Cartesian Tensors

Reference: Jeffreys *Cartesian Tensors*

1 Coordinates and Vectors

Coordinates $x_i, i = 1, 2, 3$

Unit vectors: $e_i, i = 1, 2, 3$

General vector (formal definition to follow) denoted by components e.g. $u = u_i$

Summation convention (Einstein) repeated index means summation:
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\[ u_i v_i = \sum_{i=1}^{3} u_i v_i \]

\[ u_{ii} = \sum_{i=1}^{3} u_{ii} \]

**2 Orthogonal Transformations of Coordinates**

\[ x'_i = a_{ij} x_j \]

\[ a_{ij} = \text{Transformation Matrix} \]
Position vector

\[ \mathbf{r} = x_i \mathbf{e}_i = x'_j \mathbf{e}'_j \]

\[ \Rightarrow a_{ij} x_i \mathbf{e}'_j = x_i \mathbf{e}_i \]

\[ x_i (a_{ji} \mathbf{e}'_j) = x_i \mathbf{e}_i \]

\[ \Rightarrow \mathbf{e}_i = a_{ji} \mathbf{e}'_j \]

i.e. the transformation of coordinates from the unprimed to the primed frame implies the reverse transformation from the primed to the unprimed frame for the unit vectors.

**Kronecker Delta**

\[ \delta_{ij} = 1 \text{ if } i = j \]

\[ = 0 \text{ otherwise} \]

**2.1 Orthonormal Condition:**

Now impose the condition that the primed reference is orthonormal

\[ i \cdot \mathbf{e}_j = \delta_{ij} \text{ and } \mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij} \]

Use transformation

\[ \mathbf{e}_i \cdot \mathbf{e}_j = a_{ki} \mathbf{e}'_k \cdot a_{lj} \mathbf{e}'_l \]

\[ = a_{ki} a_{lj} \delta_{kl} \]

\[ = a_{ki} a_{kj} \]

NB the last operation is an example of the substitution property of the Kronecker Delta.

Since \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \), then the orthonormal condition on \( a_{ij} \) is
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\[ a_{ki}a_{kj} = \delta_{ij} \]

In matrix notation:

\[ a^T a = I \]

Also have

\[ i_k a_{jk} = aa^T = \delta_{ij} \]

### 2.2 Reverse transformations

\[(x'_i = a_{ij}x_j) \Rightarrow a_{ik}x'_i = a_{ik}a_{ij}x_j = \delta_{kj}x_j = x_k\]

\[\therefore x_k = a_{ik}x'_i \Rightarrow x_i = a_{ji}x'_j\]

i.e. the reverse transformation is simply given by the transpose.

Similarly,

\[ e'_i = a_{ij}e_j \]

### 2.3 Interpretation of \( a_{ij} \)

Since

\[ e'_i = a_{ij}e_j \]

then the \( a_{ij} \) are the components of \( e'_i \) wrt the unit vectors in the unprimed system.
3 Scalars, Vectors & Tensors

3.1 Scalar (f):

\[ f(x'_i) = f(x_i) \]

Example of a scalar is \( f = r^2 = x_i x_i \). Examples from fluid dynamics are the density and temperature.

3.2 Vector (u):

Prototype vector: \( x_i \)

General transformation law:

\[ x'_i = a_{ij} x_j \Rightarrow u'_i = a_{ij} u_j \]

3.2.1 Gradient operator

Suppose that \( f \) is a scalar. Gradient defined by

\[ (\text{grad } f)_i = (\nabla f)_i = \frac{\partial f}{\partial x_i} \]

Need to show this is a vector by its transformation properties.

\[ \frac{\partial f}{\partial x'_i} = \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial x'_i} \]

Since,

\[ x_j = a_{kj} x'_k \]

then
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\[ \frac{\partial x_j}{\partial x'_i} = a_{kj}\delta_{ki} = a_{ij} \]

and

\[ \frac{\partial f}{\partial x'_i} = a_{ij}\frac{\partial f}{\partial x_j} \]

Hence the gradient operator satisfies our definition of a vector.

### 3.2.2 Scalar Product

\[ \mathbf{u} \cdot \mathbf{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \]

is the scalar product of the vectors \( u_i \) and \( v_i \).

