{\text{Source}} I_\nu(x', y') B(x - x', y - y') dx' dy'$$

where (x, y) represent angular coordinates in the plane of the sky.The beam is usually defined to be a Gaussian with unit amplitude. The extent of the beam is almost always quoted in terms of its FWHM and this defines the resolution of the image. Let us represent the beam by the elliptical Gaussian

$$B(x, y) = \exp\left[-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right)\right].$$

(a) Show that the FWHM of the beam is given by

$$(\theta_x, \theta_y) = \sqrt{8 \ln 2}(\sigma_x, \sigma_y)$$

(b) We define the beam area by:

$$A = \int B(x, y) dx dy$$

Show that for a smoothly varying surface brightness,

$$F_\nu(x, y) \approx I_\nu(x, y) \times A$$

and therefore,

$$I_\nu(x, y) \approx \frac{F_\nu(x, y)}{A}.$$

(c) Show that, for a Gaussian beam,

$$A = 2\pi\sigma_x\sigma_y = \frac{\pi}{4 \ln 2}\theta_x\theta_y \approx 1.133\theta_x\theta_y$$

(d) Show that

$$A \approx 2.66 \times 10^{-11}\left(\frac{\theta_x}{1\,\text{arcsec}}\right)\left(\frac{\theta_y}{1\,\text{arcsec}}\right)\text{Steradian}$$

(e) We may represent an unresolved source by a surface brightness of the form:

$$I_\nu(x, y) = A_\nu \delta(x)\delta(y)$$

Show that A_ν is the flux density of the source and that this is also the peak flux density per beam.

6. (10 marks) *Practical determination of surface brightness.*
The following figure (from Dreher et al., 1987, ApJ, **316**, 611) of the famous radio galaxy Cygnus A, shows contours of surface brightness in units of Jy/beam
Estimate the surface brightness of the points labelled A and B in units of $\text{W m}^{-2}\ \text{Hz}^{-1}\ \text{Sr}^{-1}$.

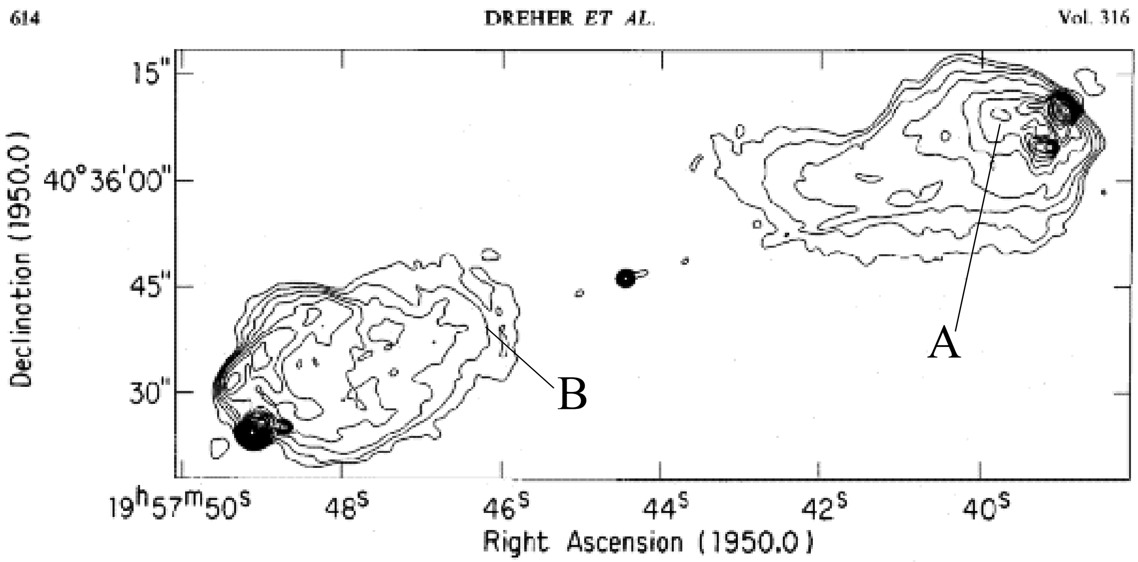


Fig. 1 - Cygnus A with 1” resolution. The contour levels are at 0.075, 0.2, 0.5, 1, 2, 3, 4, 5, 7.5, 10, 15, 20, 30,...,80, 90% of the maximum brightness of 14.5 Jy/beam.

