Cosmological Space-Times

Lecture notes compiled by Geoff Bicknell based primarily on:

Sean Carroll: *An Introduction to General Relativity*

plus additional material

Metric of special relativity

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \\ &= -dx^{0^2} + dx^2 + dy^2 + dz^2 \\ &= \eta_{\mu\nu} dx^\mu dx^\nu \\ \text{where} \quad \eta_{\mu\nu} &= \text{Minkowski tensor} = \text{diag}\left[-1, 1, 1, 1\right] \\ \mu,\ \nu &= 0, 1, 2, 3 \end{aligned}$$

This is the metric of four-dimensional flat space time

Generalised by Einstein in his 1916 General Theory of Relativity to:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Metric tensor

General relativity field equations

$$
\begin{aligned}
g_{\mu\nu} &\Rightarrow \text{Christoffel Symbols} \\
\Gamma^{\sigma}_{\mu\nu} &= \frac{1}{2} g^{\sigma\rho} \left(g_{\nu\rho,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho} \right) \\
g^{\sigma\rho} &= \text{Inverse of } g_{\mu\nu}
\end{aligned}
$$

The Christoffel symbols appear in the equations of test particles: Geodesics of space time - and also in generalised (covariant) derivatives

Riemann curvature tensor:

$$R^{\rho}_{\sigma\mu\nu} = \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

Tensors derived by contraction over indices

Ricci tensor

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Einstein's field equations

$$G_{\mu\nu} = \Lambda g_{\mu\nu} + \frac{8\pi G}{c^4} T_{\mu\nu}$$

Newton's Constant of gravitation

Cosmological constant
"Dark energy"

Matter tensor

Matter tensor

$$T_{\mu\nu} = (\rho c^2 + p) U^\mu U^\nu + p g_{\mu\nu}$$

4-velocity of matter $U^\mu = \frac{dx^\mu}{ds} = \frac{1}{c}\frac{dx^\mu}{d\tau}$

Metric of the Universe

Homogeneity and isotropy => Geometry invariant under translations and rotations

=> Maximally symmetric space time

$$ds^2 = -c^2 dt^2 + a^2(t) \left[e^{2\beta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2 \right]$$

Spatial part of metric:

$$d\sigma^2 = a^2(t) \left[e^{2\beta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

When $e^{2\beta} = 1$

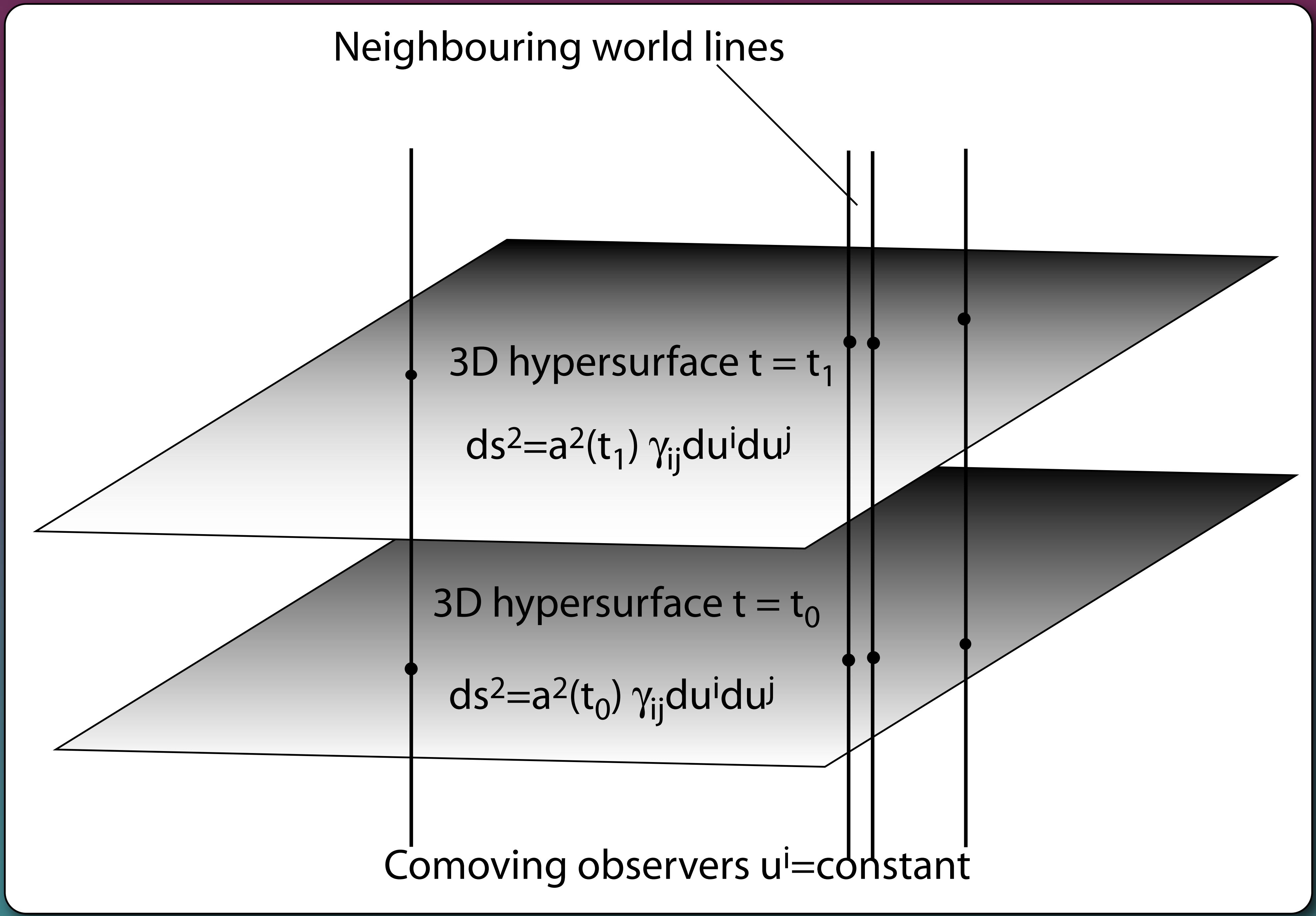
$$d\sigma^2 = a^2(t)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right]$$

