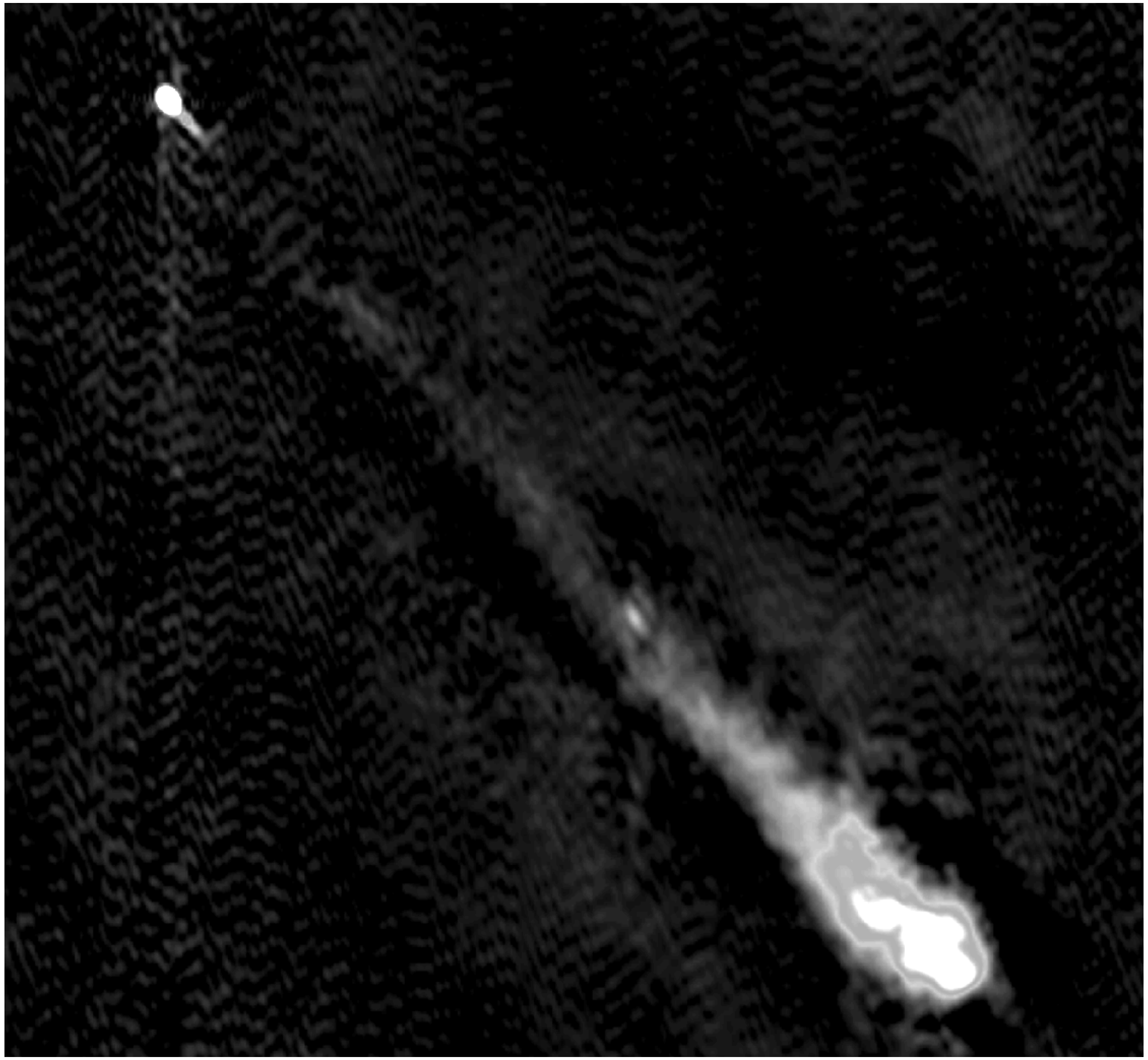
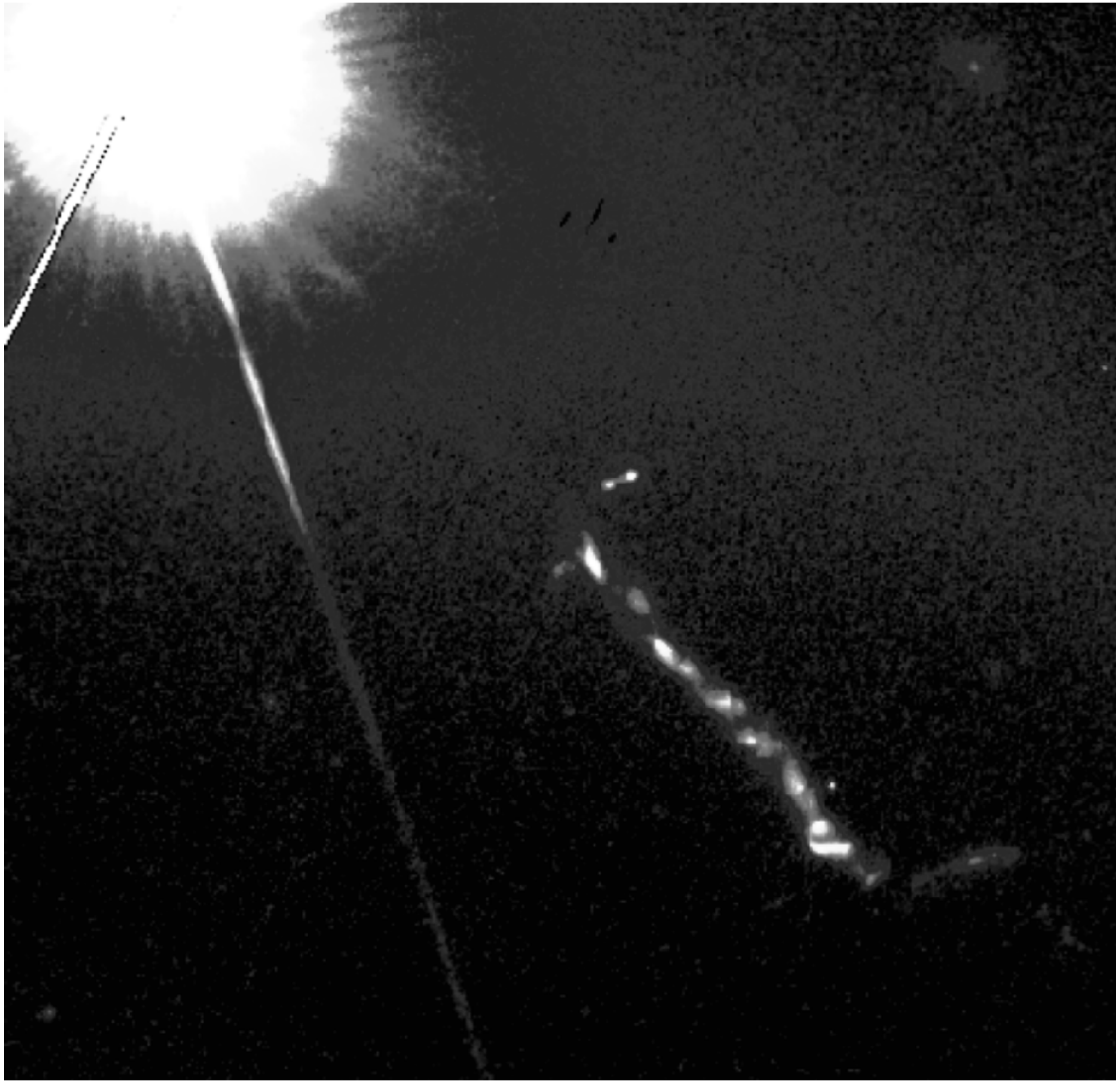
Emission and Absorption