7. *(5 marks)* <u>*The Fourier transform of the Lienard Weichert potentials.*</u> Show that, in the context of the Lienard–Weichert potentials,

$$\frac{d}{dt'}\left[\frac{\boldsymbol{n}\times(\boldsymbol{n}\times\beta(t'))}{1-\beta(t')\cdot\boldsymbol{n}}\right] = \frac{\boldsymbol{n}\times[(\boldsymbol{n}-\beta(t'))\times\dot{\beta}(t')]}{[1-\beta(t')\cdot\boldsymbol{n}]^2}$$

(Remember, that for the purpose of calculating the electromagnetic field from a distant source, we may take $\boldsymbol{n} = \text{constant}$.)

8. *(10 marks)* <u>*Transformation of Stokes parameters under a rotation.*</u> Consider a set of axes (x, y) in the plane of propagation of radiation. In this coordinate system the Stokes parameters have values $(I_\omega, Q_\omega, U_\omega, V_\omega)$. We wish to determine how the Stokes parameters change under a rotation of axes.
(a) Denote with a prime the values of the Stokes parameters in the new coordinate frame. By considering the transformation of the electric field components (E_x, E_y) show that the Stokes parameters transform under a counter-clockwise rotation of ψ according to

$$I_\omega' = I_\omega$$

$$\begin{bmatrix} Q_\omega' \\ U_\omega' \end{bmatrix} = \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ -\sin 2\psi & \cos 2\psi \end{bmatrix} \begin{bmatrix} Q_\omega \\ U_\omega \end{bmatrix}$$

$$V_\omega' = V_\omega$$

(b) Show that the frame wherein $U_\omega' = 0$ is given by $\tan 2\psi = U_\omega / Q_\omega$.

9. In astrophysics it is frequently argued that a source of radiation which undergoes a fluctuation of duration Δt must have a physical diameter of order $D < c\Delta t$. This argument is based on the fact that even if all portions of the source undergo a disturbance at the same instant and for an infinitesimal period of time, the resulting signal at the observer will be smeared out over a time interval $t_{\rm min} = D/c$ because of the finite light travel time across the source. Suppose, however, that the source is an optically thick spherical shell of radius $R(t)$ that is expanding with relativistic velocity $\beta \sim 1$, $\gamma \gg 1$ and energized by a stationary point at its centre. By consideration of relativistic beaming effects show that if the observer sees a fluctuation from the shell of duration Δt at time t, the source may actually be of radius $R < 2\gamma^2 c\Delta t$, rather than the much smaller limit given by the nonrelativistic considerations. In the rest frame of the shell surface, each surface element may be treated as an isotropic emitter. (The argument has been used to show that the active regions in quasars may be much larger than $c\Delta t \sim 1$ light month across, and thus avoids much energy being crammed into so small a volume.)

10. *(5 marks)* Dimensional correctness. Show that the expression for the minimum energy magnetic field

$$B_{\rm min} = \frac{m_e}{e}\left[\frac{a+1}{2}(1+c_E)C_2^{-1}(a)\frac{c}{m_e}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a,\gamma_1,\gamma_2)\right]^{\frac{2}{a+5}}$$

is dimensionally correct. [Useful hint: Write $I_\nu \nu^\alpha$ as $(I_\nu \nu^{-3}) \times \nu^{\frac{(a+5)}{2}}$].

11. *(10 marks)* Minimum pressure. Consider the total *pressure* of a synchrotron emitting plasma

$$P_{\rm tot} = (1+c_p)\frac{\varepsilon_e}{3} + \frac{B^2}{2\mu_0}$$

(i) Derive an expression for the value of the magnetic field that minimizes the total pressure and for the corresponding total particle pressure.
(ii) What is the numerical relationship between the minimum pressure and minimum energy density solutions?

12. *(20 marks)* Minimum pressure and energy density. Obtain a copy of the paper *The Radio Galaxy IC 4296 (PKS 1333-33) I. Multifrequency Very Large Array Observations,* by Killeen, Bicknell and Ekers, ApJ, **302**, 306-336. (This is kbe.pdf on the course web-site.) Also download the spreadsheet on minimum energy calculations that can be used as template in the following.

