Electromagnetic Theory

Summary:

- Maxwell's equations
- EM Potentials
- Equations of motion of particles in electromagnetic fields
- Green's functions
- Lienard-Weichert potentials
- Spectral distribution of electromagnetic energy from an arbitrarily moving charge

1 Maxwell's equations

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Faraday's law

$$\mathrm{curl}\,\boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$ Ampere's law

$$\mathrm{div}\,\boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$ Field diverges from electric charges

$$\mathrm{div}\,\boldsymbol{B} = 0$$ No magnetic monopoles

$$\varepsilon_0 = 8.854\times 10^{-12} \text{ Farads/metre} \qquad \mu_0 = 4\pi \times 10^{-7} \text{ Henrys/metre}$$

$$\varepsilon_0 \mu_0 = \frac{1}{c^2} \qquad c = 2.998\times 10^{8} \text{ m/s} \approx 300,000 \text{ km/s}$$

Conservation of charge

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\boldsymbol{J} = 0$$

Conservation of energy

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}\right) + \mathrm{div}\left(\frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}\right) = -(\boldsymbol{J} \cdot \boldsymbol{E})$$

Electromagnetic energy density — Poynting flux — - Work done on particles by EM field

Poynting Flux

This is defined by

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} \qquad S_i = \frac{\varepsilon_{ijk} E_j B_k}{\mu_0}$$

Conservation of momentum

Momentum density — Maxwell's stress tensor

$$\frac{\partial}{\partial t}\left(\frac{S_i}{c^2}\right) - \frac{\partial M_{ij}}{\partial x_j} = -\rho E_i - (\boldsymbol{J} \times \boldsymbol{B})_i$$

Rate of change of momentum due to EM field acting on matter

Electric part — Magnetic part

$$M_{ij} = \varepsilon_0\left(E_i E_j - \frac{1}{2}E^2\delta_{ij}\right) + \left(\frac{B_i B_j}{\mu_0} - \frac{B^2}{2\mu_0}\delta_{ij}\right)$$

= –Flux of i cpt. of EM momentum in j direction

2 Equations of motion

Charges move under the influence of an electromagnetic field according to the (relativistically correct) equation:

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + v \times \boldsymbol{B}) = q\left(\boldsymbol{E} + \frac{\boldsymbol{p} \times \boldsymbol{B}}{\gamma m}\right)$$

Momentum and energy of the particle are given by:

$$\boldsymbol{p} = \gamma m \boldsymbol{v} \qquad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$E = \gamma m c^2 \qquad E^2 = p^2 c^2 + m^2 c^4$$

3 Electromagnetic potentials

3.1 Derivation

$$\mathrm{div}\boldsymbol{B} = 0 \Rightarrow \boldsymbol{B} = \mathrm{curl}\boldsymbol{A}$$

$$\mathrm{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \Rightarrow \mathrm{curl}\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\mathrm{grad}\phi$$

$$\Rightarrow E = -\mathrm{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

Summary:

$$\boldsymbol{E} = -\operatorname{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$$

3.2 Potential equations

Equation for the vector potential *A*

Substitute into Ampere's law:

$$\operatorname{curl}\operatorname{curl}\boldsymbol{A} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\left[-\operatorname{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t}\right]$$

$$\left[\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A}\right] + \frac{1}{c^2}\frac{\partial}{\partial t}\operatorname{grad}\phi + \operatorname{grad}\operatorname{div}\boldsymbol{A} = \mu_0 \boldsymbol{J}$$

Equation for the scalar potential ϕ

Exercise:

Show that

$$\nabla^2\phi + \operatorname{div}\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\rho}{\varepsilon_0}$$

3.3 Gauge transformations

The vector and scalar potentials are not unique. One can see that the same equations are satisfied if one adds certain related terms to ϕ and $\boldsymbol{A}$, specifically, the gauge transformations

$$\boldsymbol{A}' = \boldsymbol{A} - \mathrm{grad}\psi \qquad \phi' = \phi + \frac{\partial\psi}{\partial t}$$

leaves the relationship between $\boldsymbol{E}$ and $\boldsymbol{B}$ and the potentials intact. We therefore have some freedom to specify the potentials. There are a number of gauges which are employed in electromagnetic theory.

Coulomb gauge

$$\mathrm{div}\boldsymbol{A} = 0$$

Lorentz gauge

$$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \mathrm{div}\boldsymbol{A} = 0$$

$$\Rightarrow \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \nabla^2\boldsymbol{A} = \mu_0\boldsymbol{J}$$

Temporal gauge

$$\phi = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\mathrm{div}\boldsymbol{A} = -\frac{\rho}{\varepsilon_0}$$

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \mathrm{curl}\ \mathrm{curl}\boldsymbol{A} = \mu_0 \boldsymbol{J}$$

The temporal gauge is the one most used when Fourier transforming the electromagnetic equations. For other applications, the Lorentz gauge is often used.

4 Electromagnetic waves

For waves in free space, we take

$$\boldsymbol{E} = \boldsymbol{E_0}\exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$$

$$\boldsymbol{B} = \boldsymbol{B}_0\exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$$

and substitute into the free-space form of Maxwell's equations, viz.,

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \mathrm{curl}\,\boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\mathrm{div}\,\boldsymbol{E} = 0 \qquad \mathrm{div}\,\boldsymbol{B} = 0$$

This gives:

$$i\boldsymbol{k} \times \boldsymbol{E}_0 = i\omega \boldsymbol{B}_0 \Rightarrow \boldsymbol{B}_0 = \frac{\boldsymbol{k} \times \boldsymbol{E}_0}{\omega}$$

$$i\boldsymbol{k} \times \boldsymbol{B}_0 = -\frac{1}{c^2} i\omega \boldsymbol{E}_0 \Rightarrow \boldsymbol{k} \times \boldsymbol{B}_0 = -\frac{\omega}{c^2}\boldsymbol{E}_0$$

$$i\boldsymbol{k} \cdot \boldsymbol{E}_0 = 0 \Rightarrow \boldsymbol{k} \cdot \boldsymbol{E}_0 = 0$$

$$i\boldsymbol{k} \cdot \boldsymbol{B}_0 = 0 \Rightarrow \boldsymbol{k} \cdot \boldsymbol{B}_0 = 0$$

We take the cross-product with $\boldsymbol{k}$ of the equation for $\boldsymbol{B}_0$:

$$\boldsymbol{k} \times \boldsymbol{B}_0 = \boldsymbol{k} \times \frac{(\boldsymbol{k} \times \boldsymbol{E}_0)}{\omega} = \frac{(\boldsymbol{k} \cdot \boldsymbol{E}_0)\boldsymbol{k} - k^2 \boldsymbol{E}_0}{\omega} = -\frac{\omega}{c^2}\boldsymbol{E}_0$$

and since $\boldsymbol{k} \cdot \boldsymbol{E}_0 = 0$

$$\left(\frac{\omega^2}{c^2} - k^2\right)\boldsymbol{E}_0 = \boldsymbol{0} \Rightarrow \omega = \pm ck$$

the well-known dispersion equation for electromagnetic waves in free space. The - sign relates to waves travelling in the opposite direction, i.e.

