Summary of the Equations of Fluid Dynamics

Reference:

Fluid Mechanics, L.D. Landau & E.M. Lifshitz

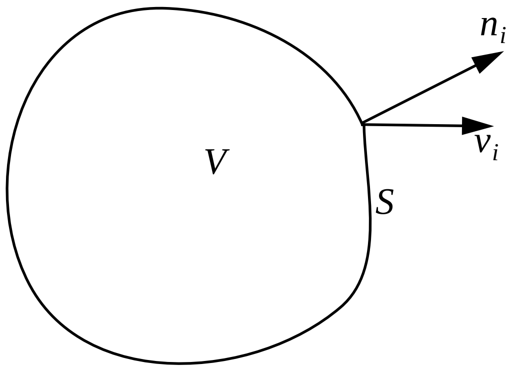
1 Introduction

Emission processes give us diagnostics with which to estimate important parameters, such as the density, and magnetic field, of an astrophysical plasma. Fluid dynamics provides us with the capability of understanding the transport of mass, momentum and energy. Normally one spends more than a lecture on Astrophysical Fluid Dynamics since this relates to many areas of astrophysics. In following lectures we are going to consider one principal application of astrophysical fluid dynamics – accretion discs. Note also that magnetic fields are not included in the following. Again a full treatment of magnetic fields warrants a full course.

2 The fundamental fluid dynamics equations

The equations of fluid dynamics are best expressed via conservation laws for the conservation of mass, momentum and energy.

2.1 Conservation of mass



Control volume for assessing conservation of mass.

Consider the rate of change of mass within a fixed volume. This changes as a result of the mass flow through the bounding surface.

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho v_i n_i \, dS$$

Using the divergence theorem,

$$\frac{\partial}{\partial t}\int_V \rho \, dV + \int_V \frac{\partial}{\partial x_i}(\rho v_i) \, dV = 0$$

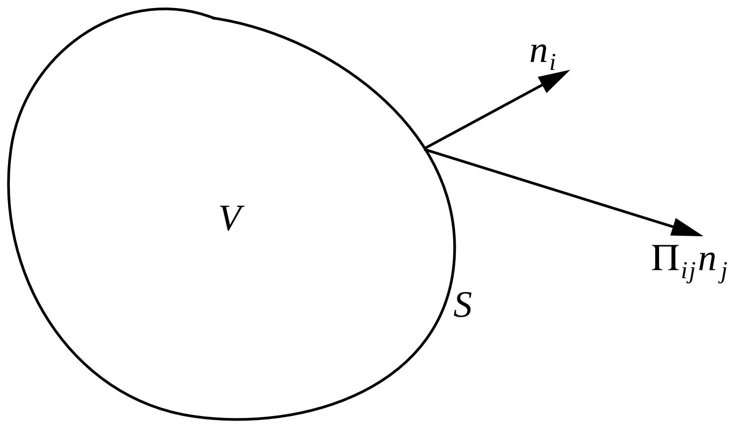
$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) \right) dV = 0$$

The continuity equation

Since the volume is arbitrary,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

2.2 Conservation of momentum



Consider now the rate of change of momentum within a volume. This decreases as a result of the flux of momentum through the bounding surface and increases as the result of body forces (in our case gravity) acting on the volume. Let

$$\Pi_{ij} = \text{Flux of } i \text{ component of momentum in the } j \text{ direction}$$

and

$$f_i = \text{Body force per unit mass}$$

then

$$\frac{\partial}{\partial t}\int_V \rho v_i dV = -\int_S \Pi_{ij} n_j dS + \int_V \rho f_i dV$$

There is an equivalent way of thinking of Π_{ij}, which is often useful, and that is, $\Pi_{ij} n_j dS$ is the i^{th} component of the force exerted on the fluid exterior to S by the fluid interior to S.

Again using the divergence theorem,

$$\int_V \left(\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial \Pi_{ij}}{\partial x_j} \right) dV = \int_V \rho f_i dV$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial \Pi_{ij}}{\partial x_j} = \rho f_i$$

Gravity

For gravity we use the gravitational potential

$$f_i = -\frac{\partial \phi_G}{\partial x_i}$$

For a single gravitating object of mass M

$$\phi_G = -\frac{GM}{r}$$

and for a self-gravitating distribution

$$\nabla^2 \phi_G = 4\pi G \rho$$

$$\Rightarrow \phi_G = -G \int_{V'} \frac{\rho(x_i')}{|x_i - x_i'|} d^3x'$$

where G is Newton's constant of gravitation.