which is the normal metric of flat space modified by the scale factor a(t)

The scale factor informs us how the universe is expanding

In this space-time metric the coordinates are <u>comoving coordinates</u>, i.e. as the Universe expands the spatial coordinates of galaxies remain constant

Space-time geometry



Geometry of 3D hypersurfaces

$$d\sigma^2 = a^2(t)\ \left[e^{2\beta(r)}dr^2 + r^2 d\Omega^2\right]$$

Maximally symmetric spaces (consequence of homogeneity and isotropy)

Characterised by

$$
\begin{aligned}
{}^{(3)}R_{ijkl} &= k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}) \\
\Rightarrow R_{ij} &= 2k\gamma_{ij}
\end{aligned}
$$

Spatial metric

$$\gamma_{ij} = \mathrm{diag}(e^{2\beta(r)}, r^2, r^2 \sin^2\theta)$$

Equations for metric tensor

$$
\begin{aligned}
{}^{(3)}R_{11} &= e^{-2\beta}\left(r\frac{\partial\beta}{\partial r} - 1\right) + 1 = 2k\gamma_{11} = 2ke^{2\beta} \\
{}^{(3)}R_{22} &= e^{-2\beta}\left(r\frac{\partial\beta}{\partial r} - 1\right) + 1 = 2kr^2 \\
{}^{(3)}R_{33} &= \left[e^{-2\beta}\left(r\frac{\partial\beta}{\partial r} - 1\right) + 1\right]\sin^2\theta = 2kr^2\sin^2\theta
\end{aligned}
$$

Solution

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

$$\begin{aligned} e^{2\beta} &= \frac{1}{1-kr^2} \\ d\sigma^2 &= a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \end{aligned}$$

Coordinate transformation

$$\begin{aligned} r'^2 &= |k|r^2 \\ \Rightarrow r' &= |k|^{1/2} r \\ \Rightarrow d\sigma^2 &= \frac{a^2(t)}{|k|}\left[\frac{dr'^2}{1-\mathrm{sgn}(k)r'^2} + r'^2 d\Omega^2\right] \end{aligned}$$

Absorb $|k|^{1/2}$ into a(t); k = -1, 0, 1

$$\begin{aligned} e^{2\beta} &= \frac{1}{1-kr^2} \\ d\sigma^2 &= a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right] \\ k &= -1, 0, +1 \end{aligned}$$

k=0

$$d\sigma^2 = a^2(t)\left[dr^2+r^2d\Omega^2\right]$$

Expanding flat space

k=-1

$$d\sigma^2 = a^2(t)\left[\frac{dr^2}{1+r^2} + r^2 d\Omega^2\right]$$

New radial variable

$$\begin{aligned} d\chi &= \frac{dr}{(1+r^2)^{1/2}} \\ \Rightarrow \chi &= \sinh^{-1} r \\ r &= \sinh\chi \end{aligned}$$