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} + \omega t)]$$

We restrict ourselves here to the positive sign. The magnetic field is given by

$$\boldsymbol{B}_0 = \frac{\boldsymbol{k} \times \boldsymbol{E}_0}{\omega} = \frac{1}{c}\left(\frac{\boldsymbol{k}}{k} \times \boldsymbol{E}_0\right) = \frac{\kappa \times \boldsymbol{E}_0}{c}$$

The Poynting flux is given by

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0}$$

and we now take the real components of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} = \boldsymbol{E}_0 \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \qquad \boldsymbol{B} = \frac{\boldsymbol{\kappa} \times \boldsymbol{E}_0}{c} \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$

where $\boldsymbol{E}_0$ is now real, then

$$\begin{aligned}\boldsymbol{S} &= \frac{\boldsymbol{E}_0 \times (\boldsymbol{\kappa} \times \boldsymbol{E}_0)}{\mu_0 c} \cos^2(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= c\varepsilon_0 (\boldsymbol{E}_0^2 \boldsymbol{\kappa} - (\boldsymbol{\kappa} \cdot \boldsymbol{E}_0)\boldsymbol{E}_0) \cos^2(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \\ &= c\varepsilon_0 E_0^2 \boldsymbol{\kappa} \cos^2(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\end{aligned}$$

The average of $\cos^2(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$ over a period $(T = 2\pi/\omega)$ is $1/2$ so that the time-averaged value of the Poynting flux is given by:

$$\langle \boldsymbol{S} \rangle = \frac{c\varepsilon_0}{2} E_0^2 \boldsymbol{\kappa}$$

5 Equations of motion of particles in a uniform magnetic field

An important special case of particle motion in electromagnetic fields occurs for $\boldsymbol{E} = 0$ and $\boldsymbol{B} = \text{constant}$. This is the basic configuration for the calculation of cyclotron and synchrotron emission.

In this case the motion of a relativistic particle is given by:

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{v} \times \boldsymbol{B}) = \frac{q}{\gamma m}(\boldsymbol{p} \times \boldsymbol{B})$$

Conservation of energy

There are a number of constants of the motion. First, the energy:

$$\boldsymbol{p} \cdot \frac{d\boldsymbol{p}}{dt} = 0$$

and since

$$E^2 = p^2 c^2 + m^2 c^4$$

then

$$E\frac{dE}{dt} = c^2 p\frac{dp}{dt} = c^2\left(\boldsymbol{p} \cdot \frac{d\boldsymbol{p}}{dt}\right) = 0 \text{ here.}$$

There for $E = \gamma mc^2$ is conserved and γ is constant - our first constant of motion.

Parallel component of momentum

The component of momentum along the direction of $\boldsymbol{B}$ is also conserved:

$$\frac{d}{dt}\left(\boldsymbol{p} \cdot \frac{\boldsymbol{B}}{B}\right) = \frac{d\boldsymbol{p}}{dt} \cdot \frac{\boldsymbol{B}}{B} = \frac{q}{\gamma m}\frac{\boldsymbol{B}}{B} \cdot (\boldsymbol{p} \times \boldsymbol{B}) = 0$$

$$\Rightarrow p_{||} = \gamma m v_{||} \text{ is conserved}$$

where $p_{\|}$ is the component of momentum parallel to the magnetic field.

We write the total magnitude of the velocity

$$v = c\beta$$

and since γ is constant, so is v and we put

$$v_{\|} = v\cos\alpha$$

where α is the *pitch angle* of the motion, which we ultimately show is a helix.

Perpendicular components

Take the z–axis along the direction of the field, then the equations of motion are:

$$\frac{d\boldsymbol{p}}{dt} = \frac{q}{\gamma m}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ p_x & p_y & p_{\|} \\ 0 & 0 & B \end{vmatrix}$$

In component form:

$$\frac{dp_x}{dt} = \frac{q}{\gamma m} p_y B = \eta \Omega_B p_y$$

$$\frac{dp_y}{dt} = -\frac{q}{\gamma m} p_x B = -\eta \Omega_B p_x$$

$$\Omega_B = \frac{|q| B}{\gamma m} = \text{Gyrofrequency}$$

$$\eta = \frac{|q|}{q} = \text{Sign of charge}$$

A quick way of solving these equations is to take the second plus i times the first:

$$\frac{d}{dt}(p_x + i p_y) = -i\eta\Omega_B (p_x + i p_y)$$

This has the solution

$$p_x + i p_y = A \exp(i\phi_0) \exp[-i\eta\Omega_B t]$$

$$\Rightarrow p_x = A\cos(\eta\Omega_B t + \phi_0) \qquad p_y = -A\sin(\eta\Omega_B t + \phi_0)$$

The parameter ϕ_0 is an arbitrary phase.

Positively charged particles:

$$p_x = A\cos(\Omega_B t + \phi_0) \qquad p_y = -A\sin(\Omega_B t + \phi_0)$$

Negatively charged particles (in particular, electrons):

$$p_x = A\cos(\Omega_B t + \phi_0) \qquad p_y = A\sin(\Omega_B t + \phi_0)$$

We have another constant of the motion:

$$p_x^2 + p_y^2 = A^2 = p^2 \sin^2\alpha = p_\perp^2$$

where $p_\perp$ is the component of momentum perpendicular to the magnetic field.

Velocity

The velocity components are given by:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \frac{1}{\gamma m}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = c\beta\begin{bmatrix} \sin\alpha\cos(\Omega_B t + \phi_0) \\ -\eta\sin\alpha\sin(\Omega_B t + \phi_0) \\ \cos\alpha \end{bmatrix}$$

Position

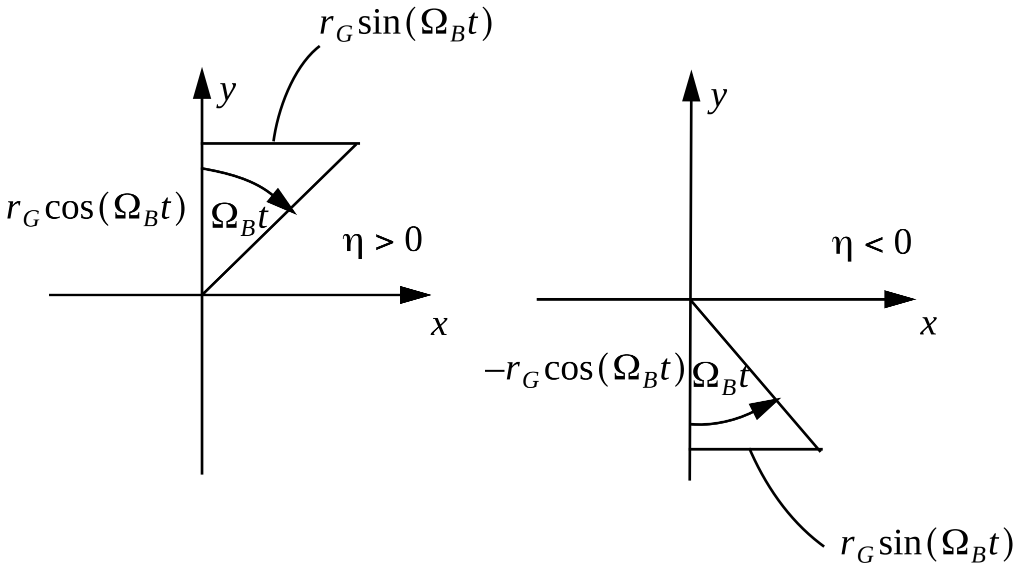
Integrate the above velocity components:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{c\beta\sin\alpha}{\Omega_B}\sin(\Omega_B t + \phi_0) \\ \eta\frac{c\beta\sin\alpha}{\Omega_B}\cos(\Omega_B t + \phi_0) \\ c\beta t\cos\alpha \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

This represents motion in a helix with

$$\text{Gyroradius} = r_G = \frac{c\beta\sin\alpha}{\Omega_B}$$

The motion is clockwise for $\eta > 0$ and anticlockwise for $\eta < 0$.



In vector form, we write:

$$\boldsymbol{x} = \boldsymbol{x}_0 + r_G \left[\sin(\Omega_B t + \phi_0) \;\; \eta \cos(\Omega_B t + \phi_0) \;\; 0 \right]$$
$$+ c\beta t \cos\alpha \left[0 \;\; 0 \;\; 1 \right]$$

The parameter $\boldsymbol{x}_0$ represents the location of the guiding centre of the motion.