1 Motivation - the Quasar 3C 273



A MERLIN radio image of the quasar 3C 273.

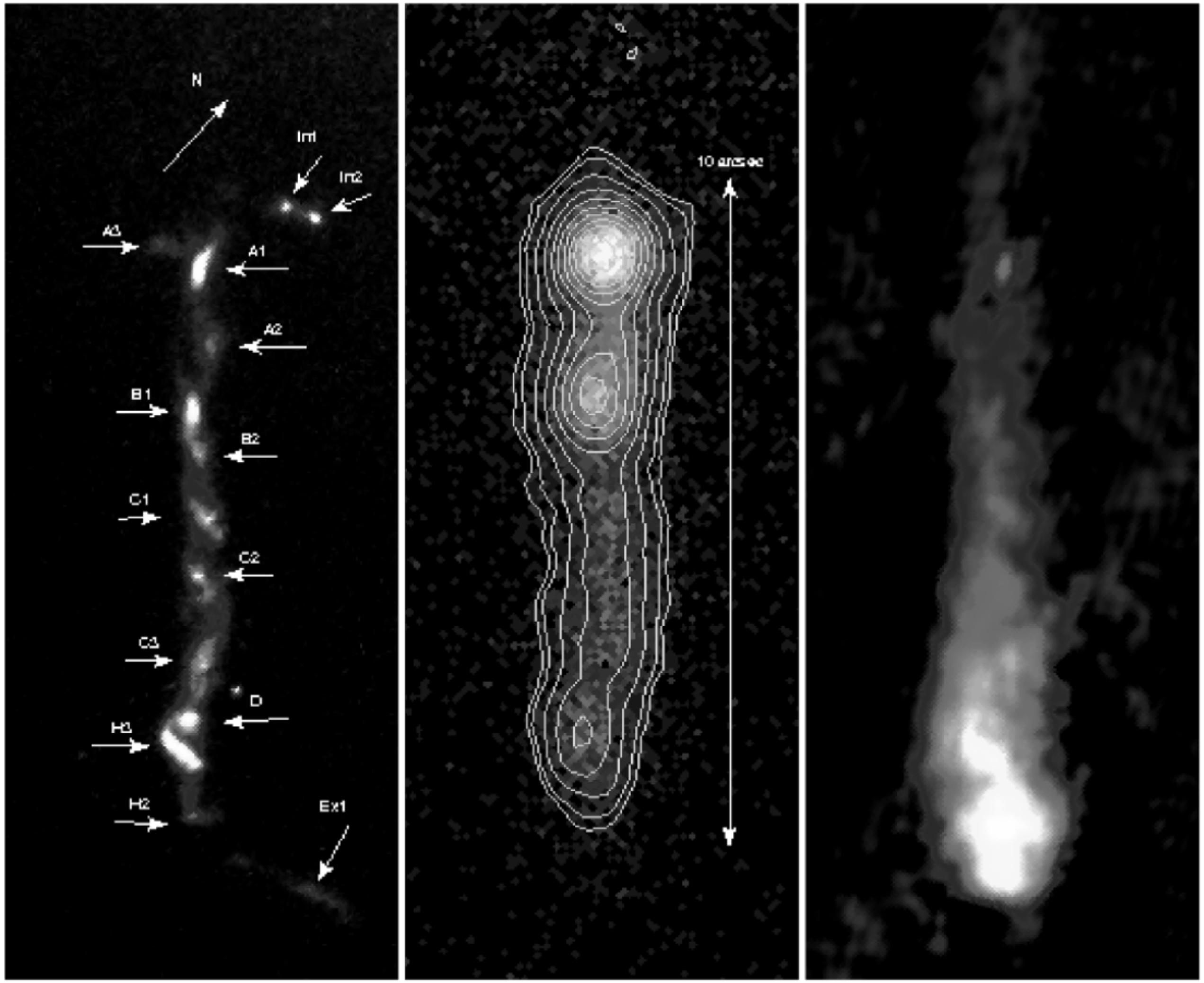
This shows the point source at the nucleus and the jet.



HST optical image of 3C273

Note the very strong central point source and the less luminous jet.

Objects such as 3C273 radiate as much energy from a region the size of the solar system as the entire galaxy.



Set of 3 images of the jet of 3C273.

