High Energy Astrophysics

Solutions to Assignment 6

25. (i) In order to calculate the free-free absorption coefficient we use Kirchhoff's law for a thermal plasma:

$$\alpha_\nu = \frac{c^2}{2h\nu^3}\left[e^{\frac{h\nu}{kt}} - 1\right] j_\nu$$

In the Rayleigh-Jeans limit ($h\nu \ll kT$) the factor

$$\frac{1}{\nu^3}\left[e^{\frac{h\nu}{kT}} - 1\right] e^{-\frac{h\nu}{kt}} \approx \frac{h}{kT}\nu^{-2}$$

After combining various constants (note that Planck's constant disappears):

$$\alpha_\nu(Z) = \frac{1}{24\pi^3}\left(\frac{\pi}{6}\right)^{1/2} \frac{Z^2 e^6}{\varepsilon_0^3 m_e^{3/2}(kT)^{3/2} c} g(\nu, T) n_e n_i(Z) \nu^{-2}$$

(ii) Inserting numerical values:

$$\varepsilon_0 = 8.85\times10^{-12} \text{ Farads m}^{-1} \qquad e = 1.60\times10^{-19} \text{ C} \qquad c = 3.0\times10^{8} \text{ m s}^{-1}$$

$$k = 1.38\times10^{-23} \text{ J K}^{-1} \qquad m_e = 9.11\times10^{-31} \text{ kg}$$

yields:

$$\alpha_\nu(Z) = 1.77\times10^{-12}[10.4 + 0.276\ln(T^3/\nu^2 Z^2)] Z^2 n_e n_i(Z) T^{-1.5} \nu^{-2}$$

(iii) The absorption coefficient from all species is

$$\alpha_\nu = \sum_Z \alpha_\nu(Z) = 1.77\times10^{-12} n_e T^{-1.5} \nu^{-2}$$

$$\times \left\{ \sum_Z [10.4 + 0.276\ln(T^3/\nu^2)] Z^2 n_i(Z) + \sum_Z 0.276\ln(1/Z^2) Z^2 n_i(Z) \right\}$$

If we just count the contributions from Hydrogen and Helium, then

$$\sum_Z Z^2 n_i(Z) = n_H + 4n_{He} = 1.4 n_H$$

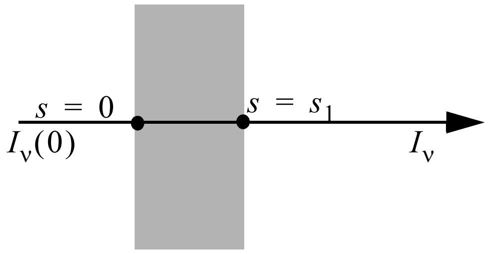
$$\sum_Z \ln(1/Z^2) Z^2 n_i(Z) = -\ln 4 \times n_{He} = -0.1\ln 4 \times n_H = -0.139 n_H$$

$$n_e = n_H + 2n_{He} = 1.2 n_H$$

Therefore,

$$\alpha_\nu = 1.77\times10^{-12} n_H^2 T^{-1.5} \nu^{-2}$$
$$\times 1.2 \times \{1.4 \times [10.4 + 0.276\ln(T^3/\nu^2)] - 0.276 \times 0.139\}$$
$$= 2.1\times10^{-12} \times [14.5 + 0.386\ln(T^3/\nu^2)] n_H^2 T^{-1.5}\nu^{-2}$$
$$= 3.05\times10^{-11}[1 + 0.0267\ln(T^3/\nu^2)] n_H^2 T^{-1.5}\nu^{-2}$$

(iv) Consider a power-law radiation field incident upon a slab in which the emissivity may be taken to be zero. The radiative transfer equation through the slab is given by:



$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$

implying that:

$$I_\nu(s) = I_\nu(0)\exp{-\alpha_\nu s_1}$$

where s_1 is the path length through the slab. Consider a typical temperature $T \sim 10^4$ for the absorbing plasma and a typical radio frequency of order 10^9 Hz. Putting $T_4 = T/10^4\mathrm{K}$ and $\nu_9 = \nu/10^9$ Hz, then the absorption coefficient is given by:

$$\alpha_\nu \approx 3.05\times10^{-11}[1 + 0.0267\ln(T^3/\nu^2)] n_H^2 T^{-1.5}\nu^{-2}$$
$$\approx 3.05\times10^{-11}[0.63 + 0.0267\ln(T_4^3/\nu_9^2)] n_H^2 T^{-1.5}\nu^{-2}$$
$$= 1.92\times10^{-11}[1 + 0.042\ln(T_4^3/\nu_9^2)] n_H^2 T^{-1.5}\nu^{-2}$$
$$\approx 1.92\times10^{-35}[1 + 0.042\ln(T_4^3/\nu_9^2)] n_H^2 T_4^{-1.5}\nu_9^{-2}$$

Since the logarithmic term varies slowly, then at microwave frequencies, the variation of the absorption coefficient with frequency is dominated by the ν_9^{-2} term. For a uniform slab, the optical depth is:

$$\tau_\nu \approx \alpha_\nu s \approx 9.4\times10^{-16}[1 + 0.042\ln(T_4^3/\nu_9^2)] n_H^2 T_4^{-1.5}\nu_9^{-2}(s_1/\mathrm{kpc})$$

and

$$I_\nu = I_\nu(0)\exp(-\tau_\nu)$$

The optical depth increases with decreasing frequency and is of the form $\exp{-a\nu_9^{-2}}$. Hence for frequencies below the point at which the optical depth becomes unity, the spectrum cuts off very rapidly.

For example, suppose that the optical depth is unity at $\nu_9 = 1$, and that $T_4 = 1$ then,

$$\tau_\nu = [1 + 0.367\ln(1/\nu_9^2)]\nu_9^{-2} = [1 - 0.367\ln(\nu_9^2)]\nu_9^{-2}$$

Consider an input spectrum with

$$I_\nu(0) = A\nu_9^{-\alpha}$$

then the emergent spectrum will be:

$$I_\nu = A\nu_9^{-\alpha}\exp\{-[1+0.367\ln(\nu_9^2)]\nu_9^{-2}\}$$

A plot of this spectrum is shown at the right for $\alpha = 0.6, 0.7, 0.8$. The important feature of the spectrum is that the power-law does not continue indefinitely for low frequencies – it turns over quite abruptly. This signature is often one that is sought by observers in ascertaining the importance of free-free absorption.

26 (i) We have seen above that the optical depth of a free-free absorbed spectrum in a region of constant density is given by:

$$\tau_\nu \approx \alpha_\nu s \approx 9.4\times10^{-16}[1-0.042\ln(T_4^3/\nu_9^2)]n_H^2 T_4^{-1.5}\nu_9^{-2}(s_1/\mathrm{kpc})$$

Hence if the turnover at a frequency of a GHz is to be attributed to free-free absorption, then $\tau_\nu \approx 1$ at $\nu_9 \approx 1$. This implies that:

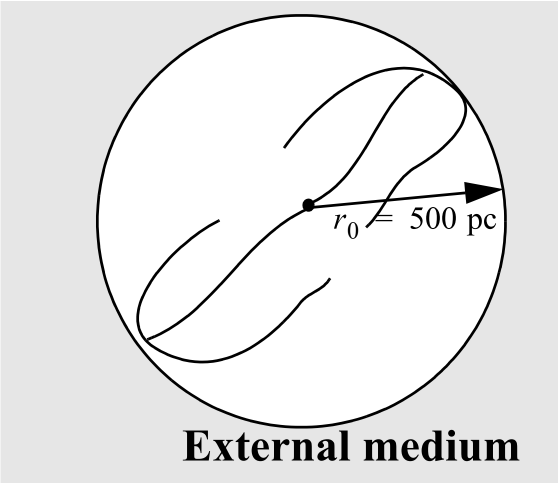
$$9.4\times10^{-16} n_H^2 T_4^{-1.5}\nu_9^{-2}(s_1/\mathrm{kpc}) \approx 1$$

$$\Rightarrow n_H \approx (9.4\times10^{-16})^{-0.5} = 3.2\times10^7\ \mathrm{m}^{-3} = 32\ \mathrm{cm}^{-3}$$

for the assigned parameters, $T_4 = 1, \nu_9 = 1$.