6 Green's functions

Green's functions are widely used in electromagnetic and other field theories. Qualitatively, the idea behind Green's functions is that they provide the solution for a given differential equation corresponding to a point source. A solution corresponding to a given source distribution is then constructed by adding up a number of point sources, i.e. by integration of the point source response over the entire distribution.

6.1 Green's function for Poisson's equation

A good example of the use of Green's functions comes from Poisson's equation, which appears in electrostatics and gravitational potential theory. For electrostatics:

$$\nabla^2\phi(\boldsymbol{x}) = -\frac{\rho_e(\boldsymbol{x})}{\varepsilon_0}$$

where ρ_e is the electric charge density.

In gravitational potential theory:

$$\nabla^2\phi(\boldsymbol{x}) = 4\pi G\rho_m(\boldsymbol{x})$$

where $\rho_m(\boldsymbol{x})$ is the mass density.

The Green's function for the electrostatic case is prescribed by:

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x'}) = -\frac{\delta(\boldsymbol{x} - \boldsymbol{x'})}{\varepsilon_0}$$

where $\delta(\boldsymbol{x} - \boldsymbol{x}')$ is the three dimensional delta function. When there are no boundaries, this equation has the solution

$$G(\boldsymbol{x}, \boldsymbol{x'}) = G(\boldsymbol{x} - \boldsymbol{x'})$$

$$G(\boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0 r}$$

The general solution of the electrostatic Poisson equation is then

$$\phi(\boldsymbol{x}) = \int_{\text{space}} G(\boldsymbol{x} - \boldsymbol{x'})\rho_{\text{e}}(\boldsymbol{x}')d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\text{space}} \frac{\rho_{\text{e}}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3x'$$

For completeness, the solution of the potential for a gravitating mass distribution is:

$$\phi(\boldsymbol{x}) = -G\int_{\text{space}} \frac{\rho_m(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}d^3x'$$

6.2 Green's function for the wave equation

In the Lorentz gauge the equation for the vector potential is:

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J}$$

and the equation for the electrostatic (scalar) potential is

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(t, \boldsymbol{x}) - \nabla^2 \phi(t, \boldsymbol{x}) = \frac{\rho}{\varepsilon_0}$$

These equations are both examples of the wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi(t, \boldsymbol{x}) - \nabla^2 \psi(t, \boldsymbol{x}) = S(t, \boldsymbol{x})$$

When time is involved a "point source" consists of a source which is concentrated at a point for an instant of time, i.e.

$$S(\boldsymbol{x}, t) = A\delta(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x}')$$

where A is the strength of the source, corresponds to a source at the point $\boldsymbol{x} = \boldsymbol{x}'$ which is switched on at $t = t'$.

In the case of no boundaries, the Green's function for the wave equation satisfies:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(t - t', \boldsymbol{x} - \boldsymbol{x'}) = \delta(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x'})$$

The relevant solution (the retarded Green's function) is:

$$G(t, \boldsymbol{x}') = \frac{1}{4\pi r}\delta\left(t - \frac{r}{c}\right)$$

so that

$$G(t - t', \boldsymbol{x} - \boldsymbol{x}') = \frac{1}{4\pi|x - x'|}\delta\left(t - t' - \frac{|x - x'|}{c}\right)$$

The significance of the delta function in this expression is that a point source at $(t', \boldsymbol{x}')$ will only contribute to the field at the point $(t, \boldsymbol{x})$ when

$$t = t' + \frac{|x - x'|}{c}$$

i.e. at a later time corresponding to the finite travel time $\frac{|x - x'|}{c}$ of a pulse from the point $\boldsymbol{x}'$. Equivalently, a disturbance which arrives at the point $t, \boldsymbol{x}$ had to have been emitted at a time

$$t' = t - \frac{|x - x'|}{c}$$

The time $t - \frac{|x - x'|}{c}$ is known as the retarded time.

The general solution of the wave equation is

$$\begin{aligned}\psi(t, \boldsymbol{x}) &= \int_{-\infty}^{\infty} dt' \int_{\text{space}} G(t - t', \boldsymbol{x} - \boldsymbol{x}')S(t', \boldsymbol{x}')d^3x' \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dt' \int_{\text{space}} \frac{S(t', \boldsymbol{x}')}{|x - x'|}\delta\left(t - \frac{|x - x'|}{c}\right)d^3x'\end{aligned}$$

6.3 The vector and scalar potential

Using the above Green's function, the vector and scalar potential for an arbitrary charge and current distribution are:

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dt' \int_{\text{space}} \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|/c)\boldsymbol{J}(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'$$

$$\phi(t, \boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dt' \int_{\text{space}} \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|/c)\rho_{\text{e}}(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'$$

7 Radiation from a moving charge – the Lienard-Weichert potentials

7.1 Deduction from the potential of an arbitrary charge distribution

The current and charge distributions for a moving charge are:

$$\rho(t, \boldsymbol{x}) = q\delta^3(\boldsymbol{x} - \boldsymbol{X}(t))$$

$$\boldsymbol{J}(t, \boldsymbol{x}) = q\boldsymbol{v}\delta^3(\boldsymbol{x} - \boldsymbol{X}(t))$$

where $\boldsymbol{v}$ is the velocity of the charge, q, and $\boldsymbol{X}(t)$ is the position of the charge at time t.The charge q is the relevant parameter in front of the delta function since

$$\int_{\text{space}} \rho(t, \boldsymbol{x}) d^3x = q \int_{\text{space}} \delta^3(\boldsymbol{x} - \boldsymbol{X}(t)) d^3x = q$$

Also, the velocity of the charge

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{X}(t)}{dt} = \dot{\boldsymbol{X}}(t)$$

so that

$$\rho(t, \boldsymbol{x}) = q\delta^3(\boldsymbol{x} - \boldsymbol{X}(t))$$
$$\boldsymbol{J}(t, \boldsymbol{x}) = q\dot{\boldsymbol{X}}(t)\delta^3(\boldsymbol{x} - \boldsymbol{X}(t))$$

With the current and charge expressed in terms of spatial delta functions it is best to do the space integration first.

We have

$$\int_{\text{space}} \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|/c)\rho_e(t', \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}' = \int_{\text{space}} \frac{\delta(t - t' - |\boldsymbol{x} - \boldsymbol{x}'|/c)\delta^3(\boldsymbol{x}' - \boldsymbol{X}(t'))}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'$$
$$= \frac{q\delta\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}\right)}{|\boldsymbol{x} - \boldsymbol{X}(t')|}$$

One important consequence of the motion of the charge is that the delta function resulting from the space integration is now a more complicated function of t', because it depends directly upon t' and indirectly though the dependence on $\boldsymbol{X}(t')$. The delta-function will now only contribute to the time integral when

$$t' = t - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}$$

The retarded time is now an implicit function of $(t, \boldsymbol{x})$, through $\boldsymbol{X}(t')$. However, the interpretation of t' is still the same, it represents the time at which a pulse leaves the source point, $\boldsymbol{X}(t')$ to arrive at the field point $(t, \boldsymbol{x})$.

We can now complete the solution for $\phi(t, \boldsymbol{x})$ by performing the integration over time:

$$\phi(t, \boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{q\delta\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}\right)}{|\boldsymbol{x} - \boldsymbol{X}(t')|} dt'$$

This equation is not as easy to integrate as might appear because of the complicated dependence of the delta-function on t'.

7.2 Aside on the properties of the delta function

The following lemma is required.