Left: HST

Middle: Chandra X-ray

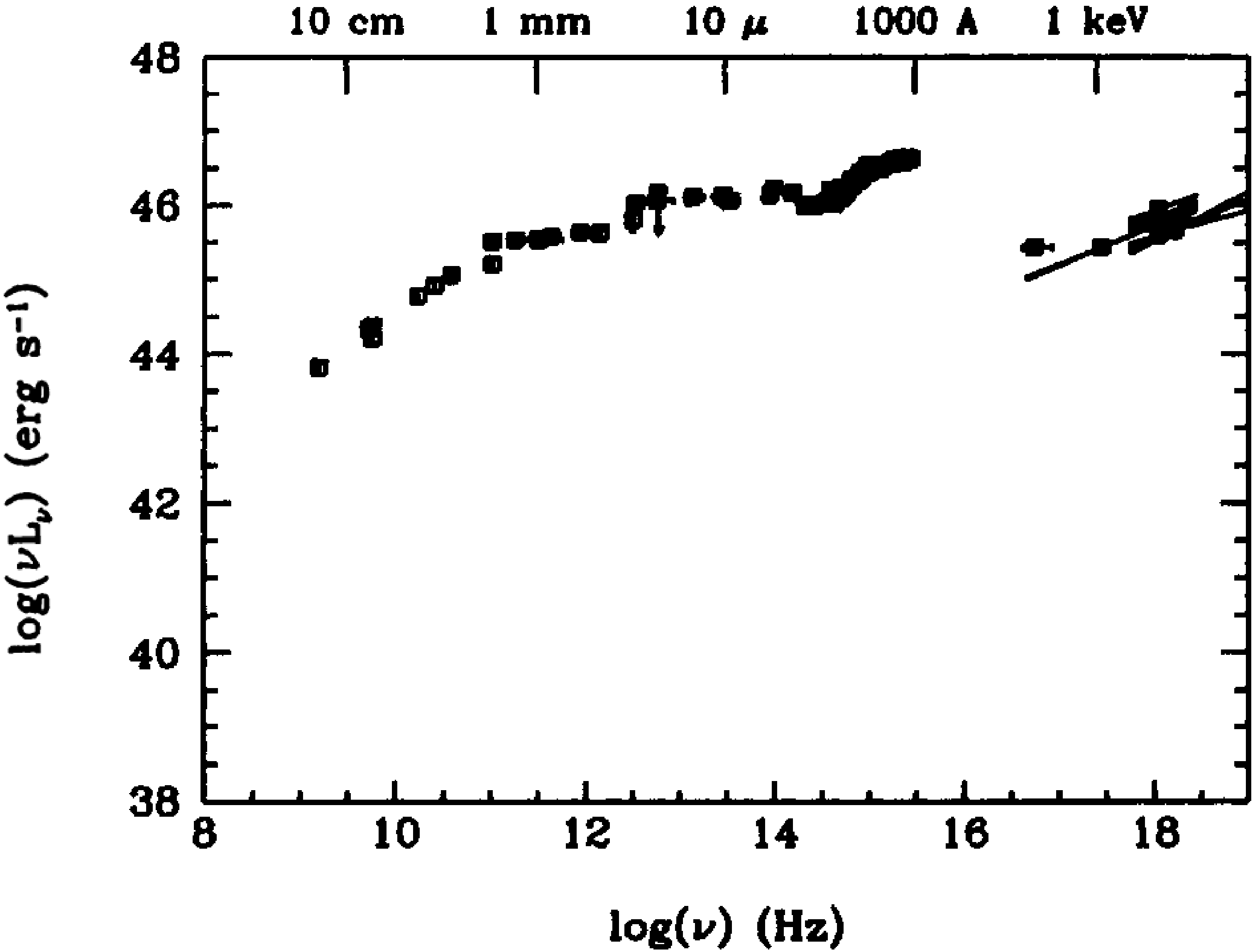
Right: Merlin radio

Credits:

Optical: NASA/STScI

X-ray: NASA/CXC

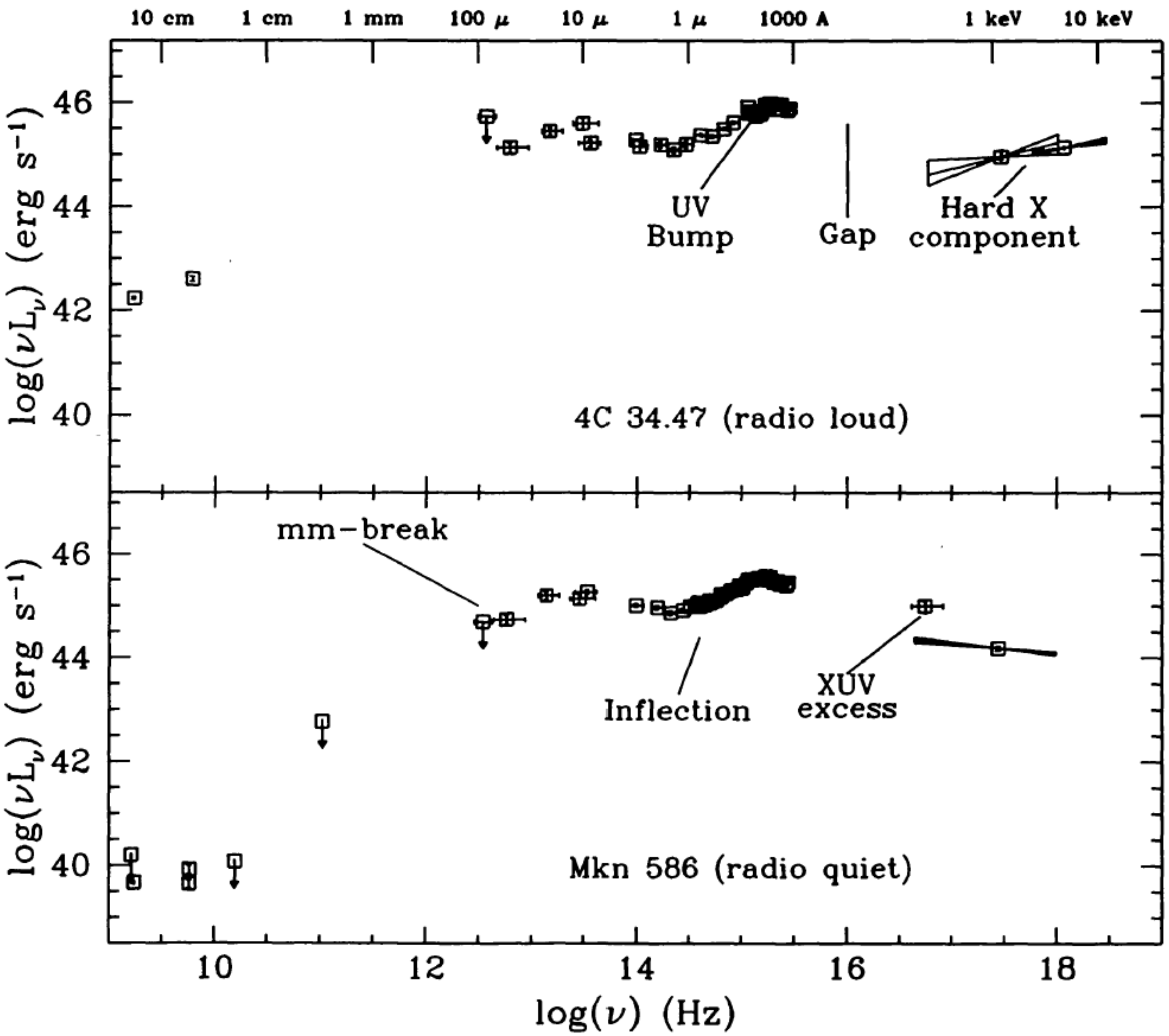
Radio: MERLIN



Integrated spectral energy distribution of 3C 273 from 10^8 to 10^{19} Hz. From Elvis et al. 1994, ApJS, 95, 1

Note the rising spectrum in the radio.