(ii) When the number density varies with distance through the slab, then

$$\tau_\nu = 9.4\times10^{-16}[1-0.042\ln(T_4^3/\nu_9^2)]T_4^{-1.5}\nu_9^{-2}\int_0^{s_1} n_H^2\, d(s_1/\mathrm{kpc})$$



For the case in point:

$$n_H = n_0\left(\frac{r}{r_0}\right)^{-\delta} \Rightarrow n_H^2 = n_0^2\left(\frac{r}{r_0}\right)^{-2\delta}$$

$$\int_{r_1}^{\infty} n_0^2\left(\frac{r}{r_0}\right)^{-2\delta} d\left(\frac{r}{r_0}\right) = \left(\frac{n_0^2}{2\delta-1}\right)\left(\frac{r_1}{r_0}\right)^{-(2\delta-1)}$$

Hence,

$$n_0^2 = (2\delta-1)(9.4\times10^{-16})^{-1}\left(\frac{r_1}{r_0}\right)^{2\delta-1}$$

$$\Rightarrow n_0 \approx 1.9\times10^7\ \mathrm{m}^{-3} = 19\ \mathrm{cm}^{-3}$$

for $r_1 = 0.5$ kpc.

27. The absorption coefficients for a synchrotron source are given by:

$$\alpha_\nu^{(i)} = -\frac{\sqrt{3}}{32\pi^2}\left(\frac{1}{\nu^2}\right)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0\sin\theta)\int_{\gamma_1}^{\gamma_2}\gamma^2\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right][F(x)\pm G(x)]d\gamma$$

$$\propto -\int_{\gamma_1}^{\gamma_2}\gamma^2\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right][F(x)\pm G(x)]d\gamma$$

Hence, for the absorption coefficient to be positive, we require

$$I_\pm = -\int_{\gamma_1}^{\gamma_2}\gamma^2\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right][F(x)\pm G(x)]d\gamma > 0$$

where

$$x = \left(\frac{2}{3}\frac{\omega}{\Omega_0\sin\theta}\right)\gamma^{-2}$$

We can note one circumstance in which $I_\pm$ would be negative and that is, if $\frac{d}{d\gamma}[\gamma^{-2}N(\gamma)] > 0$. However, this would constitute an extremely unusual nonthermal distribution – one in which $N(\gamma)$ increased more quickly than γ^2. The question to be resolved is, are there any other possible distributions that would give $I_\pm < 0$? For example, would it be possible for *some* part of the distribution to have $\gamma^{-2}N(\gamma)$ increasing making $I_\pm < 0$?

We address this question, by integrating the above expression by parts:

$$I_\pm = \left[-\gamma^2\frac{N(\gamma)}{\gamma^2}(F(x)\pm G(x))\right]_{\gamma_1}^{\gamma_2} + \int_{\gamma_1}^{\gamma_2}\frac{N(\gamma)}{\gamma^2}\frac{d}{d\gamma}\{\gamma^2[F(x)\pm G(x)]\}d\gamma$$

$$= N(\gamma_1)[F(x_1)\pm G(x_1)] - N(\gamma_2)[F(x_2)\pm G(x_2)] + \int_{\gamma_1}^{\gamma_2}\frac{N(\gamma)}{\gamma^2}\frac{d}{d\gamma}\{\gamma^2[F(x)\pm G(x)]\}d\gamma$$

The parameters γ_1 and γ_2 are the lower and upper cutoff Lorentz factors in the particle distribution. Define:

$$
\begin{aligned}
T_1 &= N(\gamma_1)[F(x_1) \pm G(x_1)] \\
T_2 &= -N(\gamma_2)[F(x_2) \pm G(x_2)] \\
T_3 &= \int_{\gamma_1}^{\gamma_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{d\gamma} \{\gamma^2 [F(x) \pm G(x)]\} d\gamma
\end{aligned}
$$

T_1 is obviously positive. T_2 could give a negative contribution and the sign of T_3 is unknown at present.