Metric of each 3D hypersurface

$$d\sigma^2 = a^2(t)\left[d\chi^2 + \sinh^2\chi d\Omega^2\right]$$

k=+1

$$d\sigma^2 = a^2(t)\left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right]$$

New radial variable

$$\begin{aligned} d\chi &= \frac{dr^2}{(1-r^2)^{1/2}} \\ \Rightarrow \chi &= \sin^{-1} r \\ r &= \sin\chi \end{aligned}$$

Metric of each 3D hypersurface

$$d\sigma^2 = a^2(t)\left[d\chi^2 + \sin^2\chi d\Omega^2\right]$$

Summary of 3D metrics

$$d\sigma^2 = a^2(t)\left[d\chi^2 + S^2(\chi)d\Omega^2\right]$$

$$\begin{aligned} S(\chi) &= \chi & k &= 0 \\ S(\chi) &= \sinh(\chi) & k &= -1 \\ S(\chi) &= \sin\chi & k &= +1 \end{aligned}$$

What geometry do these metrics represent?

k=0 => Metric of an expanding flat space

k=+1

Consider a 3-sphere embedded in a 4-dimensional Euclidean space (not space-time)

Let the equation of the sphere in (w,x,y,z) space be:

$$w^2 + x^2 + y^2 + z^2 = a^2$$

The metric of the 4-dimensional space is:

$$d\sigma^2 = dw^2 + dx^2 + dy^2 + dz^2$$

Metric of the surface of the 3-sphere

Consider the following set of spherical polars in 4-space; these provide a parametric description of the surface of the 3-sphere which has radius a(t). There are 3 angular parameters.

$$\begin{aligned} w &= a\cos\chi \\ z &= a\sin\chi\cos\theta \\ x &= a\sin\chi\sin\theta\cos\phi \\ y &= a\sin\chi\sin\theta\sin\phi \end{aligned}$$

We now determine the metric of the surface of the sphere by determining the differentials of the coordinates w, x, y and z.

Euclidean metric restricted to 3-sphere

These are the differentials of w,x,y,z in terms of the polar angles

$$
\begin{aligned}
dw &= -a\sin\chi d\chi \\
dz &= a\cos\chi\cos\theta\, d\chi - a\sin\chi\sin\theta\, d\theta \\
dx &= a\cos\chi\sin\theta\cos\phi\, d\chi + a\sin\chi\cos\theta\cos\phi\, d\theta - a\sin\chi\sin\theta\sin\phi\, d\phi \\
dx &= a\cos\chi\sin\theta\sin\phi\, d\chi + a\sin\chi\cos\theta\sin\phi\, d\theta + a\sin\chi\sin\theta\cos\phi\, d\phi
\end{aligned}
$$

This gives:

$$dw^2 + dx^2 + dy^2 + dz^2 = a^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

which is the spatial part of the space-time metric

Conclusions for k=+1:

1. The 3-space of this metric can be thought of a as a 3-sphere of radius a(t) embedded in a 4 dimensional Euclidean space