2 Radio-loud and radio quiet



Spectra of a typical radio loud quasar and a typical radio quiet object.

Note that apart from the radio emission the major differences are:

- The slopes of the hard X-ray component
- The XUV excess in the radio quiet object

3 Absorption in general

For every emission process, there is an absorption process, wherein an electromagnetic wave can affect the energy of particles thereby losing energy itself. We have dealt with the emission of synchrotron radiation in some detail. The corresponding process of synchrotron self-absorption whereby synchrotron emitting particles can absorb the radiation they emit is an important process in very compact sources. If it is present, it can be used to estimate the magnetic field in a source.

There are two ways of calculating the absorption coefficient.

1. Calculate the dielectric tensor of the plasma and then use this tensor to calculate the effect on an electromagnetic wave.This method is valuable when more information is required such as when treat-

ing the radiative transfer of polarised radiation.

2. Use a remarkable set of generic relations discovered by Einstein in order to relate absorption to emission for any process. This method leads quickly to an expression for the absorption coefficient and we shall use it here since it is of general interest.

4 Radiative transfer in a thermal gas

This section is important as a prelude to the treatment of the Einstein coefficients and is also important for our discussion of the emission from accretion disks.

4.1 The source function

Consider the radiative transfer equation:

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu$$

Divide through by α_ν

$$\frac{1}{\alpha_\nu}\frac{dI_\nu}{ds} = \frac{j_\nu}{\alpha_\nu} - I_\nu$$

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu$$

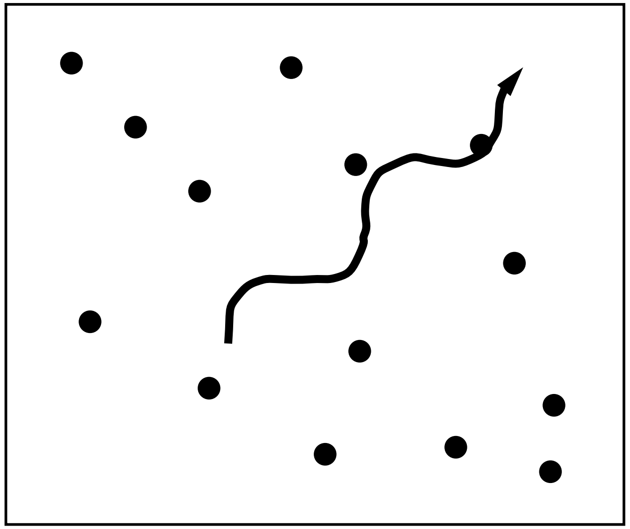
where

$$S_\nu = \frac{j_\nu}{\alpha_\nu}$$

is the *source function.*

4.2 Thermodynamic equilibrium

Now consider the situation where the matter and radiation are in thermodynamic equilibrium.



Matter and radiation in an enclosure.

In this case we know that the equilibrium radiation field is described by the Planck function, viz.

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1}$$

Since $\frac{dI_\nu}{ds} = 0$ inside the enclosure, the source function is given by

$$S_\nu = B_\nu.$$

The surface brightness (specific intensity) of a black body is given by B_ν.

4.3 Kirchhoff's law

In a thermal plasma in which the matter is in thermal equilibrium, but not necessarily with the radiation, the coefficients of emission and absorption are functions of the temperature only so that the source function is given by

$$S_\nu = B_\nu(T)$$

This holds irrespective of whether the matter and radiation are in thermal equilibrium or not. This then relates absorption to emission via

$$\alpha_\nu = [B_\nu(T)]^{-1}\varepsilon_\nu = \frac{c^2}{2h\nu^3}\left[e^{\frac{h\nu}{kT}} - 1\right]\varepsilon_\nu$$

This relationship is known as ***Kirchhoff's law***.

For a thermal plasma, in which the matter is in thermal equilibrium, Kirchhoff's law is sufficient to characterise the source function and the absorption. There are two important cases where Kirchhoff's law is insufficient:

1. The plasma is thermal but the levels of the various atoms are not in thermal equilibrium.

2.The plasma is nonthermal, i.e. no component of it is described in terms of a single temperature.

In both of these cases, one is required to go one step beyond Kirchhoff's law to the ***Einstein relations***.

5 Properties of blackbody radiation

5.1 Energy density

Recall the expression for the energy density per unit frequency

$$u_\nu = \frac{4\pi}{c} J_\nu \qquad J_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu d\Omega$$

The total energy density is:

$$\varepsilon_{\text{rad}} = \int_0^\infty u_\nu d\nu = \frac{4\pi}{c}\int_0^\infty B_\nu(T)d\nu$$

$$= \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{\left[\exp\left(\frac{h\nu}{kt}\right) - 1\right]}d\nu$$

$$= \frac{8\pi k^4 T^4}{h^3 c^3}\int_0^\infty \frac{(h\nu/kt)^3}{\left[\exp\left(\frac{h\nu}{kt}\right) - 1\right]}d\left(\frac{h\nu}{kt}\right)$$

The integral is $\pi^4/15$ so that

$$\varepsilon_{\rm rad} = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 = a T^4$$

$$a = \frac{8\pi^5 k^4}{15 h^3 c^3}$$

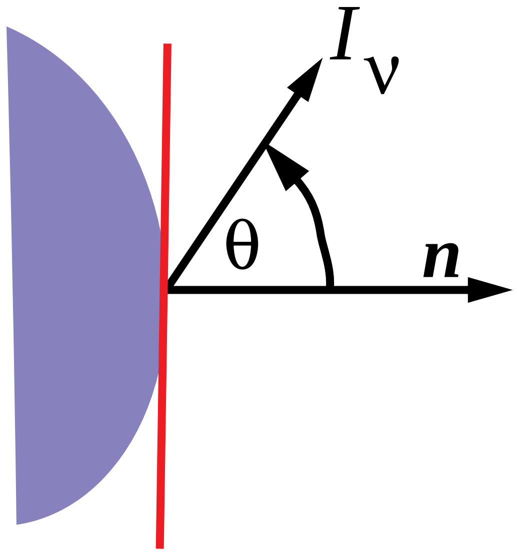
This is known as Stefan's Law.