Let us examine T_2. This is determined by the values of $N(\gamma)$, $F(x)$ and $G(x)$ at the upper cutoff. Let us assume that γ_2 is large enough that we are in the asymptotic regime $x \to 0$ for F and G. Since,

$$
[F(x), G(x)] \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)2^{1/3}} x^{1/3} \qquad x = \left(\frac{2\omega}{3\Omega_0 \sin\theta}\right)\gamma^{-2}
$$

then

$$
N(\gamma_2)F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)2^{1/3}} \left(\frac{2\omega}{3\Omega_0 \sin\theta}\right)^{1/3} N(\gamma_2)\gamma_2^{-2/3}
$$

Unless $N(\gamma)$ increases faster than $\gamma^{2/3}$ as $\gamma \to \infty$, then this term will be very small and will vanish if $\gamma_2 \to \infty$. Since $F(x) - G(x) \to 0$ as $x \to 0$ then the difference between the contributions of T_2 to the two absorption coefficients will tend to zero. A distribution in which $N(\gamma)$ increases as $\gamma \to \infty$ would be very unphysical.

To examine the term T_3 change the variable of integration to x using:

$$
\gamma = \frac{2}{3}\left(\frac{\omega}{\Omega_0 \sin\theta}\right)^{1/2} x^{-1/2} \Rightarrow T_3 = \frac{4}{9}\left(\frac{\omega}{\Omega_0 \sin\theta}\right) \int_{x_1}^{x_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{dx}\left[\frac{F(x) \pm G(x)}{x}\right] dx
$$

Now $x_1 > x_2$ so that the positivity of the absorption coefficient depends upon

$$
J_{\pm} = \int_{x_2}^{x_1} \frac{N(\gamma)}{\gamma^2} \frac{d}{dx}\left[\frac{F(x) \pm G(x)}{x}\right] dx < 0
$$

Since $\gamma^{-2}N(\gamma) > 0$, then $J_{\pm} < 0$ if

$$
\frac{d}{dx}\left[\frac{F(x) \pm G(x)}{x}\right] < 0
$$

The functions $F(x)$ and $G(x)$ are given by:

$$
F(x) = x\int_x^{\infty} K_{5/3}(t)dt \qquad G(x) = xK_{2/3}(x)
$$

Hence

$$\frac{d}{dx}\left[\frac{F(x)}{x}\right] = -K_{5/3}(x) \qquad \frac{d}{dx}\left[\frac{G(x)}{x}\right] = \frac{d}{dx}[K_{2/3}(x)]$$

Therefore,

$$\frac{d}{dx}\left[\frac{F(x) \pm G(x)}{x}\right] = -K_{5/3}(x) \pm \frac{d}{dx}[K_{2/3}(x)]$$

At this stage, one can simply use Maple to plot the functions to show that both of these functions are negative. One can also be a tad more sophisticated and use the recurrence properties of the Bessel functions (see, for example Abramowitz and Stegun, Handbook of Mathematical Functions) to show that:

$$\frac{d}{dx}[K_{2/3}(x)] = -\frac{1}{2}K_{5/3}(x) - \frac{1}{2}K_{1/3}(x)$$

Hence,

$$-K_{5/3}(x) + \frac{d}{dx}[K_{2/3}(x)] = -\frac{3}{2}K_{5/3}(x) - \frac{1}{2}K_{1/3}(x)$$

$$-K_{5/3}(x) - \frac{d}{dx}[K_{2/3}(x)] = -\frac{1}{2}K_{5/3}(x) + \frac{1}{2}K_{1/3}(x)$$

Both $K_{5/3}(x), K_{1/3}(x) > 0$ and $K_{5/3}(x) > K_{1/3}(x)$. Hence, the integrals, $J_{\pm}$ are negative, irrespective of the distribution function $N(\gamma)$.

Thus for reasonable electrons distributions, the terms T_1 and T_3 are positive. T_2 could be negative. However, this is unlikely for a reasonable electron distribution. Hence, we conclude that a synchrotron maser is unlikely.