(i) Use the spreadsheet to estimate the minimum pressure in the western jet at $25''$ and $50''$ distance from the core.
(ii) If the jet is confined by a thermal gas in pressure equilibrium with the jet and if the temperature of the gas is approximately $10^{7\circ}$ K, then estimate the minimum number density of the thermal plasma at both positions

13. (15marks) Synchrotron cooling and pitch angle scattering

(i) It is generally assumed that electrons are scattered in pitch angle, on timescales that are short compared to the radiative time scale. (This is why the pitch angle distribution is taken to be isotropic.) If the pitch-angle distribution for an electron is uniform (in solid angle), show that, on timescales long compared to the pitch-angle scattering time,

$$\langle \sin^2\alpha \rangle = \frac{2}{3}$$

(ii) Hence show that the mean energy loss rate is given by

$$\frac{d\gamma}{dt} = -\frac{2}{3}\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_0}\right)\gamma^2$$

(iii) The synchrotron cooling timescale, $t_{\rm syn}$, is defined as the time over which an electron loses half of its energy as a result of synchrotron losses. Show that, for an electron with initial Lorentz factor, γ_0, and subject to efficient pitch angle scattering:

$$t_{\rm syn} = \frac{3}{2}\frac{\mu_0 m_e c}{\sigma_T}B^{-2}\gamma_0^{-1} = \frac{3}{2}\frac{m_e}{\varepsilon_0 c\sigma_T}B^{-2}\gamma_0^{-1} \,.$$

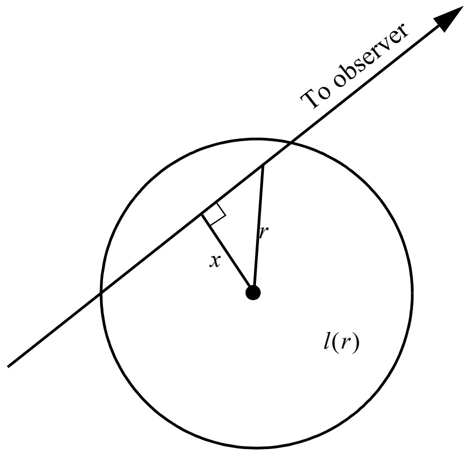
(iv) Show that the mean critical frequency of electrons with Lorentz factor γ_0 is

$$\nu_0 = \frac{3}{16}\frac{eB}{m_e}\gamma_0^2$$

(v) Hence show that the synchrotron cooling timescale expressed in terms of the mean critical frequency is given by:

$$t_{\rm syn} = \frac{3^{3/2}}{8}\frac{(em_e)^{1/2}}{\varepsilon_0 c\sigma_T}B^{-3/2}\nu_0^{-1/2}$$

(vi) Estimate the synchrotron cooling timescale for (a) GHz electrons in a 1 nT magnetic field (b) Optically emitting electrons in a 10 nT magnetic field (c) X-ray emitting electrons in a $1\,\mu$T field.



14. *(10 marks) This question introduces the Abel integral which pops up in many different astrophysical contexts, including Galactic Dynamics and X-ray emission from extended hot haloes of elliptical galaxies. Indeed, one finds the Abel integral in contexts in which one needs to calculate a surface brightness or specific intensity from a spherical emitting region.*

(a) Consider starlight from a spherical galaxy. For many purposes we approximate the source of the starlight by a smoothed luminosity density $l(r)$ where r is the distance from the centre of the galaxy. The luminosity generally refers to the luminosity integrated over a specific band, e.g. the broadband B or V filters often used by extragalactic astronomers.

Show that the surface brightness of the starlight from the galaxy is given as a function of the projected radius x by

$$\Sigma(x) = 2\int_x^{\infty} \frac{l(r) r dr}{(r^2 - x^2)^{1/2}}$$

(The integral in this expression is known as the Abel integral and the inversion of the integral to estimate the luminosity density from the surface brightness is an interesting mathematical problem.)

(b) The mass density of an elliptical galaxy within a few core radii of the core is often expressed in the Hubble form

$$\rho = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-3/2}$$

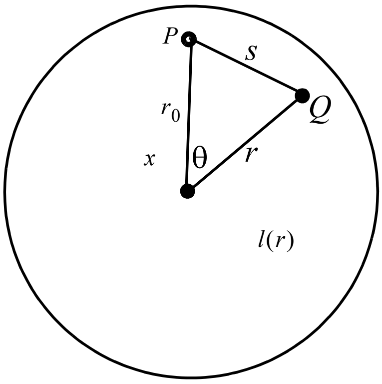
where r_c is the core radius. Assuming that the mass to light ratio in a given band (M/L) is constant, show that the surface brightness is given by:

$$\Sigma(x) = \frac{2 l_0 r_c}{1 + \frac{x^2}{r_c^2}}$$

where l_0 is the central luminosity density.