5.2 Flux from the surface of a black body

The total (frequency integrated) flux is given by:

$$F = \int_0^\infty \pi B_\nu(T) d\nu = \frac{c}{4} \times \frac{4\pi}{c} \int_0^\nu B_\nu(T) d\nu = \frac{ac}{4} T^4 = \sigma T^4$$

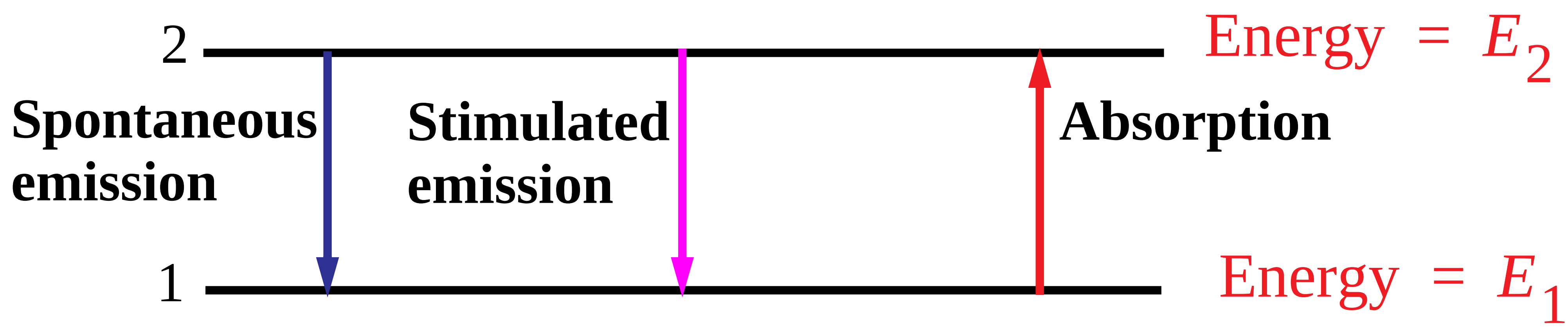
where $\sigma = \dfrac{ac}{4}$ is the Stefan-Boltzmann constant.



$$F_\nu = \int_0^{2\pi} d\phi \int_0^{\pi} B_\nu(T) \cos\theta \sin\theta \, d\theta = \pi B_\nu(T)$$

6.1 Definition of the Einstein coefficients

These relations are arrived at in a similar manner to Kirchhoff's law via the inclusion of a new process – *stimulated emission*. Einstein found it necessary to include stimulated emission in the analysis of radiation processes. Neglecting it led to inconsistencies.



Processes in a 2-level atom

The above diagram refers to a 2-level atom with energy levels E_1 and E_2.[1] Emission corresponds to a transition of an electron from level 2 to level 1, with the emission of a photon with frequency given by

$$h\nu_{21} = E_2 - E_1$$

where h is Planck's constant. The transition is not exactly sharp as indicated. Absorption is the reverse process, whereby a photon with energy $= E_2 - E_1$ causes a transition from level 2 to level 1. In general the emitted and absorbed radiation is described by a profile function $\phi(\nu)$ sharply peaked on $\nu = \nu_{21}$. This function expresses the fact that in emission there is a range of photon energies resulting

1.Generalisation to a multilevel atom is trivial since we consider levels in pairs.

from the transition. In absorption, photons with a frequency slightly different from ν_{21} also cause a transition. We assume that $\phi(\nu)$ for emission and absorption are equal. This is an adequate assumption for the present purposes.

The description of the various terms is as follows:

Spontaneous emission

This is the emission that occurs in the absence of a radiation field. This is what we calculate from the Quantum Mechanics of the atom in question or in the case of continuum radiation, this is what we calculate from the application of electromagnetic theory.

Let the probability that an atom makes a spontaneous transition from level 2 to 1 in time dt, emitting a photon within solid angle $d\Omega$ be given by $A_{21} d\Omega dt$. Another way of saying this is that the probability per unit time is $A_{21} d\Omega$.

Now let N_2 be the number of atoms per unit volume in level 2. The contribution to the emissivity from spontaneous emission is then

$$j_\nu^{\text{spont}} = N_2 h\nu_{21} \phi(\nu) A_{21}$$

Stimulated emission

This is another component of the emission which occurs as a result of the radiation field. The presence of a photon field stimulates the production of additional photons. The stimulated photons have the same direction and polarisation as the original photons.

Let $B_{21} I_\nu d\Omega dt$ be the probability of stimulated emission in time dt into solid angle $d\Omega$ and let the number of atoms per unit volume in level 2 be N_2. The contribution to the emissivity from stimulated emission is then

$$j_\nu^{\mathrm{stim}} = N_2 h\nu_{21} \phi(\nu) B_{21} I_\nu$$

Note that the stimulated photons are emitted in the same direction as the incident photons and with the same polarisation.

Absorption

Absorption occurs when a photon interacts with the atom and causes a transition from level 1 to level 2. Let $B_{12}I_\nu d\Omega dt$ be the probability that an atom absorbs a photon from solid angle $d\Omega$ in time dt. Then the absorption is given by

$$\alpha_\nu I_\nu = N_1 B_{12} h\nu_{21}\phi(\nu) I_\nu.$$

The coefficients A_{21}, B_{21} and B_{12} are the ***Einstein coefficients***.

Differences in notation and approach.

The Einstein coefficients are sometimes described in terms of the mean intensity. This is valid when the emissivity is isotropic. They are also sometimes defined in terms of the photon energy density in which case there is a factor of $4\pi / c$ difference in the definition.

In many treatments of the Einstein relations (including the original paper), it is assumed, either implicitly or explicitly, that the profile function, $\phi(\nu)$ is a delta function.