**Exercise:**

Show that \( \mathbf{u} \cdot \mathbf{v} \) is a scalar.

### 3.3 Tensor

Prototype second rank tensor \( x_i x_j \)

General definition:

\[ \gamma_{ij} = a_{ik}a_{jl}T_{kl} \]

**Exercise:**

Show that \( u_i v_j \) is a second rank tensor if \( u_i \) and \( v_j \) are vectors.

**Exercise:**

\[ i, j = \frac{\partial u_i}{\partial x_j} \]
is a second rank tensor. (Introduces the comma notation for partial derivatives.) In dyadic form this is written as grad \( u \) or \( \nabla u \).

### 3.3.1 Divergence

**Exercise:**

Show that the quantity

\[
\nabla \cdot \nu = \text{div } \nu = \frac{\partial v_i}{\partial x_i}
\]

is a scalar.

### 4 Products and Contractions of Tensors

It is easy to form higher order tensors by multiplication of lower rank tensors, e.g. \( ijk = T_{ij}u_k \) is a third rank tensor if \( T_{ij} \) is a second rank tensor and \( u_k \) is a vector (first rank tensor). It is straightforward to show that \( T_{ijk} \) has the relevant transformation properties.

Similarly, if \( T_{ijk} \) is a third rank tensor, then \( T_{ijj} \) is a vector. Again the relevant transformation properties are easy to prove.

### 5 Differentiation following the motion

This involves a common operator occurring in fluid dynamics. Suppose the coordinates of an element of fluid are given as a function of time by

\[
x_i = x_i(t)
\]
The velocities of elements of fluid at all spatial locations within a given region constitute a vector field, i.e. \( v_i = v_i(x_j, t) \)

If we follow the trajectory of an element of fluid, then on a particular trajectory \( x_i = x_i(t) \). The acceleration of an element is then given by:

\[
f_i = \frac{dv_i}{dt} = \frac{d}{dt} v(x_j(t), t) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} dx_j = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}
\]

Exercise: Show that \( f_i \) is a vector.

6 The permutation tensor \( \varepsilon_{ijk} \)

\[
\varepsilon_{ijk} = 0 \quad \text{if any of } i, j, k \text{ are equal}
\]

\[
= 1 \quad \text{if } i, j, k \text{ unequal and in cyclic order}
\]

\[
= -1 \quad \text{if } i, j, k \text{ unequal and not in cyclic order}
\]

E.g.

\[
\varepsilon_{112} = 0 \quad \varepsilon_{123} = 1 \quad \varepsilon_{321} = -1
\]
Is $\varepsilon_{ijk}$ a tensor?

In order to show this we have to demonstrate that $\varepsilon_{ijk}$, when defined the same way in each coordinate system has the correct transformation properties.

Define

$$
\varepsilon'_{ijk} = \varepsilon_{lmn} a_{il} a_{jm} a_{kn}
$$

$$
= \varepsilon_{123} a_{i1} a_{j2} a_{k3} + \varepsilon_{312} a_{i3} a_{j1} a_{k2} + \varepsilon_{231} a_{i2} a_{j1} a_{k2} + \varepsilon_{213} a_{i2} a_{j1} a_{k3} + \varepsilon_{321} a_{i3} a_{j2} a_{k1} + \varepsilon_{132} a_{i1} a_{j3} a_{k2}
$$

$$
= a_{i1} (a_{j2} a_{k3} - a_{j3} a_{k2}) - a_{i2} (a_{j1} a_{k3} - a_{j3} a_{k2})
+ a_{i3} (a_{j1} a_{k2} - a_{j2} a_{k1})
\nonumber
$$

In view of the interpretation of the $a_{ij}$, the rows of this determinant represent the components of the primed unit vectors in the unprimed system. Hence:

$$
\varepsilon'_{ijk} = e'_i \cdot e'_j \times e'_k
$$

This is zero if any 2 of $i, j, k$ are equal, is +1 for a cyclic permutation of unequal indices and -1 for a noncyclic permutation of unequal indices. This is just the definition of $\varepsilon'_{ijk}$. Thus $\varepsilon_{ijk}$ transforms as a tensor.
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### 6.1 Uses of the permutation tensor

#### 6.1.1 Cross product

Define

\[ c_i = \varepsilon_{ijk} a_j b_k \]

then

\[
\begin{align*}
c_1 &= \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = a_2 b_3 - a_3 b_2 \\
c_2 &= \varepsilon_{231} a_3 b_1 + \varepsilon_{213} a_1 b_3 = a_3 b_1 - a_1 b_3 \\
c_3 &= \varepsilon_{312} a_1 b_2 + \varepsilon_{321} a_2 b_1 = a_1 b_2 - a_2 b_1
\end{align*}
\]

These are the components of \( \mathbf{c} = \mathbf{a} \times \mathbf{b} \).