(c) Show that the central surface brightness is $2 l_0 r_c$ and that the surface brightness falls to half its central value at $x = r_c$.

15. *(15 marks)* Mean intensity and energy density in a spherical galaxy. Consider a point P at $r = r_0$ in a spherical galaxy. In some cases the inverse Compton emission from starlight in such a galaxy is an important source of high energy emission. In order to calculate the mean intensity at this point ($r = r_0$) we need to take into account the intensity along rays such as QP shown in the figure.



(a) Show that the mean intensity at P is given by the following integral over all the rays coming into P.

$$J_\nu(r_0) = \frac{1}{4\pi}\int j_\nu(r) ds d\Omega$$

where Ω is the solid angle centred on P and $j_\nu(r)$ is the averaged volume emissivity of starlight.

(b) Now show that the mean intensity can be expressed as:

$$J_\nu(r_0) = \frac{1}{4\pi}\int \frac{j_\nu}{s^2} dV$$

where dV is the volume element centred at Q.

(c) Hence show that

$$J_\nu(r_0) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} \frac{r^2 j_\nu(r) \sin\theta}{(r^2 + r_0^2 - 2 r r_0 \cos\theta)} dr$$

(d) Then show that

$$J_\nu(r_0) = \frac{1}{2}\int_0^{\infty} j_\nu(r)\left(\frac{r}{r_0}\right) \ln\left|\frac{1 + r/r_0}{1 - r/r_0}\right| dr$$

16. (*10 marks*) Energy density at the centre of a spherical galaxy. Show that at the centre of a spherical galaxy, the mean intensity is given by the central surface brightness, Σ_ν, (at that frequency) and that, at the centre of the galaxy, the energy density of starlight per unit frequency (u_ν) and integrated over the entire spectrum (u) are given by:

$$u_\nu = \frac{1}{2c}\Sigma_\nu \qquad u = \frac{1}{2c}\Sigma_{\mathrm{Bol}}$$

where Σ_{Bol} is the bolometric surface brightness.

17. *(10 marks)* The energy density of starlight in an elliptical galaxy. In a classic paper, Sargent et al. 1978 (ApJ, **221**,731) used velocity dispersions and stellar dynamics[1] to estimate the central mass and luminosity density in the Galaxy M87. This was the first paper to provide dynamical evidence for a black hole in an elliptical galaxy. In their paper, they report that

1. **You might like to consult Binney & Tremaine *Galactic Dynamics* for the theoretical stellar dynamics background.**

the central surface brightness of M87 in the V-band is 5.5×10^{3} solar luminosities pc^{-2}. "Solar luminosities" refers here to the luminosity of the Sun in V-band. The absolute magnitude of the Sun in V-band is 4.84, its bolometric absolute magnitude is 4.77 and the bolometric luminosity of the Sun is 3.83×10^{26} W.

Use these numbers to estimate the energy density of starlight in the V-band in the centre of M87. Express your answer in both $\mathrm{J\ m^{-3}}$ and $\mathrm{eV\ m^{-3}}$.

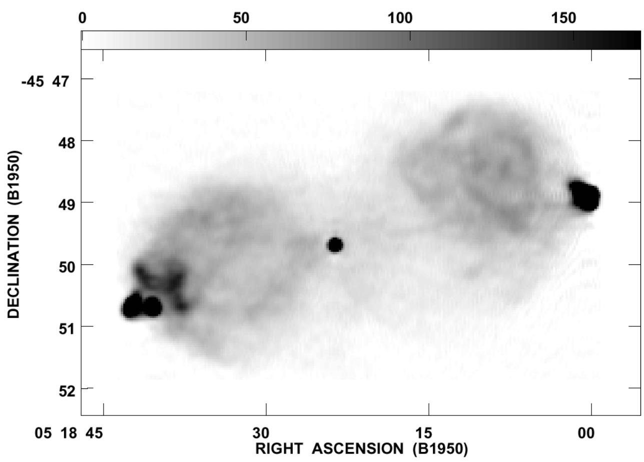
18. *(10 marks)* Energy density in a uniform emitting blob of plasma. Consider a spherical blob of plasma of radius, R, in which the emissivity, j_ν, is independent of radius and isotropic.