6.2 The radiative transfer equation in terms of the Einstein coefficients

Taking into account spontaneous and stimulated emissivity, the total emissivity is

$$j_\nu = N_2 h\nu_{21}\phi(\nu)A_{21} + N_2 h\nu_{21}\phi(\nu)B_{21}I_\nu$$

Hence the radiative transfer equation is:

$$\begin{aligned}\frac{dI_\nu}{ds} &= j_\nu - \alpha_\nu I_\nu \\ &= N_2 h\nu_{21}\phi(\nu)A_{21} \\ &\quad + N_2 h\nu_{21}\phi(\nu)B_{21}I_\nu - N_1 B_{12} h\nu_{21}\phi(\nu)I_\nu\end{aligned}$$

Nett absorption

The stimulated emission term has the same form as the absorption term and we therefore incorporate it into the absorption term as a negative absorption, giving:

$$\frac{dI_\nu}{ds} = N_2 h\nu_{21}\phi(\nu)A_{21} - [N_1 B_{12} - N_2 B_{21}]h\nu_{21}\phi(\nu)I_\nu$$

so that the absorption coefficient

$$\alpha_\nu^* = [N_1 B_{12} - N_2 B_{21}]h\nu_{21}\phi(\nu)$$

6.3 Derivation of the Einstein relations

We proceed similarly to deriving the ***Kirchhoff relations***. When radiation and matter are in thermal equilibrium, then

$$I_\nu = B_\nu(T) = \frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1}$$

and $\frac{dI_\nu}{ds} = 0$ in our blackbody cavity implies that

$$N_2 h\nu_{21}\phi(\nu)A_{21} = [N_1 B_{12} - N_2 B_{21}]h\nu_{21}\phi(\nu)B_\nu$$

Cancelling out common factors:

$$N_2 A_{21} = [N_1 B_{12} - N_2 B_{21}] \frac{2h\nu_{21}^3}{c^2} \left[e^{\frac{h\nu_{21}}{kT}} - 1 \right]^{-1}$$

We know that when a system is in thermodynamic equilibrium, the population of the various energy levels is given by:

$$N \propto g \exp\left[-\frac{E}{kT} \right]$$

where g is the statistical weight.

Hence,

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\frac{\exp\left[-\frac{E_2}{kt}\right]}{\exp\left[-\frac{E_1}{kT}\right]} = \frac{g_2}{g_1}\exp\left[-\frac{(E_2 - E_1)}{kT}\right] = \frac{g_2}{g_1}\exp\left[-\frac{h\nu_{21}}{kT}\right]$$

The equation for radiative equilibrium can be written:

$$A_{21} = \left[\frac{N_1}{N_2}B_{12} - B_{21}\right]\frac{2h\nu_{21}^3}{c^2}\left[e^{\frac{h\nu_{21}}{kT}} - 1\right]^{-1}$$

$$\Rightarrow A_{21} = \left[\frac{g_1}{g_2}\frac{B_{12}}{B_{21}}e^{\frac{h\nu_{21}}{kT}} - 1\right]B_{21} \times \frac{2h\nu_{21}^3}{c^2}\left[e^{\frac{h\nu_{21}}{kT}} - 1\right]^{-1}$$

These relationships are independent of the temperature if and only if

$$\frac{g_1}{g_2}\frac{B_{12}}{B_{21}} = 1 \qquad A_{21} = B_{21}\frac{2h\nu_{21}^3}{c^2}$$

That is,

$$A_{21} = \frac{2h\nu_{21}^3}{c^2} B_{21}$$

$$B_{12} = \frac{g_2}{g_1} B_{21}$$

These are the Einstein relations. They have been derived for the special case where the matter and the radiation are all in thermal equilibrium. However, they represent general relationships between emission and absorption coefficients which are valid for all situations.

Important features

- The Einstein relations are independent of the temperature so that they are applicable to nonthermal as well as thermal distributions.

- The Einstein relations would be impossible without the presence of stimulated emission represented by the coefficient B_{21}. If $B_{21} = 0$ then both spontaneous emission and absorption coefficients are zero.

Convenient representation

Finally, for convenience, the relations are usually represented without the 21 subscript on the frequency:

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21}$$

$$B_{12} = \frac{g_2}{g_1} B_{21}$$

Given any one coefficient, the others can be determined and the complete emission and absorption properties of the plasma can be specified.

6.4 Special cases

The emissivity and absorption coefficient are given by

$$j_\nu = N_2 h\nu_{21}\phi(\nu)A_{21}$$

$$\alpha^*_\nu = [N_1 B_{12} - N_2 B_{21}]h\nu_{21}\phi(\nu)$$

Hence the source function,

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{N_2 A_{21}}{[N_1 B_{12} - N_2 B_{21}]} = \left[\frac{N_1 B_{12}}{N_2 B_{21}} - 1\right]^{-1}\frac{A_{21}}{B_{21}}$$

$$= \frac{2h\nu^3}{c^2}\left[\frac{N_1 g_2}{N_2 g_1} - 1\right]^{-1}$$

where we have used the Einstein relations.

The absorption coefficient

$$\alpha_\nu^* = N_2 B_{21} h\nu_{21}\phi(\nu)\left[\frac{N_1 g_2}{N_2 g_1} - 1\right]$$

Local thermodynamic equilibrium => Kirchhoff's Law

In LTE we know that

$$\frac{N_1 g_2}{N_2 g_1} = \exp\left(\frac{h\nu}{kT}\right)$$

so that for LTE, we recover Kirchhoff's law,

$$S_\nu = \frac{2h\nu^3/c^2}{[e^{h\nu/kT} - 1]}$$

Non Local Thermodynamic Equilibrium (Non LTE)

Non Local Thermodynamic Equilibrium is a situation in which

$$\frac{N_1 g_2}{N_2 g_1} \neq \exp\left[\frac{\Delta E}{kt}\right]$$

where ΔE is the difference in energy between any two levels.

Masers and lasers

If

$$\frac{N_1 g_2}{N_2 g_1} - 1 < 0$$

i.e.

$$N_1 < \frac{g_1}{g_2} N_2$$

then, the absorption coefficient is negative. Radiation in this case is amplified, not absorbed. A population for which this is the case is called an inverted population because there are more particles in the upper level than in the higher (modulo the statistical weight.) Inverted populations like this in the laboratory give ***lasers***. In astronomical contexts, they give ***masers***. In LTE and in may other contexts, $N_2 < (g_1/g_2)N_1$ so that the situations which give rise to masers are somewhat unusual. They arise from ***pumping*** of the upper level by some source of radiation, usually in the infrared. The velocities of masers have proven of great importance in the detection of black holes.

7 The Einstein relations for continuum radiation

The Einstein relations can be generalised to polarised continuum radiation provided one makes the correct identification of the relevant number densities. Since we are dealing with continuum radiation there are no discrete energy levels and the emitted photons do not have discrete energies, i.e. there are no emission lines. We therefore consider the distribution of particles in ***momentum space*** and make an appropriate identification of the populations of the levels which we have previously called N_1 and N_2.