Expressions for Π_{ij}

The momentum flux is composed of a bulk part plus a part resulting from the motion of particles moving with respect to the centre of mass velocity of the fluid (v_i). For a perfect fluid (an approximation often used in astrophysics), we take p to be the isotropic pressure, then

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij}$$

The equations of motion are then:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p\delta_{ij}) = -\rho \frac{\partial \phi_G}{\partial x_i}$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi_G}{\partial x_i}$$

There is also another useful form for the momentum equation derived using the continuity equation.

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = v_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial}{\partial x_j}(\rho v_j) + \rho v_j \frac{\partial v_i}{\partial x_j}$$

$$= v_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) \right] + \left[\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} \right]$$

$$= \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j}$$

Hence, another form of the momentum equation is:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi_G}{\partial x_i}$$

On dividing by the density

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \phi_G}{\partial x_i}$$

Differentiation following the motion

This is a good place to introduce differentiation following the motion. For a function $f(x_i, t)$, the variation of f following the motion of a fluid element which has coordinates

$$x_i = x_i(t)$$

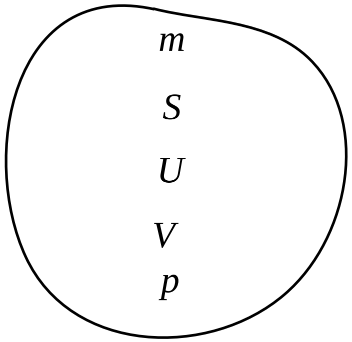
is given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i}$$

Hence, the momentum equation can be written compactly as

$$\rho\frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \phi_G}{\partial x_i}$$

2.3 Thermodynamics



Element of fluid and the variables used to describe its state.

Before going on to consider the consequences of the conservation of energy, we consider the thermodynamics of a *comoving* volume element. (See the figure at left.)

Define:

m = Mass of element

ε = Internal energy density per unit volume

P = pressure (as above)

s = entropy per unit mass

T = temperature (in degrees Kelvin)

We have the following quantities for the volume element:

$$U = \text{Total internal energy} = \frac{m\varepsilon}{\rho}$$

$$S = \text{Entropy} = ms$$

$$V = \text{volume} = \frac{m}{\rho}$$

The second law of thermodynamics tell us that the change in entropy of a mass of gas is related to changes in other thermodynamic variables as follows:

$$kTdS = dU + pdV$$

$$\Rightarrow kTd(ms) = d\left(\frac{m\varepsilon}{\rho}\right) + pd\left(\frac{m}{\rho}\right)$$

$$\Rightarrow kTds = d\left(\frac{\varepsilon}{\rho}\right) + pd\left(\frac{1}{\rho}\right) = \frac{1}{\rho}d\varepsilon - \frac{(\varepsilon + p)}{\rho^2}d\rho$$

$$\Rightarrow \rho kTds = d\varepsilon - \frac{(\varepsilon + p)}{\rho}d\rho$$

Specific enthalpy

A commonly used thermodynamic variable is the specific enthalpy:

$$h = \frac{\varepsilon + p}{\rho}$$

In terms of the specific enthalpy, the equation

$$kTds = d\left(\frac{\varepsilon}{\rho}\right) + pd\left(\frac{1}{\rho}\right)$$

becomes

$$kTds = d\left(\frac{\varepsilon + p}{\rho}\right) - d\left(\frac{p}{\rho}\right) + pd\left(\frac{1}{\rho}\right) = dh - \frac{dp}{\rho}$$

For a parcel of fluid following the motion, we obtain, after dividing by the time increment of a volume element,

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + p)}{\rho}\frac{d\rho}{dt}$$

$$kT\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt}$$

The fluid is *adiabatic* when there is no transfer of heat in or out of the volume element:

$$kT\frac{ds}{dt} = 0 \Rightarrow \begin{aligned} \frac{d\varepsilon}{dt} - \frac{(\varepsilon + p)}{\rho}\frac{d\rho}{dt} &= 0 \\ \frac{dh}{dt} - \frac{1}{\rho}\frac{dp}{dt} &= 0 \end{aligned}$$

The quantities ds, $d\varepsilon$,dp etc. are perfect differentials, and these relationships are valid relations from point to point within the fluid. Two particular relationships we shall use in the following are:

$$\rho kT\frac{\partial s}{\partial t} = \frac{\partial \varepsilon}{\partial t} - h\frac{\partial \rho}{\partial t}$$

$$\rho kT\frac{\partial s}{\partial x_i} = \rho\frac{\partial h}{\partial x_i} - \frac{\partial p}{\partial x_i}$$

