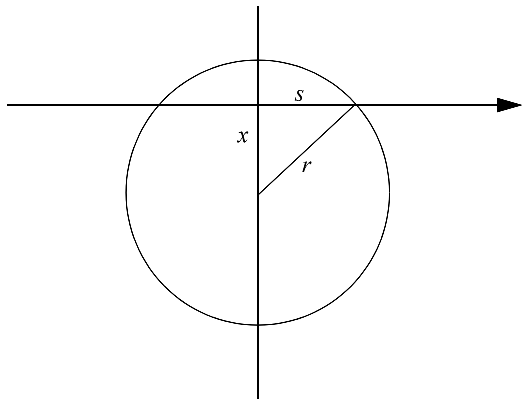
High Energy Astrophysics

Solutions to Assignment 4

14 (i) Luminosity per unit area of stars along a ray as shown in the diagram:



$$\Sigma(x) = \int_{-\infty}^{\infty} l(r)ds$$

$$s = \sqrt{r^2 - x^2} \qquad ds = \frac{rdr}{\sqrt{r^2 - x^2}}$$

Replace integral from $-\infty$ to ∞ by twice integral from 0 to ∞ and make the integration variable r.

$$\Sigma(x) = 2\int_{x}^{\infty} \frac{l(r)rdr}{\sqrt{r^2 - x^2}}$$

In extragalactic astronomy, $\Sigma(x)$ is called the surface brightness.
(ii) Take the luminosity density

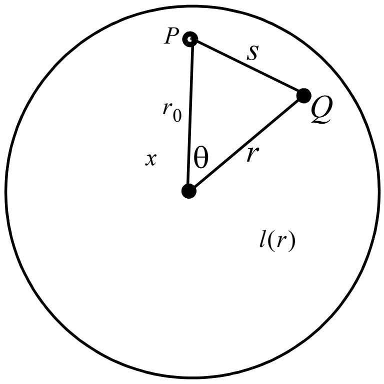
$$l(r) = \frac{l_0}{\left(1 + \frac{r^2}{r_c^2}\right)^{3/2}}$$

Surface brightness given by

$$\Sigma(x) = 2l_0\int_{0}^{\infty} \frac{1}{\left(1 + \frac{r^2}{r_c^2}\right)^{3/2}} \frac{rdr}{(r^2 - x^2)^{1/2}} = \frac{2l_0}{\left(1 + \frac{x^2}{r_c^2}\right)}$$

The last integral can be worked out using either *Maple* or *Mathematica.*
(iii) Obviously $\Sigma(0) = 2l_0 r_c$ and at $r = r_c$, $\Sigma(x) = l_0 r_c$ and this is half the central value.

15. (a)The mean intensity at P is given by



$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$$

and since $I_\nu = \int j_\nu ds$ along each ray, then

$$J_\nu = \frac{1}{4\pi}\int j_\nu ds d\Omega$$

(b) We can write the above integral as

$$J_\nu = \frac{1}{4\pi}\int \frac{j_\nu}{s^2} s^2 ds d\Omega$$

and since $s^2 ds d\Omega$ is an element of volume at Q, then

$$J_\nu = \frac{1}{4\pi}\int \frac{j_\nu}{s^2} dV$$

(c) Now make use of the cosine rule for a triangle: $s^2 = r^2 + r_0^2 - 2rr_0\cos\theta$ to obtain

$$J_\nu = \frac{1}{4\pi}\int \frac{j_\nu}{r^2 + r_0^2 - 2rr_0\cos\theta} dV$$

(d) The volume element at Q can be expressed in terms of polar coordinates by $dV = r^2\sin\theta dr d\theta d\phi$ and the integral can be expressed in the form:

$$J_\nu(r_0) = \frac{1}{4\pi}\int_0^\infty j_\nu(r)\int_0^\pi \frac{r^2\sin\theta}{r^2 + r_0^2 - 2rr_0\cos\theta}\int_0^{2\pi} d\phi d\theta dr$$

The integral over ϕ is easy so that

$$J_\nu(r_0) = \frac{1}{2}\int_0^\infty j_\nu(r)\int_0^\pi \frac{r^2\sin\theta}{r^2 + r_0^2 - 2rr_0\cos\theta} d\theta$$

The integral over θ is also straightforward:

$$\int_0^\pi \frac{r^2\sin\theta}{r^2 + r_0^2 - 2rr_0\cos\theta} d\theta = \frac{r}{r_0}\ln\left|\frac{r + r_0}{r - r_0}\right| = \frac{r}{r_0}\ln\left|\frac{1 + \frac{r}{r_0}}{1 - \frac{r}{r_0}}\right|$$

and the mean intensity becomes

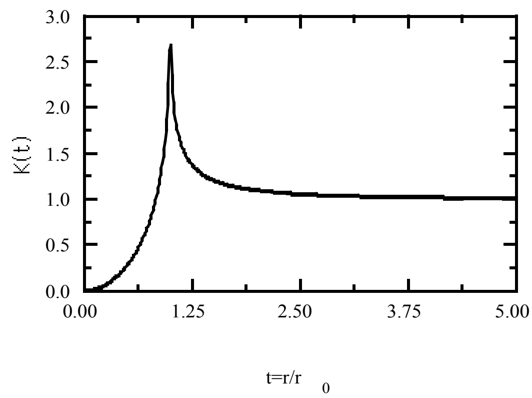
$$J_\nu(r_0) = \int_0^\infty K\left(\frac{r}{r_0}\right) j_\nu(r) dr$$

where

$$K(t) = \frac{t}{2}\ln\left|\frac{t + 1}{t - 1}\right|$$

The photon energy density is

$$u_\nu(r_0) = \frac{4\pi}{c} J_\nu(r_0) = \frac{4\pi}{c}\int_0^\infty K\left(\frac{r}{r_0}\right) j_\nu(r) dr$$



The function $K(t)$ is plotted at the left.

16. At the centre of a galaxy $r_0 = 0$ and $t \to \infty$. Hence $K(t) \to 1$ and

$$J_\nu(0) = \int_0^\infty j_\nu(r)dr$$

The emissivity is related to the luminosity density by:

$$j_\nu = \frac{1}{4\pi} l_\nu(r) \Rightarrow J_\nu = \frac{1}{8\pi} \times 2\int_0^\infty l_\nu(r)dr = \frac{1}{8\pi}\Sigma_\nu(0)$$

Hence, the energy density per unit frequency,

$$u_\nu = \frac{4\pi}{c} \times \frac{1}{8\pi}\Sigma_\nu(0) = \frac{1}{2c}\Sigma_\nu(0)$$

and the total energy density,

$$u = \frac{1}{2c}\Sigma(0)$$

17. For M87 the central surface brightness in the V-band is: $\Sigma_V = 5.5\times10^3 L_o \text{ pc}^{-2}$. The central surface brightness of M87 in the V-band, in physical units, is therefore,

$$5.5\times10^3 \times \frac{3.83\times10^{26}}{(3.1\times10^{16})^2} \times 10^{(0.4(4.77-4.84))} = 2.1\times10^{-3} \text{ W m}^{-2}$$

The energy density of starlight in the V band is then

$$u_V = \quad = \frac{1}{2c} \times 2.1\times10^{-3} = 3.4\times10^{-12} \text{ J m}^{-3} = 2.1\times10^{7} \text{ eV m}^{-3}$$