7.1 The phase-space distribution function

Remember the phase-space distribution function $f(\boldsymbol{x}, \boldsymbol{p})$ which is defined by

$$\begin{array}{c}\text{The number of particles in} \\ \text{an element of phase space}\end{array} = f(\boldsymbol{x}, \boldsymbol{p})d^3x d^3p$$

Hence,

$$\begin{array}{c}\text{The number density} \\ \text{in an element of momentum space}\end{array} = f(\boldsymbol{x}, \boldsymbol{p})d^3p$$

We use this number density in discussing the Einstein relations for continuum emission.

Relation to the number per unit energy

Taking polar coordinates in phase space, the element of volume is

$$d^3p = p^2 dp d\Omega = p^2 dp \sin\theta d\theta d\phi$$

For an isotropic distribution, in which $f(\boldsymbol{x}, \boldsymbol{p}) = f(\boldsymbol{x}, p)$, the number density of electrons is

$$n = \int_{4\pi} d\Omega \int_0^\infty f(p) p^2 dp = 4\pi \int_0^\infty p^2 f(p) dp$$

after integrating out the angular part. Hence, the number density of particles per unit momentum is

$$N(p) = 4\pi p^2 f(\boldsymbol{x}, p)$$

The relationship between $f(\boldsymbol{x}, \boldsymbol{p})$ and $N(\boldsymbol{x}, E)$ is

$$4\pi p^2 f(\boldsymbol{x}, p)dp = N(\boldsymbol{x}, E)dE$$

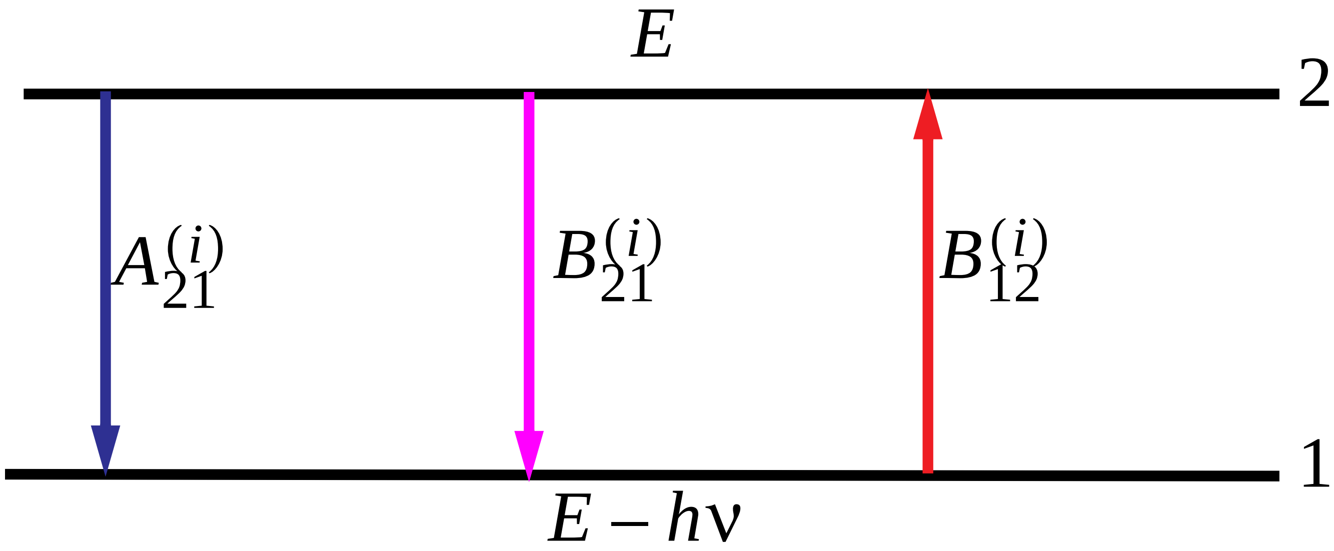
For relativistic particles, $E = cp$ and, dropping the explicit spatial dependence,

$$N(E) = \frac{4\pi}{c^3}E^2 f(E/c)$$

7.2 Einstein relations for continuum emission

7.2.1 Number densities

We consider Einstein coefficients for polarised emission and absorption of continuum photons as follows. Note that we consider each radiation mode separately.



Einstein coefficients for transitions between energy states in a plasma differing by the energy of the emitted photon, $h\nu$.

The wave vector of the emitted photon is $\boldsymbol{k} = k\boldsymbol{\kappa}$ where k is the wave number and $\boldsymbol{\kappa}$ is the direction of the photon. The momentum is $\hbar\boldsymbol{k} = (h\nu/c)\boldsymbol{\kappa}$.

Consider a plasma in which the emission of a photon of momentum $\hbar\boldsymbol{k}$ results in a change of the momentum of the emitting particle by the corresponding amount.