2.3.1 Equation of state

The above equations can be used to derive the equation of state of a gas in which the ratio of specific heats ($\gamma = c_p/c_v$) is a constant. Consider the following form of the entropy, internal energy, pressure relation:

$$\rho kTds = d\varepsilon - \frac{(\varepsilon + p)}{\rho}d\rho$$

In a perfect gas,

$$p = \frac{\rho k T}{\mu m_p}$$

where μ is the mean molecular weight and

$$p = (\gamma - 1)\varepsilon \Rightarrow \varepsilon + p = \gamma\varepsilon$$

Hence,

$$\mu m_p(\gamma - 1)\varepsilon ds = d\varepsilon - \frac{\gamma\varepsilon}{\rho} d\rho$$

$$\Rightarrow \mu m_p(\gamma - 1) ds = \frac{d\varepsilon}{\varepsilon} - \frac{\gamma}{\rho} d\rho$$

$$\Rightarrow \mu m_p(\gamma - 1)(s - s_0) = \ln\varepsilon - \gamma \ln\rho$$

$$\Rightarrow \frac{\varepsilon}{\rho^\gamma} = \exp[\mu m_p(\gamma - 1)(s - s_0)] = \exp[\mu m_p(\gamma - 1)s]$$

(We can discard the s_0 since the origin of entropy is arbitrary.)

We therefore have,

$$\begin{aligned}\varepsilon &= \exp[\mu m_p(\gamma-1)s]\times\rho^\gamma \\ p &= (\gamma-1)\exp[\mu m_p(\gamma-1)s]\times\rho^\gamma \\ &= K(s)\rho^\gamma\end{aligned}$$

The function $K(s)$ is often referred to as the pseudo-entropy. For a completely ionised monatomic gas $\gamma = 5/3$.

2.4 Conservation of energy

Take the momentum equation in the form:

$$\rho\frac{\partial v_i}{\partial t} + \rho v_j\frac{\partial v_i}{\partial x_j} = -\frac{\partial P}{\partial x_i} - \rho\frac{\partial \phi_G}{\partial x_i}$$

and take the scalar product with the velocity:

$$\rho v_i\frac{\partial v_i}{\partial t} + \rho v_j v_i\frac{\partial v_i}{\partial x_j} = -v_i\frac{\partial P}{\partial x_i} - \rho v_i\frac{\partial \phi_G}{\partial x_i}$$

$$\Rightarrow \rho\frac{\partial}{\partial t}\left(\frac{1}{2}v_i v_i\right) + \rho v_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}v_i v_i\right) = -v_i\frac{\partial P}{\partial x_i} - \rho v_i\frac{\partial \phi_G}{\partial x_i}$$

That is,

$$\rho\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) + \rho v_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}v^2\right) = -v_i\frac{\partial p}{\partial x_i} - \rho v_i\frac{\partial \phi_G}{\partial x_i}$$

Before, we used the continuity equation to move the ρ and ρv_j outside the differentiations. Now we can use the same technique to move them inside and we recover the equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_i\frac{\partial p}{\partial x_i} - \rho v_i\frac{\partial \phi_G}{\partial x_i}$$

The aim of the following is to put the right hand side into some sort of divergence form.

Consider first the term

$$\begin{aligned} -v_i\frac{\partial P}{\partial x_i} &= \rho k T v_i\frac{\partial s}{\partial x_i} - \rho v_i\frac{\partial h}{\partial x_i} \\ &= \rho k T\frac{ds}{dt} - \rho k T\frac{\partial s}{\partial t} - \rho v_i\frac{\partial h}{\partial x_i} \\ &= \rho k T\frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} + h\frac{\partial \rho}{\partial t} - \rho v_i\frac{\partial h}{\partial x_i} \end{aligned}$$

We now eliminate the $\frac{\partial \rho}{\partial t}$ term using continuity, viz

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho v_i)$$

and we obtain

$$-v_i \frac{\partial p}{\partial x_i} = \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} - h \frac{\partial}{\partial x_i}(\rho v_i) - \rho v_i \frac{\partial h}{\partial x_i}$$

$$= \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x_i}(\rho h v_i)$$

The term

$$
\begin{aligned}
-\rho v_i \frac{\partial \phi_G}{\partial x_i} &= -\frac{\partial}{\partial x_i}(\rho \phi_G v_i) + \phi_G \frac{\partial}{\partial x_i}(\rho v_i) \\
&= -\frac{\partial}{\partial x_i}(\rho \phi_G v_i) - \phi_G \frac{\partial \rho}{\partial t} \\
&= -\frac{\partial}{\partial x_i}(\rho \phi_G v_i) - \frac{\partial}{\partial t}(\rho \phi_G) + \rho \frac{\partial \phi_G}{\partial t}
\end{aligned}
$$

When the gravitational potential is constant in time,

$$
\frac{\partial \phi_G}{\partial t} = 0 \Rightarrow -\rho v_i \frac{\partial \phi_G}{\partial x_i} = -\frac{\partial}{\partial x_i}(\rho \phi_G v_i) - \frac{\partial}{\partial t}(\rho \phi_G)
$$