(a) Let $\xi = r/R$ be the dimensionaless radius of the blob and let I_ν^{peak} be its peak surface brightness. Show that the energy density inside the blob is given as a function of radius by:

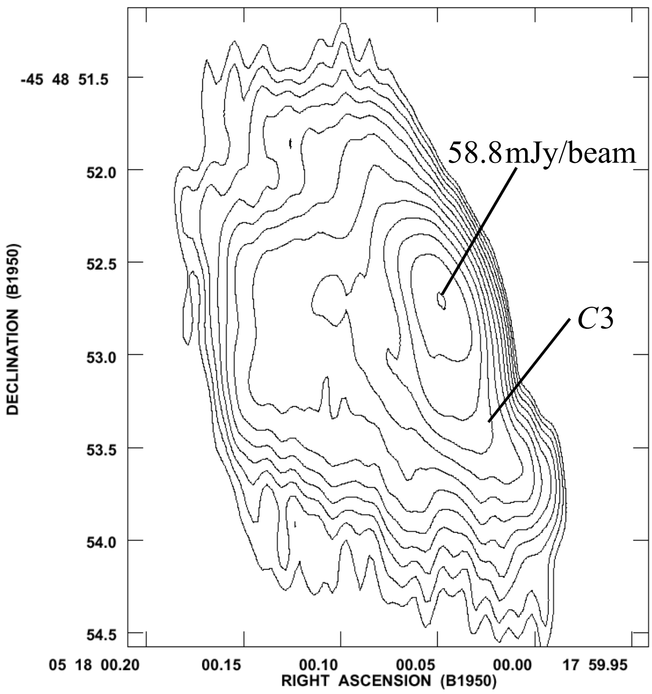
$$u_\nu(r) = \left[\frac{2\pi j_\nu R}{c}\right]\left[\frac{1}{2}(\xi^{-1}-\xi)\ln\left(\frac{1+\xi}{1-\xi}\right)+1\right]$$
$$= \frac{\pi I_\nu^{\mathrm{peak}}}{c}\left[\frac{1}{2}(\xi^{-1}-\xi)\ln\left(\frac{1+\xi}{1-\xi}\right)+1\right]$$

(b) Check the validity of this equation by expanding about the origin, $r = 0$ and independently evaluating the energy density at the origin.
(c) In order to gain an understanding of how the energy density varies with radius inside the blob, plot the function of ξ appearing in part (a).

19. *(15 marks)* Estimate of radiation energy density in a hot spot. The following image and contour map are of the radio galaxy Pictor A and are taken from the paper by Perley et al. [A&A **328**, 12-32 (1997)]. A pdf version of this paper is on the course web page.



This is a greyscale image of the entire source from Perley et al. (1997). Pictor A is an FR2 radio galaxy, similar in many respects to Cygnus A. The redshift of Pictor A is 0.0342.



This is a contour image at 2 cm wavelength of the area around the Western hotspot of Pictor A. The contours are spaced by a factor of $\sqrt{2}$ between 2.21 and 70.71 % of the peak intensity of 58.8 mJy/beam. The beam is $0.45'' \times 0.09''$.
The spectral index of the hot spot is approximately 0.7 and the spectrum breaks at approximately 10^{14} Hz.

(a) Estimate the radiation energy density in the centre of the region delineated by the 3rd contour down from the peak intensity (C3 in the diagram).

(You will have to make some assumptions in this estimate including the fact that the region around the hot spot is obviously not perfectly round.)
(b) Compare the radiation energy density in the Pictor A western hot spot to the energy density of the microwave background.

20. *(5 marks)* <u>Linear and circular polarisation.</u> (a) Show that the electric field of a linearly polarised monochromatic wave can be expressed in the form:

$$\boldsymbol{E} = (E_0 \cos\psi \cos\omega t)\boldsymbol{e}_1 + (E_0 \sin\psi \cos\omega t)\boldsymbol{e}_2$$

and give a physical interpretation for the angle ψ.
(b) Show that a linearly polarised wave may be expressed as the sum of two oppositely circularly polarised waves and determine the phase difference between the two circularly polarised components as a function of the parameters of the linearly polarised wave.