#### 6.1.2 Triple Product

In dyadic notation the triple product of three vectors is:

\[
\mathbf{t} = \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}
\]

In tensor notation this is

\[
= u_i \varepsilon_{ijk} v_j w_k = \varepsilon_{ijk} u_i v_j w_k
\]

#### 6.1.3 Curl

\[
\text{(curl } \mathbf{u})_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}
\]

e.g.

\[
\text{(curl } \mathbf{u})_1 = \varepsilon_{123} \frac{\partial u_3}{\partial x_2} + \varepsilon_{132} \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}
\]

etc.
6.1.4 The tensor $\varepsilon_{iks}\varepsilon_{mps}$

Define

$$T_{ikmp} = \varepsilon_{iks}\varepsilon_{mps}$$

**Properties:**

- If $i = k$ or $m = p$ then $T_{ikmp} = 0$.
- If $i = m$ we only get a contribution from the terms $s \neq i$ and $k \neq i, s$. Consequently $k = p$. Thus $\varepsilon_{iks} = \pm 1$ and $\varepsilon_{mps} = \varepsilon_{iks} = \pm 1$ and the product $\varepsilon_{iks}\varepsilon_{iks} = (\pm 1)^2 = 1$.
- If $i = p$, similar argument tells us that we must have $s \neq i$ and $k = m \neq i$. Hence, $\varepsilon_{iks} = \pm 1$, $\varepsilon_{mps} = \mp 1 \Rightarrow \varepsilon_{iks}\varepsilon_{mps} = -1$.

So,

$$i = m, k = p \Rightarrow 1 \text{ unless } i = k \Rightarrow 0$$

$$i = p, k = m \Rightarrow -1 \text{ unless } i = k \Rightarrow 0$$

These are the components of the tensor $\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$.

$$\therefore \varepsilon_{iks}\varepsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$$
6.1.5 Application of $\epsilon_{ik5}\epsilon_{mps}$

$$(\text{curl } (u \times v))_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}u_lv_m) = \epsilon_{ijk}\epsilon_{klm}\frac{\partial}{\partial x_j}(u_lv_m)$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial u_l}{\partial x_m}v_m + u_l\frac{\partial v_m}{\partial x_j}\right)$$

$$= \frac{\partial u_i}{\partial x_m}v_m - \frac{\partial u_j}{\partial x_j}v_i + u_i\frac{\partial v_m}{\partial x_m} - u_j\frac{\partial v_i}{\partial x_j}$$

$$= \nabla\nabla u - \nabla \cdot v + u \nabla \cdot v - v \nabla \cdot u_i$$

7 The Laplacean

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \frac{\partial^2\phi}{\partial x_i \partial x_i}$$
8 Tensor Integrals

8.1 Green’s Theorem

In dyadic form:

$$\int_V \nabla \cdot \mathbf{v} \, dV = \int_S (\mathbf{v} \cdot \mathbf{n}) \, dS$$

In tensor form:

$$\int_V \frac{\partial u_i}{\partial x_i} \, dV = \int_S u_i n_i \, dS = \text{Flux of } \mathbf{u} \text{ through } S$$

Extend this to tensors:

$$\int_V \frac{\partial T_{ij}}{\partial x_j} \, dV = \int_S T_{ij} n_j \, dS = \text{Flux of } T_{ij} \text{ through } S$$
8.2 Stoke’s Theorem

In dyadic form:

\[
\int_S (\text{curl } \mathbf{u}) \cdot \mathbf{n} dS = \int_C \mathbf{u} \cdot d\mathbf{s}
\]

In tensor form:

\[
\int_S \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} n_i dS = \int_C u_i t_i ds
\]