We define the delta-function by

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$

Some care is required in calculating $\int_{-\infty}^{\infty} f(t)\delta(g(t)-a)dt$.

Consider

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\delta(g(t)-a)dt &= \int_{-\infty}^{\infty} f(t)\delta(g(t)-a)\frac{dt}{dg}dg \\
&= \int_{-\infty}^{\infty} \frac{f(t)}{\dot{g}(t)}\delta(g-a)dg \\
&= \frac{f(g^{-1}(a))}{\dot{g}(g^{-1}(a))}
\end{aligned}$$

where $g^{-1}(a)$ is the value of t satisfying $g(t) = a$.

7.3 Derivation of the Lienard-Wierchert potentials

In the above integral we have the delta function $\delta(t - t' - |x - \boldsymbol{X}(t')|/c) = \delta(t' + |x - \boldsymbol{X}(t')|/c - t)$ so that

$$g(t') = t' + |\boldsymbol{x} - \boldsymbol{X}(t')|/c - t$$

Differentiating this with respect to t':

$$\frac{dg(t')}{dt'} = \dot{g}(t') = 1 + \frac{\partial}{\partial t'} \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}$$

To do the partial derivative on the right, express $|\boldsymbol{x} - \boldsymbol{X}(t')|^2$ in tensor notation:

$$|\boldsymbol{x} - \boldsymbol{X}(t')|^2 = x_i x_i - 2 x_i X_i(t') + X_i(t') X_i(t')$$

Now,

$$\frac{\partial}{\partial t'}|\boldsymbol{x} - \boldsymbol{X}(t')|^2 = 2|\boldsymbol{x} - \boldsymbol{X}(t')| \times \frac{\partial}{\partial t}|\boldsymbol{x} - \boldsymbol{X}(t')|$$

Differentiating the tensor expression for $|\boldsymbol{x} - \boldsymbol{X}(t')|^2$ gives:

$$\frac{\partial}{\partial t'}|\boldsymbol{x} - \boldsymbol{X}(t')|^2 = -2 x_i \dot{X}_i(t') + 2 X_i(t') \dot{X}_i(t') = -2 \dot{X}_i(t')(x_i - X_i(t'))$$

Hence,

$$2|\boldsymbol{x}-\boldsymbol{X}(t')| \times \frac{\partial}{\partial t}|\boldsymbol{x}-\boldsymbol{X}(t')| = -2\dot{X}_i(t')(x_i - X_i(t'))$$

$$\Rightarrow \frac{\partial}{\partial t'}|\boldsymbol{x}-\boldsymbol{X}(t')| = -\frac{\dot{X}_i(t')(x_i - X_i(t'))}{|\boldsymbol{x}-\boldsymbol{X}(t')|} = -\frac{\dot{\boldsymbol{X}}(t')\cdot(\boldsymbol{x}-\boldsymbol{X}(t'))}{|\boldsymbol{x}-\boldsymbol{X}(t')|}$$

The derivative of $g(t')$ is therefore:

$$\dot{g}(t') = 1 + \frac{\partial}{\partial t'}\frac{|\boldsymbol{x}-\boldsymbol{X}(t')|}{c} = 1 - \frac{\dot{\boldsymbol{X}}(t')\cdot(\boldsymbol{x}-\boldsymbol{X}(t'))/c}{|\boldsymbol{x}-\boldsymbol{X}(t')|}$$

$$= \frac{|\boldsymbol{x}-\boldsymbol{X}(t')| - \dot{\boldsymbol{X}}(t')\cdot(\boldsymbol{x}-\boldsymbol{X}(t'))/c}{|\boldsymbol{x}-\boldsymbol{X}(t')|}$$

Hence the quantity $1/(\dot{g}(t'))$ which appears in the value of the integral is

$$\frac{1}{\dot{g}(t')} = \frac{|\boldsymbol{x}-\boldsymbol{X}(t')|}{|\boldsymbol{x}-\boldsymbol{X}(t')| - \dot{\boldsymbol{X}}(t')\cdot(\boldsymbol{x}-\boldsymbol{X}(t'))/c}$$

Thus, our integral for the scalar potential:

$$\phi(t, \boldsymbol{x}) = \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} \frac{q\delta\left(t - t' - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}\right)}{|\boldsymbol{x} - \boldsymbol{X}(t')|} dt'$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{|\boldsymbol{x} - \boldsymbol{X}(t')| - \frac{\dot{\boldsymbol{X}}(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{c}} \times \frac{1}{|\boldsymbol{x} - \boldsymbol{X}(t')|}$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{x} - \boldsymbol{X}(t')| - \frac{\dot{\boldsymbol{X}}(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{c}}$$

where, it needs to be understood that the value of t' involved in this solution satisfies, the equation for retarded time:

$$t' = t - \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c}$$

We also often use this equation in the form:

$$t' + \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c} = t$$

7.4 Nomenclature and symbols

We define the *retarded* position vector:

$$\boldsymbol{r}' = \boldsymbol{x} - \boldsymbol{X}(t')$$

and the *retarded* distance

$$r' = |\boldsymbol{x} - \boldsymbol{X}(t')| \, .$$

The unit vector in the direction of the retarded position vector is:

$$\boldsymbol{n}'(t') = \frac{\boldsymbol{r}'}{r'}$$

The relativistic β of the particle is

$$\beta(t') = \frac{\dot{\boldsymbol{X}}(t')}{c}$$

7.5 Scalar potential

In terms of these quantities, therefore, the scalar potential is:

$$\begin{aligned}\phi(t, \boldsymbol{x}) &= \frac{q}{4\pi\varepsilon_0} \frac{1}{|\boldsymbol{x} - \boldsymbol{X}(t')| - \dfrac{\dot{\boldsymbol{X}}(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{c}} \\ &= \frac{q}{4\pi\varepsilon_0} \frac{1}{r' - \beta(t') \cdot \boldsymbol{r}'} \\ &= \left(\frac{q}{4\pi\varepsilon_0 r'}\right) \frac{1}{[1 - \beta(t') \cdot \boldsymbol{n}']}\end{aligned}$$

This potential shows a Coulomb-like factor times a factor $(1 - \beta(t') \cdot \boldsymbol{n}')^{-1}$ which becomes extremely important in the case of relativistic motion.

7.6 Vector potential

The evaluation of the integral for the vector potential proceeds in an analogous way. The major difference is the velocity $\dot{\boldsymbol{X}}(t')$ in the numerator.

$$\boldsymbol{A}(t, \boldsymbol{x}) = \frac{\mu_0 q}{4\pi} \frac{\dot{\boldsymbol{X}}(t')}{|\boldsymbol{x} - \boldsymbol{X}(t')| - \dfrac{\dot{\boldsymbol{X}}(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{c}}$$

$$= \frac{\mu_0 q}{4\pi r'} \frac{\dot{\boldsymbol{X}}(t')}{[1 - \beta(t') \cdot \boldsymbol{n}']}$$

Hence we can write

$$\boldsymbol{A}(t, \boldsymbol{x}) = \mu_0 \varepsilon_0 \times \frac{q}{4\pi\varepsilon_0 r'} \frac{\dot{\boldsymbol{X}}(t')}{[1 - \beta(t') \cdot \boldsymbol{n}']} = \frac{1}{c^2} \frac{q}{4\pi\varepsilon_0 r'} \frac{\dot{\boldsymbol{X}}(t')}{[1 - \beta(t') \cdot \boldsymbol{n}']}$$

$$= c^{-1} \beta(t') \phi(t, \boldsymbol{x})$$

This is useful when for expressing the magnetic field in terms of the electric field.