21. *(10 marks)* <u>Retarded time.</u> Consider the electromagnetic field at a point described by polar coordinates r, θ due to a charge q in constant rectilinear motion at velocity βc along the polar axis. Assume that the particle is at the origin $r = 0$ at $t = 0$.
(a) Show that the retarded time is given by the smaller of the solutions of the quadratic equation

$$(t - t')^2 = \frac{r^2}{c^2} - 2t'\left(\frac{r}{c}\right)\beta\cos\theta + \beta^2 t'^2$$

(b) Show that the electric and magnetic fields of a charge q in constant rectilinear motion at velocity βc are given by:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0\gamma^2}\frac{\boldsymbol{x}' - r'\beta}{(r' - \boldsymbol{x}'\cdot\beta)^3}$$

$$\boldsymbol{B} = \frac{q\mu_0 c}{4\pi\gamma^2}\frac{\beta\times\boldsymbol{x}'}{(r' - \boldsymbol{x}'\cdot\beta)^3}$$

with $\gamma = (1-\beta^2)^{-1/2}$.

22.*(10 marks)* Jet sidedness.

Consider two oppositely directed relativistic jets in a radio galaxy and assume that the jets on each side of the nucleus are identical.

(i) For optically thin jets show that the ratio of surface brightnesses of the two jets is given by

$$R = \frac{I_\nu(\text{main jet})}{I_\nu(\text{counter jet})} = \left(\frac{1+\beta\cos\theta}{1-\beta\cos\theta}\right)^{2+\alpha}$$

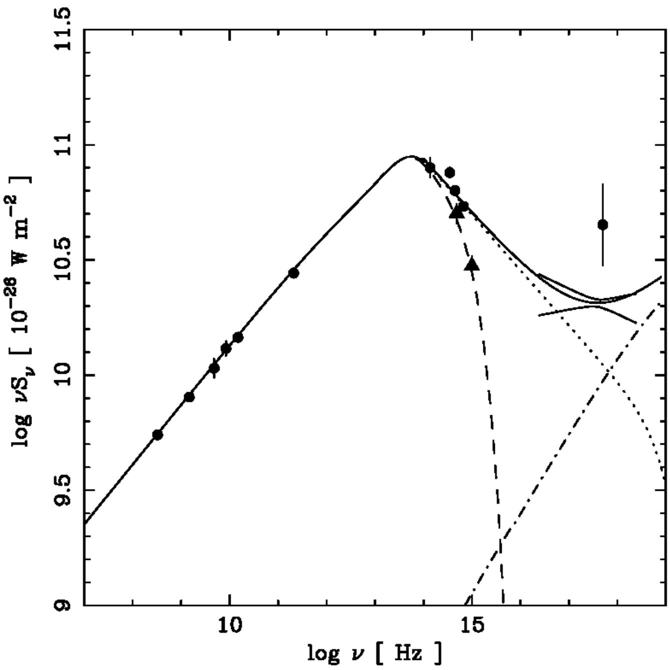
where θ is the inclination of the main (brightest) jet, to the line of sight, and α is the optically thin spectral index.

(ii) Show that the speed of the jet and the angle of inclination are related by:

$$\beta\cos\theta = \frac{R^{\frac{1}{2+\alpha}} - 1}{R^{\frac{1}{2+\alpha}} + 1}$$

(iii) The Radio Galaxy M87 has been one of the most thoroughly studied active galaxies, in many regions of the spectrum, from radio through to X–ray. You might like to have a look at a recent observational paper by Biretta, Zhou and Owen in ApJ, **447**, 582 (biretta.pdf on the course web-site) as well as the cited references. In the optical region where the spectral index is approximately 1.5, the ratio of the jet surface brightness to the surface brightness of a putative counter jet is at least 150. Assuming that there is an almost identical counter jet, determine the lower limit on the velocity and the maximum inclination of the jet to the line of sight.

23. *(20 marks)* <u>*Estimation of magnetic field from Inverse Compton emission.*</u> The plot at the right is the spectrum (plotted as fluence) of hot spot A in Pictor A appearing in a paper by Wilson, Young and Shopbell (astro-ph 0008467). The complete paper is available from the course web-site.



The bowtie represents the Chandra data point and the range of allowable slopes of the X-ray fluence. This slope does not even approximately match the slope in the radio part of the spectrum.

One possible interpretation of this is that the continuation of the synchrotron spectrum (indicated by the dotted line) also provides some contribution to the X-ray spectrum.

