14 (i) Luminosity per unit area of stars along a ray as shown in the diagram:

\[ \Sigma(x) = \int_{-\infty}^{\infty} l(r) ds \]

\[ s = \sqrt{r^2 - x^2} \quad ds = \frac{r dr}{\sqrt{r^2 - x^2}} \]

Replace integral from \(-\infty\) to \(\infty\) by twice integral from 0 to \(\infty\) and make the integration variable \(r\).

\[ \Sigma(x) = 2 \int_{0}^{\infty} l(r) r dr \]

In extragalactic astronomy, \(\Sigma(x)\) is called the surface brightness. (ii) Take the luminosity density

\[ l(r) = \frac{l_0}{\left(1 + \frac{r^2}{r_c^2}\right)^{3/2}} \]

Surface brightness given by

\[ \Sigma(x) = 2l_0 \int_{0}^{\infty} \frac{1}{\left(1 + \frac{r^2}{r_c^2}\right)^{3/2} (r^2 - x^2)^{1/2}} r dr = \frac{2l_0}{\left(1 + \frac{x^2}{r_c^2}\right)} \]

The last integral can be worked out using either Maple or Mathematica. (iii) Obviously \(\Sigma(0) = 2l_0 r_c\) and at \(r = r_c\), \(\Sigma(x) = l_0 r_c\) and this is half the central value.
15. (a) The mean intensity at P is given by

\[ J_\nu = \frac{1}{4\pi} \int I_\nu d\Omega \]

and since \( I_\nu = \int j_\nu ds \) along each ray, then

\[ J_\nu = \frac{1}{4\pi} \int j_\nu ds d\Omega \]

(b) We can write the above integral as

\[ J_\nu = \frac{1}{4\pi} \int \frac{j_\nu s^2}{dV} ds d\Omega \]

and since \( s^2 ds d\Omega \) is an element of volume at Q, then

\[ J_\nu = \frac{1}{4\pi} \int \frac{j_\nu s^2}{dV} \]

(c) Now make use of the cosine rule for a triangle: \( s^2 = r^2 + r_0^2 - 2rr_0\cos\theta \) to obtain

\[ J_\nu = \frac{1}{4\pi} \int \frac{j_\nu}{r^2 + r_0^2 - 2rr_0\cos\theta} dV \]

(d) The volume element at Q can be expressed in terms of polar coordinates by

\[ dV = r^2 \sin\theta dr d\theta d\phi \]

and the integral can be expressed in the form:

\[ J_\nu(r_0) = \frac{1}{4\pi} \int_0^\infty j_\nu(r) \int_0^\pi \frac{r^2 \sin\theta}{r^2 + r_0^2 - 2rr_0 \cos\theta} d\phi d\theta \]

The integral over \( \phi \) is easy so that

\[ J_\nu(r_0) = \frac{1}{2} \int_0^\infty j_\nu(r) \int_0^\pi \frac{r^2 \sin\theta}{r^2 + r_0^2 - 2rr_0 \cos\theta} d\theta \]

The integral over \( \theta \) is also straightforward:

\[ \int_0^\pi \frac{r^2 \sin\theta}{r^2 + r_0^2 - 2rr_0 \cos\theta} d\theta = \frac{r}{r_0} \ln \left| \frac{r + r_0}{r - r_0} \right| = \frac{r}{r_0} \ln \left| \frac{1 + \frac{r}{r_0}}{1 - \frac{r}{r_0}} \right| \]

and the mean intensity becomes

\[ J_\nu(r_0) = \int_0^\infty K\left( \frac{r}{r_0} \right) j_\nu(r) dr \]

where

\[ K(t) = \frac{t}{2} \ln \left| \frac{t + 1}{t - 1} \right| \]

The photon energy density is

\[ u_\nu(r_0) = \frac{4\pi}{c} J_\nu(r_0) = \frac{4\pi}{c} \int_0^\infty K\left( \frac{r}{r_0} \right) j_\nu(r) dr \]
16. At the centre of a galaxy $r_0 = 0$ and $t \to \infty$. Hence $K(t) \to 1$ and

$$J_\nu(0) = \int_0^\infty j_\nu(r) dr$$

The emissivity is related to the luminosity density by:

$$j_\nu = \frac{1}{4\pi} l_\nu(r) \Rightarrow J_\nu = \frac{1}{8\pi} \times 2 \int_0^\infty l_\nu(r) dr = \frac{1}{8\pi} \Sigma_\nu(0)$$

Hence, the energy density per unit frequency,

$$u_\nu = \frac{4\pi}{c} \times \frac{1}{8\pi} \Sigma_\nu(0) = \frac{1}{2c} \Sigma_\nu(0)$$

and the total energy density,

$$u = \frac{1}{2c} \Sigma(0)$$

17. For M87 the central surface brightness in the V-band is: $\Sigma_\nu = 5.5 \times 10^3 L_\odot \text{pc}^{-2}$. The central surface brightness of M87 in the V-band, in physical units, is therefore,

$$5.5 \times 10^3 \times \frac{3.83 \times 10^{26}}{(3.1 \times 10^{16})^2} \times 10^{(0.4(4.77 - 4.84))} = 2.1 \times 10^{-3} \text{ W m}^{-2}$$

The energy density of starlight in the V band is then

$$u_\nu = \frac{1}{2c} \times 2.1 \times 10^{-3} = 3.4 \times 10^{-12} \text{ J m}^{-3} = 2.1 \times 10^7 \text{ eV m}^{-3}$$