7.7 Determination of the electromagnetic field from the Lienard-Wierchert potentials

To determine the electric and magnetic fields we need to determine

$$\boldsymbol{E} = -\operatorname{grad}\phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$$

The potentials depend directly upon $\boldsymbol{x}$ and indirectly upon $\boldsymbol{x}, t$ through the dependence upon t'. Hence we need to work out the derivatives of t' with respect to both t and $\boldsymbol{x}$.

Expression for $\frac{\partial t'}{\partial t}$

Since,

$$t' + \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{c} = t$$

We can determine $\dfrac{\partial t'}{\partial t}$ by differentiation of this implicit equation.

$$\frac{\partial t'}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}(|\boldsymbol{x} - \boldsymbol{X}(t')|) = 1$$

$$\Rightarrow \frac{\partial t'}{\partial t} + \frac{(\boldsymbol{x} - \boldsymbol{X}(t'))}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \cdot \left(-\frac{\dot{\boldsymbol{X}}(t')}{c}\right)\frac{\partial t'}{\partial t} = 1$$

$$\left[1 - \frac{\dot{\boldsymbol{X}}(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{c|\boldsymbol{x} - \boldsymbol{X}(t')|}\right]\frac{\partial t'}{\partial t} = 1$$

Solving for $\partial t' / \partial t$:

$$\begin{aligned}\frac{\partial t'}{\partial t} &= \frac{|\boldsymbol{x} - \boldsymbol{X}(t')|}{|\boldsymbol{x} - \boldsymbol{X}(t')| - (\dot{\boldsymbol{X}}(t')/c) \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))} \\ &= \frac{r'}{r' - \dot{\boldsymbol{X}}(t') \cdot \boldsymbol{r}'/c}\end{aligned}$$

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \boldsymbol{\beta}(t') \cdot \boldsymbol{n}'}$$

Expression for $\dfrac{\partial t'}{\partial x_i} = \nabla t'$

Again differentiate the implicit function for t':

$$\frac{\partial t'}{\partial x_i} + \frac{(x_i - X_i(t'))}{c|\boldsymbol{x} - \boldsymbol{X}(t')|} - \frac{(x_j - X_j(t'))\dot{X}_j(t')/c}{|\boldsymbol{x} - \boldsymbol{X}(t')|} \times \frac{\partial t'}{\partial x_i} = 0$$

$$\frac{\partial t'}{\partial x_i}\left[1 - \frac{\beta_j(x_j - X_j(t'))}{|\boldsymbol{x} - \boldsymbol{X}(t')|}\right] + \frac{x_i - X_i}{c|\boldsymbol{x} - \boldsymbol{X}(t')|} = 0$$

$$\Rightarrow \frac{\partial t'}{\partial x_i}\left[\frac{|\boldsymbol{x} - \boldsymbol{X}(t')| - \beta(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}{|\boldsymbol{x} - \boldsymbol{X}(t')|}\right] = -\frac{x_i - X_i}{c|\boldsymbol{x} - \boldsymbol{X}(t')|}$$

$$\Rightarrow \frac{\partial t'}{\partial x_i} = \frac{-(x_i - X_i(t'))/c}{|\boldsymbol{x} - \boldsymbol{X}(t')| - \beta(t') \cdot (\boldsymbol{x} - \boldsymbol{X}(t'))}$$

i.e. $$\frac{\partial t'}{\partial x_i} = -\frac{x_i'}{r' - \dot{\boldsymbol{X}}(t') \cdot \boldsymbol{r}'/c} = -\frac{c^{-1} n_i'}{1 - \boldsymbol{\beta}(t') \cdot \boldsymbol{n}'}$$

or $$\nabla t' = -\frac{c^{-1}\boldsymbol{n}'}{1 - \boldsymbol{\beta}(t') \cdot \boldsymbol{n}'}$$

The potentials include explicit dependencies upon the spatial coordinates of the field point and implicit dependencies on $(t, \boldsymbol{x})$ via the dependence on t'

The derivatives of the potentials can be determined from:

$$\left.\frac{\partial\phi}{\partial x_i}\right|_t = \left.\frac{\partial\phi}{\partial x_i}\right|_{t'} + \left.\frac{\partial\phi}{\partial t'}\right|_{x_i}\frac{\partial t'}{\partial x_i}$$

$$\left.\frac{\partial A_i}{\partial t}\right|_{x_i} = \frac{\partial A_i}{\partial t'}\frac{\partial t'}{\partial t}$$

$$\left.\frac{\partial A_i}{\partial x_j}\right|_t = \left.\frac{\partial A_i}{\partial x_j}\right|_{t'} + \frac{\partial A_i}{\partial t'}\frac{\partial t'}{\partial x_j}$$

$$\varepsilon_{ijk}\left.\frac{\partial A_k}{\partial x_j}\right|_t = \varepsilon_{ijk}\left.\frac{\partial A_k}{\partial x_j}\right|_{t'} + \varepsilon_{ijk}\frac{\partial t'}{\partial x_j}\frac{\partial A_k}{\partial t'}$$

In dyadic form:

$$\nabla\phi|_t = \nabla\phi|_{t'} + \left.\frac{\partial\phi}{\partial t'}\right|_{\boldsymbol{x}}\nabla t' \qquad \left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial \boldsymbol{A}}{\partial t'}\right|_{\boldsymbol{x}}\frac{\partial t'}{\partial t}$$

$$\text{curl } \boldsymbol{A}|_t = \text{curl } \boldsymbol{A}|_{t'} + \nabla t' \times \left.\frac{\partial \boldsymbol{A}}{\partial t'}\right|_{\boldsymbol{x}}$$

Electric field

The calculation of the electric field goes as follows. Some qualifiers on the partial derivatives are omitted since they should be fairly obvious

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\phi|_{t'} - \frac{\partial\phi}{\partial t'}\nabla t' - \frac{\partial[c^{-1}\beta(t')\phi]}{\partial t'}\left(\frac{\partial t'}{\partial t}\right)$$

$$= -\nabla\phi|_{t'} - \frac{\partial\phi}{\partial t'}\left[\nabla t' + \frac{\beta}{c}\frac{\partial t'}{\partial t}\right] - \frac{\phi}{c}\dot{\beta}(t')\frac{\partial t'}{\partial t}$$

The terms

$$\nabla t' + \frac{\beta}{c}\frac{\partial t'}{\partial t} = -\frac{1}{c}\frac{(\boldsymbol{n}' - \beta)}{1 - \beta\cdot\boldsymbol{n}'} \qquad \frac{\partial t'}{\partial t} = \frac{1}{(1 - \beta\cdot\boldsymbol{n}')}$$

Other useful formulae to derive beforehand are:

$$\frac{\partial \boldsymbol{r}'}{\partial t'} = -c^{-1}\beta \qquad \frac{\partial r'}{\partial t'} = -c^{-1}\beta\cdot\boldsymbol{n}'$$

In differentiating $\frac{1}{r'(1 - \beta\cdot n')}$ it is best to express it in the form $\frac{1}{r' - \beta\cdot\boldsymbol{r}'}$.