(a) On the basis of this interpretation, estimate the magnetic field in the Pictor A hot spot. (Refer back to Q5 of Set 2 for additional data that you may need.) The electron index is another essential parameter. This is in the Wilson et al. paper (somewhere!).

(b) Compare the magnetic field estimated in this way with the minimum energy magnetic field.

(c) Wilson et al. considered another alternative, namely that the X-ray emission corresponds to the inverse Compton emission from an invisible low energy radio emitting component. Outline the physical reasons why they would consider this as an alternative. In considering this part you should consider first the possibility that the X-ray emission is the peak of the inverse Compton emission corresponding to the peak of the of the synchrotron emission at around $10^{13.7}$ Hz.

24. *(10 marks)* <u>*Synchrotron self-absorption.*</u>

(i) Show that the brightness temperature of a self-absorbed synchrotron source is given by

$$T_b \approx 2.0\times10^9 \gamma \text{ K}$$

where γ is the Lorentz factor of the radiating electrons. Hence shown that the minimum brightness temperature of a self-absorbed source is approximately 2×10^9 K.

(ii) Obtain a copy of the paper *Parsec-Scale Images of Flat-Spectrum Radio Sources in Seyfert Galaxies,* by Mundell et al. (mundell.pdf on the course web-site). In that paper, the authors present VLBA images of 5 Seyfert galaxies with flat spectrum cores which they argue are synchrotron self absorbed. However, the cores are unresolved and the *apparent* brightness temperatures are all between about 10^6 K and 10^8 K. Using the information in that paper, determine the *maximum* sizes of the flat spectrum regions, assuming that they are synchrotron self-absorbed.

25. *(15 marks)* *Free-free absorption.*

This question is about another process which can be important for the absorption of radio waves – free-free absorption or bremsstrahlung absorption. This is the absorption process corresponding to free-free or bremsstrahlung emission.

Electrons in a thermal plasma are accelerated as they encounter ions in *Coulomb Collisions,* giving rise to radiation.The emissivity for encounters between ions of atomic number Z (density $n_i(Z)$) and electrons (density n_e), is

$$\varepsilon_\nu(Z) = \frac{1}{12\pi^3}\left(\frac{\pi}{6}\right)^{1/2}\frac{Z^2e^6}{\varepsilon_0^3c^3m_e^2}\left(\frac{m_e}{kT}\right)^{1/2} g(\nu, T)n_i(Z)n_e\exp\left(-\frac{h\nu}{kT}\right)$$

where $g(\nu, T)$ is a (slowly varying) *Gaunt* factor. At radio wavelengths,

$$g(\nu, T) = \frac{\sqrt{3}}{2\pi}\left[\ln\left(\frac{128\varepsilon_0^2k^3T^3}{m_ee^4\nu^2Z^2}\right) - \gamma_E^{1/2}\right]$$

where $\gamma_E \approx 0.577.....$ is Euler's constant (not a Lorentz factor).

(i) Determine an expression for the free-free absorption coefficient resulting from ions of atomic number Z and electrons, and show that in the Rayleigh-Jeans limit, it becomes:

$$\alpha_\nu(Z) = \frac{1}{24\pi^3}\left(\frac{\pi}{6}\right)^{1/2}\frac{Z^2e^6}{\varepsilon_0^3m_e^{3/2}(kT)^{3/2}c}g(\nu, T)n_i(Z)n_e\nu^{-2} \quad \mathrm{m}^{-1}$$

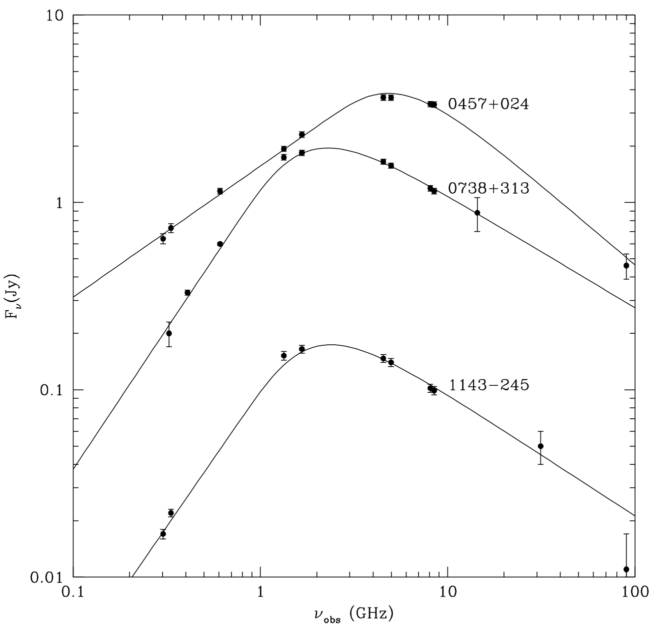
(ii) Show that numerically the above expression becomes:

$$\alpha_\nu(Z) \approx 1.760\times10^{-12}\left[10.36 + 0.2757\ln\left(\frac{T^3}{\nu^2Z^2}\right)\right]Z^2n_i(Z)n_eT^{-1.5}\nu^{-2}$$

(iii) The absorption coefficient for the entire plasma involves summing over all ions. Suppose that the plasma is completely ionised and we count contributions from Hydrogen and Helium only. Further assume that the abundance of Helium by number is 0.1 and that the Hydrogen density is n_H. Show then that the total absorption coefficient,

$$\alpha_\nu = \sum_Z \alpha_\nu(Z) \approx 2.11\times10^{-12}\left[14.4 + 0.386\ln\left(\frac{T^3}{\nu^2}\right)\right]n_H^2T^{-1.5}\nu^{-2}$$

(iv) Describe the general characteristics of a free-free absorbed spectrum for which the emissivity of the particles is a power-law.



26. Some AGN known as Gigahertz Peak Spectrum (GPS) radio sources exhibit a peak in the radio spectrum at GHz frequencies. (See the examples at the left.) It has been suggested that this may be the result of free-free absorption by dense, ionised ambient gas at a temperature of about 10^4 K. (Although it is not essential you may like to consult the paper, Unification of the Radio and *Optical Properties of Gigahertz Peak Spectrum and Compact Steep-Spectrum Radio Sources,* by Bicknell, Dopita and O'Dea – bdo.pdf on the course web-site.)
(i) Suppose that the ionised gas is uniform and has an extent of 1 kpc. Determine the ambient density.
(ii) A more realistic model may be that the ambient ionised gas is distributed as a power-law, i.e.

$$n_H = n_0\left(\frac{r}{r_0}\right)^{-\delta}$$

where n_0 is the Hydrogen density at $r_0 = 1$ kpc. Suppose that the source has a characteristic double lobed structure and that its radial extent is approximately 500 pc. Estimate n_0 for $\delta = 2$.
m.

27.*(15marks)* <u>*Synchrotron masers.*</u>
Show that a synchrotron maser in vacuum is unlikely.

28.*(5 marks)* <u>*Surface density and central density of accretion disk.*</u> Show that, for an isothermal accretion disk, the surface density, Σ and the central density, ρ_c are related by

$$\Sigma = \sqrt{\pi}\rho_c h$$

29.*(10 marks)* <u>*Accretion disc parameters.*</u> Once the viscosity parameter, α, is known, the surface density of an accretion disc may be estimated from the mass accretion rate, $\dot{M}$. (See Accretion Discs Part II). Consider an accretion disc with the following parameters: $M = 10^7$ solar masses, $\dot{M} = 0.1\dot{M}_{edd}$, $\alpha = 0.1$. Determine the following parameters at a radius of 10 gravitational radii from the central black hole:

(i) The temperature (ii) h/r (iii) The surface density (in $\mathrm{kg\ m^{-2}}$) (iv) The mid-plane disk density (in $\mathrm{kg\ m^{-3}}$) (v) The Keplerian velocity in $\mathrm{km\ s^{-1}}$ (vi) The radial inflow velocity in km s^{-1} .

30. *(20 marks*) <u>*Estimation of parameters in VLBI jets.*</u> Obtain a copy of the paper *VLBI observations of 3C 273 at 22 GHz and 43 GHz*, by Mantovani et al. from the course web-site (mantovani.pdf) and use the information in that paper, information given in lectures and any other related papers to estimate the magnetic field and particle energy density in the self-absorbed component at the base of the jet.

Some of your estimates may involve inspired assumptions and at some stages the estimates may be based upon uncertain parameters such as the size of the component. Your estimate of the required parameters should therefore make clear what assumptions are involved and in what way the estimate depends upon any unknown parameters.