Hence, the energy equation

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = -v_i \frac{\partial P}{\partial x_i} - \rho v_i \frac{\partial \phi_G}{\partial x_i}
$$

becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j\right) = \rho k T\frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x_i}(\rho h v_i)$$
$$-\frac{\partial}{\partial x_i}(\rho \phi_G v_i) - \frac{\partial}{\partial t}(\rho \phi_G)$$

Bringing terms over to the left hand side:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho \phi_G v_j\right) = \rho k T\frac{ds}{dt}$$

When the fluid is adiabatic

$$\rho k T\frac{ds}{dt} = 0$$

and we have the energy equation for a perfect fluid:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho \phi_G v_j\right) = 0$$

The total energy per unit volume is

$$E = \frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G = \text{Kinetic + internal + gravitational energy}$$

and the *energy flux* is

$$F_{E,\,i} = \frac{1}{2}\rho v^2 v_i + \rho h v_i + \rho\phi_G v_i = \rho\left(\frac{1}{2}v^2 + h + \phi\right)v_i$$

$$= \text{Kinetic + enthalpy + gravitational fluxes}$$

3 Fluids with viscosity

In most astrophysical contexts we do not have to consider molecular viscosity since it is generally small. However, we do need to consider viscosity in circumstances where it is important to discuss the means whereby energy is dissipated in a fluid.

3.1 The momentum flux in a viscous fluid

The starting point for considering viscosity is the momentum flux. We put

$$\Pi_{ij} = \rho v_i v_j + p\delta_{ij} - \sigma_{ij}$$

where, the viscous stress tensor, σ_{ij}, is given by

$$\sigma_{ij} = \eta\left(v_{i,j} + v_{j,i} - \frac{2}{3}\delta_{ij}v_{k,k}\right) + \zeta\delta_{ij}v_{k,k}$$

$$= 2\eta s_{ij} + \zeta\delta_{ij}v_{k,k}$$

The tensor

$$s_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i} - \frac{2}{3}\delta_{ij}v_{k,k}\right)$$

is the (trace-free) *shear tensor* of the fluid and

$$v_{k,k}$$

is the dilation, which is important in compressible fluids.

The complete equations of motion are therefore:

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} - \rho\frac{\partial \phi_G}{\partial x_i}$$

3.2 Energy conservation

If we now take the scalar product of the momentum equation with v_i we obtain

$$\rho\frac{\partial}{\partial t}\left(\frac{1}{2}v^2\right) + \rho v_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}v^2\right) = -v_i\frac{\partial P}{\partial x_i} - \rho v_i\frac{\partial \phi_G}{\partial x_i} + v_i\frac{\partial \sigma_{ij}}{\partial x_j}$$

that is, the same as before, but with the additional term

$$v_i\frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(v_i\sigma_{ij}) - v_{i,\,j}\sigma_{ij}$$

Hence the energy equation becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho\phi_G v_j - v_i\sigma_{ij}\right)$$

$$= \rho k T\frac{ds}{dt} - v_{i,\,j}\sigma_{ij}$$

The quantity $v_i \sigma_{ij} n_j$ is interpreted as the work done on the fluid by the viscous force; hence its appearance with terms that we associate with the energy flux. This is not the full story, however. When there is momentum transport associated with viscosity, there is also a heat flux, $\boldsymbol{q}$ which is often represented as being proportional to the temperature gradient with a heat conduction coefficient κ, i.e.

$$q_i = -\kappa \frac{\partial T}{\partial x_i}$$

We then write the full energy equation as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho\phi_G v_j - v_i \sigma_{ij} + q_j\right) \\ &= \rho k T \frac{ds}{dt} - v_{i,j}\sigma_{ij} + q_{j,j} \end{aligned}$$

Conservation of energy is expressed by:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \varepsilon + \rho\phi_G\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{2}\rho v^2 v_j + \rho h v_j + \rho\phi_G v_j - v_i \sigma_{ij} + q_j\right) = 0$$

and the entropy changes according to

$$\rho k T \frac{ds}{dt} = v_{i,j}\sigma_{ij} - q_{j,j}$$

The term $v_{i,j}\sigma_{ij}$ represents *viscous heating* and the term $q_{j,j}$ represents escape of heat from the volume resulting from the heat flux. The viscous heating term can be written:

$$\sigma_{ij}v_{i,j} = 2\eta s_{ij}s_{ij} + \zeta v_{k,k}^2$$

remembering the definition of the shear tensor:

$$s_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i} - \frac{2}{3}\delta_{ij}v_{k,k}\right)$$