With a little bit of algebra, it can be shown that

$$\nabla\phi|_{t'} = -\frac{q}{4\pi\varepsilon_0 r'^2}\frac{\boldsymbol{n}' - \beta}{(1-\beta\cdot\boldsymbol{n}')^2} \qquad \frac{\partial\phi}{\partial t'} = \frac{qc}{4\pi\varepsilon_0 r'^2}\frac{[\beta\cdot\boldsymbol{n}' - \beta^2 + c^{-1}r'\dot{\beta}\cdot n']}{(1-\beta\cdot\boldsymbol{n})^2}$$

Combining all terms:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r'^2}\frac{[(\boldsymbol{n}' - \boldsymbol{\beta})(1-\beta'^2 + c^{-1}r'\dot{\beta}\cdot\boldsymbol{n}') - c^{-1}r'\dot{\boldsymbol{\beta}}(1-\beta\cdot\boldsymbol{n}')]}{(1-\boldsymbol{\beta}\cdot\boldsymbol{n}')^3}$$

The immediate point to note here is that many of the terms in this expression decrease as r'^{-2}. However, the terms proportional to the acceleration only decrease as r'^{-1}. These are the radiation terms:

$$\boldsymbol{E}_{\text{rad}} = \frac{q}{4\pi c\varepsilon_0 r}\frac{[(\boldsymbol{n}' - \beta)\dot{\beta}\cdot\boldsymbol{n}' - \dot{\beta}(1-\dot{\beta}\cdot\boldsymbol{n}')]}{(1-\beta\cdot\boldsymbol{n}')^3}$$

Magnetic field

We can evaluate the magnetic field without going through more tedious algebra. The magnetic field is given by:

$$\boldsymbol{B} = \text{curl } \boldsymbol{A} = \text{curl } \boldsymbol{A}\big|_{t'} + \nabla t' \times \frac{\partial \boldsymbol{A}}{\partial t'}$$

$$= \text{curl } (c^{-1}\phi\beta)\big|_{t'} + \nabla t' \times \frac{\partial \boldsymbol{A}}{\partial t'}$$

where

$$\nabla t' = \frac{-c^{-1}\boldsymbol{n}'}{1 - \beta \cdot \boldsymbol{n}'} = -c^{-1}\boldsymbol{n}' \times \frac{\partial t'}{\partial t}$$

Now the first term is given by:

$$\text{curl } (c^{-1}\phi\beta) = c^{-1}\nabla\phi\big|_{t'} \times \beta$$

and we know from calculating the electric field that

$$\nabla\phi\big|_{t'} = -\frac{q}{4\pi\varepsilon_0 r'^2}\frac{\boldsymbol{n}' - \beta}{(1 - \beta \cdot \boldsymbol{n}')^2} = \xi(\boldsymbol{n}' - \beta)$$

Therefore,

$$c^{-1}\nabla\phi|_{t'} \times \beta = c^{-1}\xi(\boldsymbol{n}' - \beta) \times \beta = c^{-1}\xi(\boldsymbol{n}' - \beta) \times (\beta - \boldsymbol{n} + \boldsymbol{n}) = c^{-1}\xi(\boldsymbol{n}' - \beta) \times \boldsymbol{n}'$$

$$= c^{-1}\nabla\phi|_{t'} \times \boldsymbol{n}'$$

Hence, we can write the magnetic field as:

$$\boldsymbol{B} = c^{-1}\nabla\phi|_{t'} \times \boldsymbol{n}' - c^{-1}\boldsymbol{n}' \times \frac{\partial \boldsymbol{A}}{\partial t'}\frac{\partial t'}{\partial t} = c^{-1}\boldsymbol{n}' \times \left[-\nabla\phi|_{t'} - \frac{\partial \boldsymbol{A}}{\partial t}\right]$$

Compare the term in brackets with

$$\boldsymbol{E} = -\nabla\phi|_{t'} + \left.\frac{\partial\phi}{\partial t'}\right|_{\boldsymbol{x}} \nabla t' - \frac{\partial \boldsymbol{A}}{\partial t}$$

Since $\nabla t' \propto \boldsymbol{n}'$, then

$$\boldsymbol{B} = c^{-1}(\boldsymbol{n}' \times \boldsymbol{E})$$

This equation holds for both radiative and non-radiative terms.

Poynting flux

The Poynting flux is given by:

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = \frac{\boldsymbol{E} \times (\boldsymbol{n}' \times \boldsymbol{E})}{c\mu_0} = c\varepsilon_0[E^2\boldsymbol{n}' - (\boldsymbol{E} \cdot \boldsymbol{n}')\boldsymbol{E}]$$

We restrict attention to the radiative terms in which $\boldsymbol{E}_{\mathrm{rad}} \propto r'^{-1}$

For the radiative terms,

$$\boldsymbol{E}_{\mathrm{rad}} \cdot \boldsymbol{n}' = \frac{q}{4\pi c\varepsilon_0 r}\frac{[(1-\dot{\beta} \cdot \boldsymbol{n}')(\boldsymbol{n}' \cdot \dot{\beta}) - \boldsymbol{n}' \cdot \dot{\beta}(1-\dot{\beta} \cdot \boldsymbol{n}')]}{(1-\beta \cdot \boldsymbol{n}')^3} = 0$$

so that the Poynting flux,

$$\boldsymbol{S} = c\varepsilon_0 E^2 \boldsymbol{n}'$$

This can be understood in terms of equal amounts of electric and magnetic energy density ($(\varepsilon_0/2)E^2$) moving at the speed of light in the direction of $\boldsymbol{n}'$. This is a very important expression when it comes to calculating the spectrum of radiation emitted by an accelerating charge.

8 Radiation from relativistically moving charges

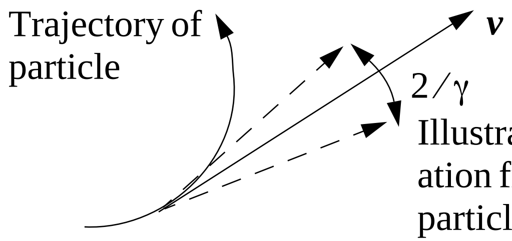


Illustration of the beaming of radiation from a relativistically moving particle.

Note the factor $(1 - \beta \cdot \boldsymbol{n}')^{-3}$ in the expression for the electric field. When $1 - \beta' \cdot \boldsymbol{n}' \approx 0$ the contribution to the electric field is large; this occurs when $\beta' \cdot \boldsymbol{n}' \approx 1$, i.e. when the angle between the velocity and the unit vector from the retarded point to the field point is approximately zero.

We can quantify this as follows: Let θ be the angle between $\beta(t')$ and $\boldsymbol{n}'$, then

$$
\begin{aligned}
1 - \beta' \cdot \boldsymbol{n}' &= 1 - |\beta'|\cos\theta \approx 1 - \left(1 - \frac{1}{2\gamma^2}\right)\left(1 - \frac{1}{2}\theta^2\right) \\
&= 1 - \left(1 - \frac{1}{2\gamma^2} - \frac{\theta^2}{2}\right) \\
&= \frac{1}{2\gamma^2} + \frac{\theta^2}{2} = \frac{1}{2\gamma^2}(1 + \gamma^2\theta^2)
\end{aligned}
$$

So you can see that the minimum value of $1 - \beta' \cdot \boldsymbol{n}'$ is $1/(2\gamma^2)$ and that the value of this quantity only remains near this for $\theta \sim 1/\gamma$. This means that the radiation from a moving charge is beamed into a narrow cone of angular extent $1/\gamma$. This is particularly important in the case of synchrotron radiation for which $\gamma \sim 10^4$ (and higher) is often the case.

9 The spectrum of a moving charge

9.1 Fourier representation of the field

Consider the transverse electric field, $\boldsymbol{E}(t)$, resulting from a moving charge, at a point in space and represent it in the form:

$$\boldsymbol{E}(t) = E_1(t)\boldsymbol{e}_1 + E_2(t)\boldsymbol{e}_2$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are appropriate axes in the plane of the wave. (Note that in general we are not dealing with a monochromatic wave, here.)

The Fourier transforms of the electric components are:

$$E_\alpha(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} E_\alpha(t)\,dt \qquad E_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} E_\alpha(\omega)\,d\omega$$

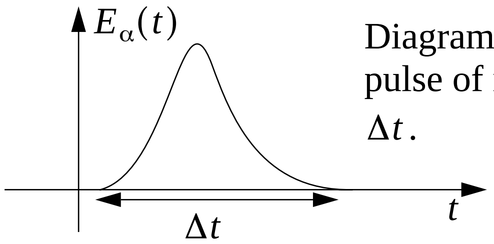
The condition that $E_\alpha(t)$ be real is that

$$E_\alpha(-\omega) = E^*_\alpha(\omega)$$

Note: We do not use a different symbol for the Fourier transform, e.g. $\tilde{E}_\alpha(\omega)$. The transformed variable is indicated by its argument.