2. The 3-sphere is expanding

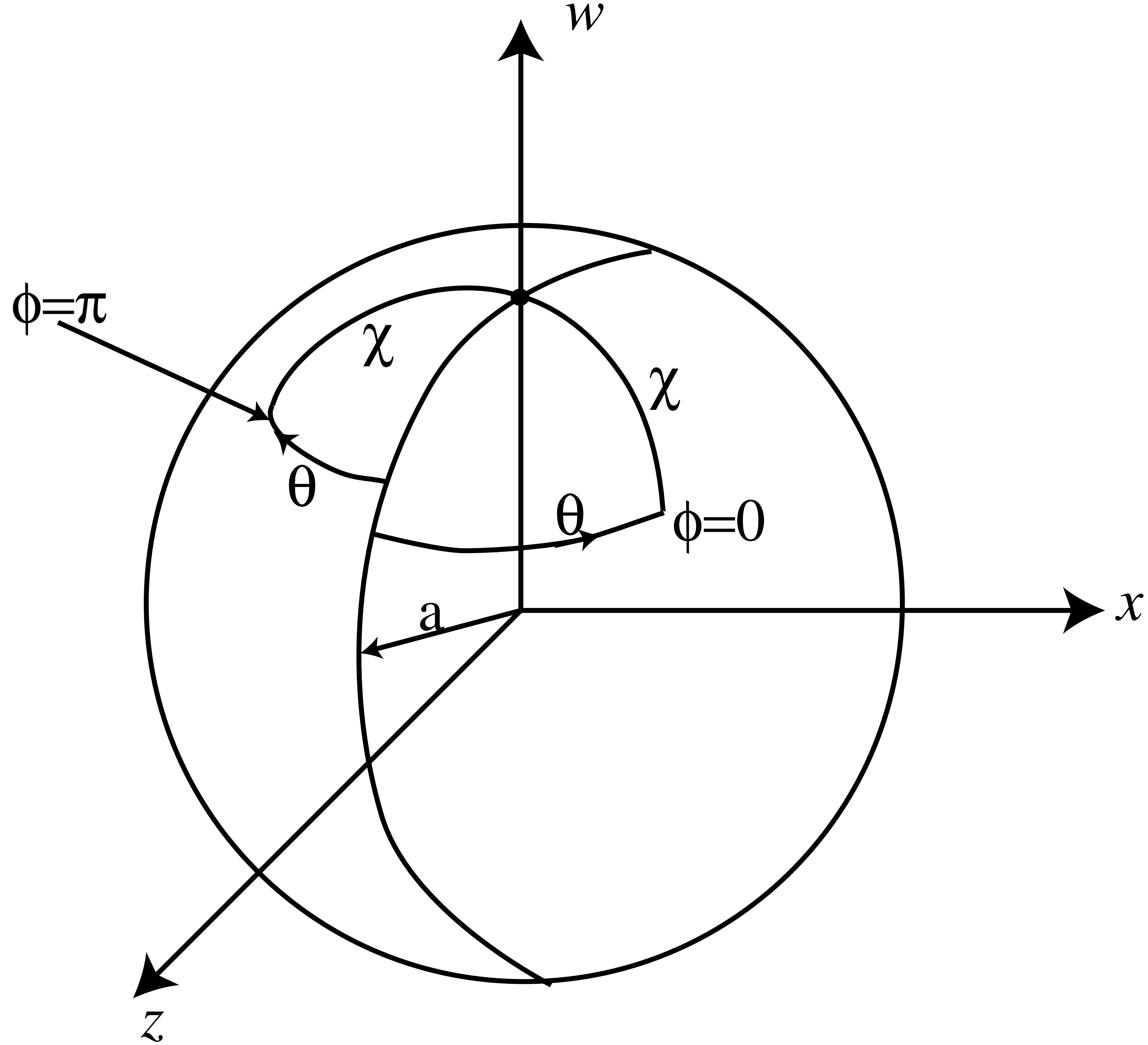
3. Since a 3-sphere is closed the k=1 metric represents a closed

Universe

Embedding diagram

Consider section y=0:

$$y = a\sin\chi\sin\theta\sin\phi = 0$$
$$\Rightarrow \phi = 0 \quad \text{or} \quad \pi$$



Embedding of a 3-sphere in a 4-dimensional Euclidean space

$$\phi = 0 \quad \text{section}$$
$$w = a\cos\chi$$
$$z = a\sin\chi\cos\theta$$
$$x = a\sin\chi\sin\theta$$

$$\phi = \pi \quad \text{section}$$
$$w = a\cos\chi$$
$$z = a\sin\chi\cos\theta$$
$$x = -a\sin\chi\sin\theta$$

The case k = -1

Consider the equation of a 3-hyperboloid embedded in a 4-dimensional Euclidean space:

$$w^2 - x^2 - y^2 - z^2 = a^2$$

We can parametrically express this in terms of hyperspherical polars

$$\begin{aligned} w &= a\cosh\chi \\ z &= a\sinh\chi\cos\theta \\ x &= a\sinh\chi\sin\theta\cos\phi \\ y &= a\sinh\chi\sin\theta\sin\phi \end{aligned}$$

Differentials:

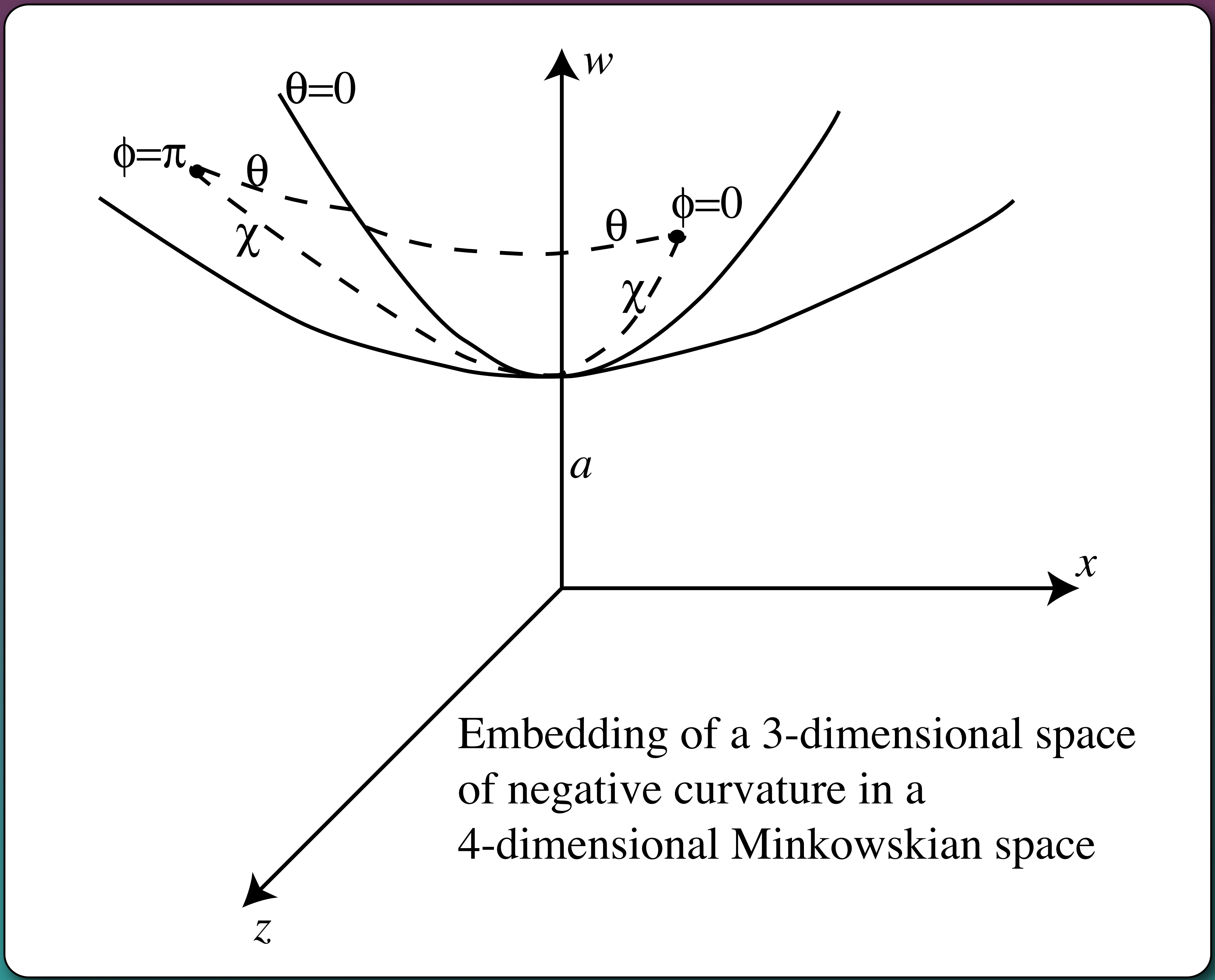
$$\begin{aligned}
dw &= a\sinh\chi\,d\chi \\
dz &= a\cosh\chi\cos\theta\,d\chi - a\sinh\chi\sin\theta d\theta \\
dx &= a\cosh\chi\sin\theta\cos\phi d\chi + a\sinh\chi\cos\theta\cos\phi\,d\theta - a\sinh\chi\sin\theta\sin\phi \\
dy &= a\cosh\chi\sin\theta\sin\phi d\chi + a\sinh\chi\cos\theta\sin\phi\,d\theta + a\sinh\chi\sin\theta\cos\phi
\end{aligned}$$

Metric restricted to 3-hyperboloid

$$\begin{aligned}
d\sigma^2 &= dw^2 + dx^2 + dy^2 + dz^2 \\
&= a^2(t)\left[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right]
\end{aligned}$$

Embedding: y = 0 section

$$y = 0 \quad \Rightarrow \quad \sin\phi = 0$$
$$\Rightarrow \quad \phi = 0 \quad \text{or} \quad \pi$$



Summary

The metric of the expanding Universe can be expressed in one of the 3 following ways:

$$ds^2 = -c^2dt^2 + a^2(t)\left[d\chi^2 + \chi^2 d\Omega^2\right]$$

k=0 Infinite flat Universe

$$ds^2 = -c^2dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi d\Omega^2\right]$$

k=1 Finite closed Universe

$$ds^2 = -c^2dt^2 + a^2(t)\left[d\chi^2 + \sinh^2\chi d\Omega^2\right]$$

k=-1 Infinite, open Universe