9.2 Spectral power in a pulse

Outline of the following calculation



Diagrammatic representation of a pulse of radiation with a duration Δt.

- Consider a pulse of radiation
- Calculate total energy per unit area in the radiation.
- Use Fourier transform theory to calculate the spectral distribution of energy.
- Show this can be used to calculate the spectral *power* of the radiation.

The energy per unit time per unit area of a pulse of radiation is given by:

$$\frac{dW}{dtdA} = \text{Poynting Flux} = (c\varepsilon_0)E^2(t) = (c\varepsilon_0)[E_1^2(t) + E_2^2(t)]$$

where E_1 and E_2 are the components of the electric field wrt (so far arbitrary) unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in the plane of the wave.

The total energy per unit area in the α–component of the pulse is

$$\frac{dW_{\alpha\alpha}}{dA} = (c\varepsilon_0)\int_{-\infty}^{\infty} E_\alpha^2(t)dt$$

From Parseval's theorem,

$$\int_{-\infty}^{\infty} E_\alpha^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|E_\alpha(\omega)\right|^2 d\omega$$

The integral from $-\infty$ to ∞ can be converted into an integral from 0 to ∞ using the reality condition. For the negative frequency components, we have

$$E_\alpha(-\omega) \times E_\alpha^*(-\omega) = E_\alpha^*(\omega) \times E_\alpha(\omega) = \left|E_\alpha(\omega)\right|^2$$

so that

$$\int_{-\infty}^{\infty} E_\alpha^2(t)dt = \frac{1}{\pi}\int_0^{\infty} \left|E_\alpha(\omega)\right|^2 d\omega$$

The total energy per unit area in the pulse, associated with the α component, is

$$\frac{dW_{\alpha\alpha}}{dA} = c\varepsilon_0 \int_{-\infty}^{\infty} E_\alpha^2(t)dt = \frac{c\varepsilon_0}{\pi} \int_0^{\infty} |E_\alpha(\omega)|^2 d\omega$$

(The reason for the $\alpha\alpha$ subscript is evident below.)

[Note that there is a difference here from the Poynting flux for a pure monochromatic plane wave in which we pick up a factor of $1/2$. That factor results from the time integration of $\cos^2\omega t$ which comes from, in effect, $\int_0^{\infty} |E_\alpha(\omega)|^2 d\omega$. This factor, of course, is not evaluated here since the pulse has an arbitrary spectrum.]

We identify the spectral components of the contributors to the Poynting flux by:

$$\frac{dW_{\alpha\alpha}}{d\omega dA} = \frac{c\varepsilon_0}{\pi} |E_\alpha(\omega)|^2$$

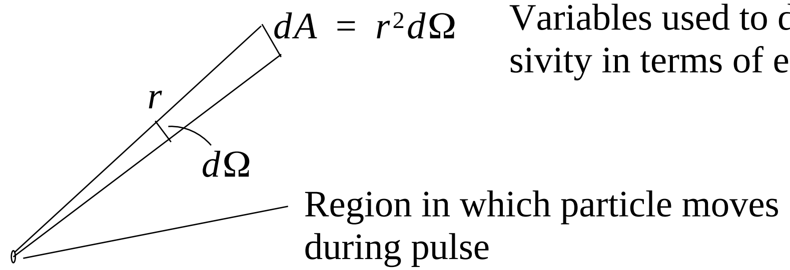
The quantity $\frac{dW_{\alpha\alpha}}{d\omega dA}$ represents the energy per unit area per unit circular frequency in the *entire pulse*, i.e. we have accomplished our aim and determined the *spectrum* of the pulse.

We can use this expression to evaluate the power associated with the pulse. Suppose the pulse repeats with period T, then we define the power associated with component α by:

$$\frac{dW_{\alpha\alpha}}{dAd\omega dt} = \frac{1}{T}\frac{dW}{dAd\omega} = \frac{c\varepsilon_0}{\pi T}\left|E_\alpha(\omega)\right|^2$$

This is equivalent to integrating the pulse over, say several periods and then dividing by the length of time involved.

9.3 Emissivity



Variables used to define the emissivity in terms of emitted power.

Consider the surface dA to be located a long distance from the distance over which the particle moves when emitting the pulse of radiation. Then $dA = r^2 d\Omega$ and

$$\frac{dW_{\alpha\alpha}}{dAd\omega dt} = \frac{1}{r^2}\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} \Rightarrow \frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} = r^2\frac{dW_{\alpha\alpha}}{dAd\omega dt}$$

The quantity

$$\frac{dW_{\alpha\alpha}}{d\Omega d\omega dt} = \frac{c\varepsilon_0 r^2}{\pi T}|E_\alpha(\omega)|^2 = \frac{c\varepsilon_0 r^2}{\pi T}E_\alpha(\omega)E_\alpha^*(\omega) \qquad \text{(Summation not implied)}$$

is the emissivity corresponding to the $\boldsymbol{e}_\alpha$ component of the pulse.

9.4 Relationship to the Stokes parameters

We generalise our earlier definition of the Stokes parameters for a plane wave to the following:

$$I_\omega = \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) + E_2(\omega)E_2^*(\omega)]$$

$$Q_\omega = \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) - E_2(\omega)E_2^*(\omega)]$$

$$U_\omega = \frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) + E_1(\omega)E_2^*(\omega)]$$

$$V_\omega = \frac{1}{i}\frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) - E_1(\omega)E_2^*(\omega)]$$

The definition of I_ω is equivalent to the definition of specific intensity in the Radiation Field chapter.

Also note the appearance of ***circular*** frequency resulting from the use of the Fourier transform.

We define a ***polarisation tensor*** by:

$$I_{\alpha\beta,\,\omega} = \frac{1}{2}\begin{bmatrix} I_\omega + Q_\omega & U_\omega - iV_\omega \\ U_\omega + iV_\omega & I_\omega - Q_\omega \end{bmatrix} = \frac{c\varepsilon_0}{\pi T} E_\alpha(\omega) E^*_\beta(\omega)$$

We have calculated above the emissivities,

$$\frac{dW_{\alpha\alpha}}{d\Omega\, d\omega\, dt} = \frac{c\varepsilon_0 r^2}{\pi T} E_\alpha(\omega) E^*_\alpha(\omega) \qquad \text{(Summation not implied)}$$

corresponding to $E_\alpha E^*_\alpha$. More generally, we define:

$$\frac{dW_{\alpha\beta}}{d\Omega\, d\omega\, dt} = \frac{c\varepsilon_0}{\pi T} r^2 E_\alpha(\omega) E^*_\beta(\omega)$$

and these are the emissivities related to the components of the polarisation tensor $I_{\alpha\beta}$.

In general, therefore, we have

$$\frac{dW_{11}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(I_\omega + Q_\omega)$$

$$\frac{dW_{22}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(I_\omega - Q_\omega)$$

$$\frac{dW_{12}}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(U_\omega - iV_\omega)$$

$$\frac{dW_{21}}{d\Omega d\omega dt} = \frac{dW_{12}^*}{d\Omega d\omega dt} \rightarrow \text{Emissivity for } \frac{1}{2}(U_\omega + iV_\omega)$$

Consistent with what we have derived above, the total emissivity is

$$\varepsilon_\omega^I = \frac{dW_{11}}{d\Omega d\omega dt} + \frac{dW_{22}}{d\Omega d\omega dt}$$

and the emissivity into the Stokes Q is

$$\varepsilon_\omega^Q = \frac{dW_{11}}{d\Omega d\omega dt} - \frac{dW_{22}}{d\Omega d\omega dt}$$

Also, for Stokes U and V:

$$\epsilon_\omega^U = \frac{dW_{12}}{d\Omega d\omega dt} + \frac{dW_{12}^*}{d\Omega d\omega dt}$$

$$\epsilon_\omega^V = i\left(\frac{dW_{12}}{d\Omega d\omega dt} - \frac{dW_{12}^*}{d\Omega d\omega dt}\right)$$

Note the factor of r^2 in the expression for $dW_{\alpha\beta}/d\Omega d\omega dt$. In the expression for the $\boldsymbol{E}$–vector of the radiation field

$$\boldsymbol{E} = \frac{q}{4\pi c\epsilon_0 r}\frac{[(\boldsymbol{n}' - \beta)(\boldsymbol{n}' \cdot \dot{\beta}) - \dot{\beta}(1 - \dot{\beta} \cdot \boldsymbol{n}')]}{(1 - \beta \cdot \boldsymbol{n}')^3}$$

$\boldsymbol{E} \propto 1/r$. Hence $r\boldsymbol{E}$ and consequently $r^2 E_\alpha E_\beta^*$ are independent of r, consistent with the above expressions for emissivity.

The emissivity is determined by the solution for the electric field.

10 Fourier transform of the Lienard-Wierchert radiation field

The emissivities for the Stokes parameters obviously depend upon the Fourier transform of

$$r\boldsymbol{E}(t) = \frac{q}{4\pi c\varepsilon_0}\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^3}$$

where the prime means evaluation at the retarded time t' given by

$$t' = t - \frac{r'}{c} \qquad r' = |\boldsymbol{x} - \boldsymbol{X}(t')|$$

The Fourier transform involves an integration wrt t. We transform this to an integral over t' as follows:

$$dt = \frac{\partial t}{\partial t'}dt' = \frac{1}{\partial t' / \partial t}dt' = (1 - \beta' \cdot \boldsymbol{n}')dt'$$

using the results we derived earlier for differentiation of the retarded time. Hence,

$$r\boldsymbol{E}(\omega) = \frac{q}{4\pi c\varepsilon_0}\int_{-\infty}^{\infty}\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^3}e^{i\omega t}(1 - \beta' \cdot \boldsymbol{n}')dt'$$

$$= \frac{q}{4\pi c\varepsilon_0}\int_{-\infty}^{\infty}\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2}e^{i\omega t}dt'$$

The next part is

$$e^{i\omega t} = \exp\left[i\omega\left(t' + \frac{r'}{c}\right)\right]$$

Since

$$r' = |\boldsymbol{x} - \boldsymbol{X}(t')| \approx \boldsymbol{x} \quad \text{when} \quad \boldsymbol{x} \gg \boldsymbol{X}(t')$$

then we expand r' to first order in $\boldsymbol{X}$. Thus,

$$r' = |x_j - X_j(t')| \qquad \frac{\partial r'}{\partial X_i} = \frac{-(x_i - X_i(t'))}{r'} = -\frac{x_i}{r} \text{ at } X_i = 0$$

$$r' = r + \left.\frac{\partial r'}{\partial X_i}\right|_{X_j = 0} \times X_i(t') = r - \frac{x_i}{r'} X_i(t') = r - n_i X_i = r - \boldsymbol{n} \cdot \boldsymbol{X}(t')$$

Note that it is the unit vector $\boldsymbol{n} = \frac{\boldsymbol{r}}{r}$ which enters here, rather than the retarded unit vector $\boldsymbol{n}'$

Hence,

$$\exp(i\omega t) = \exp\left[i\omega\left(t' + \frac{r}{c} - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] = = \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] \times \exp\left[\frac{i\omega r}{c}\right]$$

The factor $\exp\left[\frac{i\omega r}{c}\right]$ is common to all Fourier transforms $rE_\alpha(\omega)$ and when one multiplies by the complex conjugate it gives unity. This also shows why we expand the argument of the exponential to first order in $\boldsymbol{X}(t')$ since the leading term is eventually unimportant.

The remaining term to receive attention in the Fourier Transform is

$$\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2}$$

We first show that we can replace $\boldsymbol{n}'$ by $\boldsymbol{n}$ by also expanding in powers of $X_i(t')$.

$$\begin{aligned}
n_i' &= \frac{x_i - X_i(t')}{|x_j - X_j(t')|} = \frac{x_i}{r} + \left[\frac{\partial}{\partial X_j}\frac{x_i - X_i}{|x_j - X_j(t')|}\right]_{X_j = 0} \times X_j \\
&= \frac{x_i}{r} + \left[-\delta_{ij} + \frac{(x_i - X_i)(x_j - X_j)}{|x_j - X_j(t')|^3}\right]_{X_j = 0} \times X_j \\
&= n_i + \left[-\delta_{ij} + \frac{x_i x_j}{r^2}\right]\frac{X_j}{r}
\end{aligned}$$

So the difference between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is of order $\boldsymbol{X}/r$, i.e. of order the ratio the dimensions of the distance the particle moves when emitting a pulse to the distance to the source. This time however, the leading term does not cancel out and we can safely neglect the terms of order $\boldsymbol{X}/r$. Hence we put,

$$\frac{\boldsymbol{n}' \times [(\boldsymbol{n}' - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n}')^2} = \frac{\boldsymbol{n} \times [(\boldsymbol{n} - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n})^2}$$

It is straightforward (exercise) to show that

$$\frac{d}{dt'}\frac{\boldsymbol{n} \times (n \times \beta')}{1 - \beta' \cdot \boldsymbol{n}'} = \frac{\boldsymbol{n} \times [(\boldsymbol{n} - \beta') \times \dot{\beta}']}{(1 - \beta' \cdot \boldsymbol{n})^2}$$

Hence,

$$rE(\omega) = \frac{q}{4\pi c\varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} \frac{d}{dt'}\frac{\boldsymbol{n} \times (\boldsymbol{n} \times \beta')}{1 - \beta' \cdot \boldsymbol{n}} \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] dt'$$

One can integrate this by parts. First note that

$$\left.\frac{\boldsymbol{n} \times (\boldsymbol{n} \times \beta')}{1 - \beta' \cdot \boldsymbol{n}}\right|_{-\infty}^{\infty} = 0$$

since we are dealing with a pulse. Second, note that,

$$\frac{d}{dt'}\exp\left[i\omega\left(t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right] = \exp\left[i\omega\left(t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right] \times i\omega[1 - \beta'\cdot\boldsymbol{n}]$$

and that the factor of $[1 - \beta'\cdot\boldsymbol{n}']$ cancels the remaining one in the denominator. Hence,

$$r\boldsymbol{E}(\omega) = \frac{-i\omega q}{4\pi c\varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} \boldsymbol{n}\times(\boldsymbol{n}\times\beta')\exp\left[i\omega\left(t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right)\right]dt'$$

In order to calculate the Stokes parameters, one selects a coordinate system ($\boldsymbol{e}_1$ and $\boldsymbol{e}_2$) in which this is as straightforward as possible. The motion of the charge enters through the terms involving $\beta(t')$ and $\boldsymbol{X}(t')$ in the integrand.

Remark

The feature associated with radiation from a relativistic particle, namely that the radiation is very strongly peaked in the direction of motion, shows up in the previous form of this integral via the factor $(1 - \beta\cdot\boldsymbol{n}')^{-3}$. This dependence is not evident here. However, when we proceed to evaluate the integral in specific cases, this dependence resurfaces.