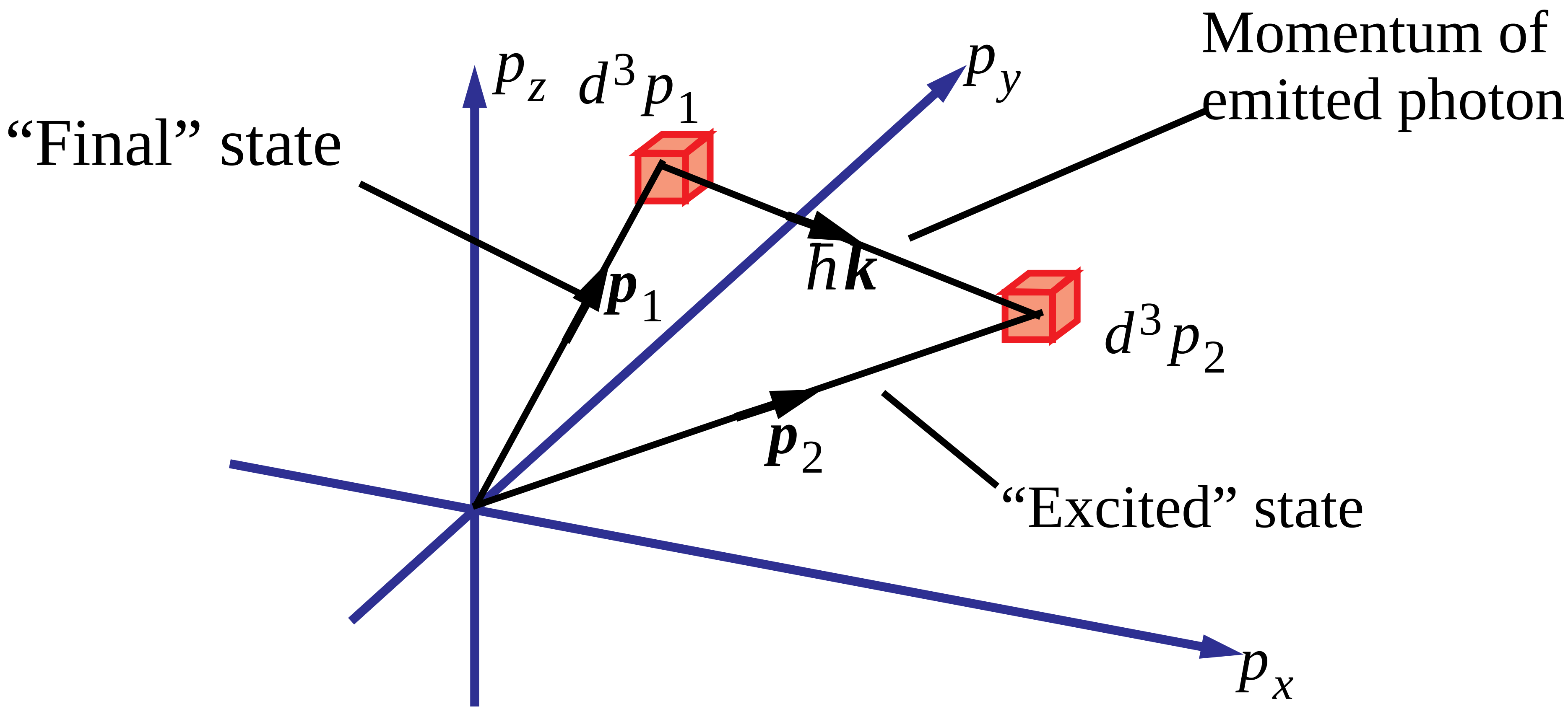
The relevant number densities are

$$N_1 = \Delta N(\boldsymbol{p} - \hbar\boldsymbol{k}) \qquad N_2 = \Delta N(\boldsymbol{p})$$

where

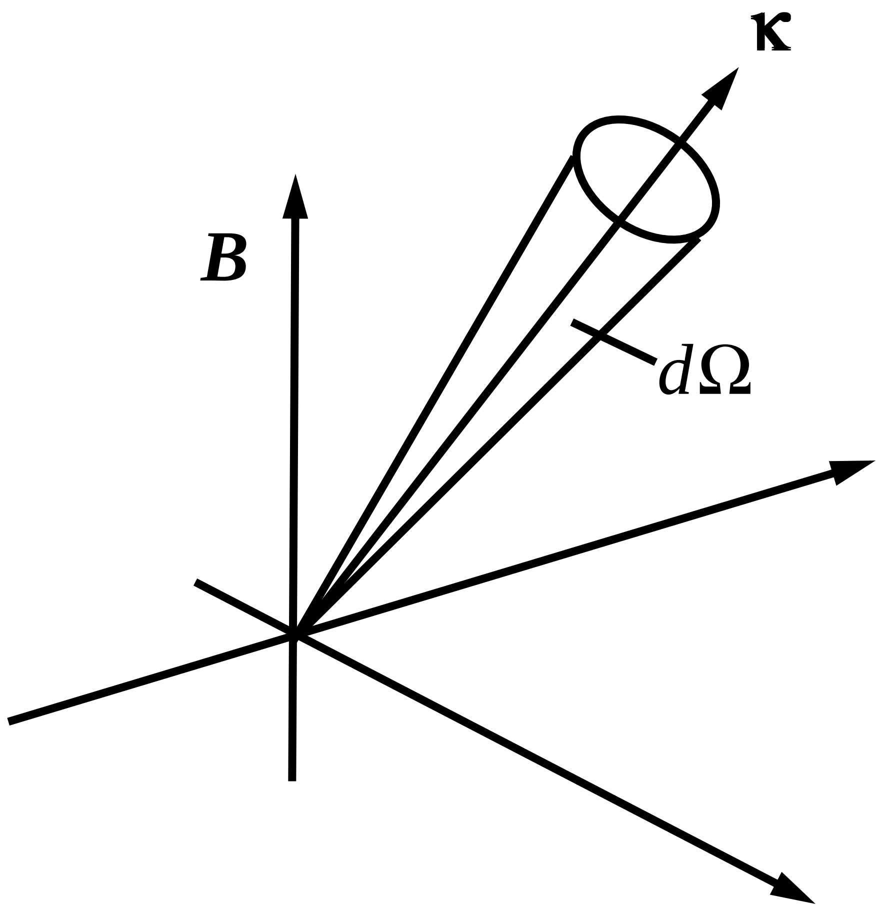
$$\Delta N(\boldsymbol{p}) = f(\boldsymbol{x}, \boldsymbol{p})d^3p$$

In focusing on $\Delta N(\boldsymbol{p})$ we are considering pairs of particle momenta which are separated in momentum by the momentum of the photon. They are separated in energy by $h\nu$.



Momentum of
emitted photon
p_z
d^3p_1
p_y
"Final" state
$\hbar\boldsymbol{k}$
$\boldsymbol{p}_1$
d^3p_2
$\boldsymbol{p}_2$
"Excited" state
p_x

7.2.2 The Einstein coefficients for polarised continuum radiation



Illustrating the definition of the Einstein coefficients for the emission and absorption of radiation in a particular direction given by the unit vector $\boldsymbol{\kappa}$.

The Einstein coefficients are defined by:

$$A_{21}^{(i)} d\nu d\Omega = \text{Probability per unit time for spontaneous emission of a photon in mode } i \text{ in the ranges } d\nu \text{ and } d\Omega$$

$$B_{12}^{(i)} I_\nu d\nu d\Omega = \text{Probability per unit time for the absorption of a photon in mode } i \text{ in the ranges } d\nu \text{ and } d\Omega$$

$$B_{21}^{(i)} I_\nu d\nu d\Omega = \text{Probability per unit time for stimulated emission of a photon in mode } i \text{ in the ranges } d\nu \text{ and } d\Omega$$

For polarised emission the Einstein relations are:

$$B_{21}^{(i)} = B_{12}^{(i)} = \left(\frac{c^2}{h\nu^3}\right) A_{21}^{(i)}$$

The factor of 2 difference is the result of the Planck function for thermal emission being halved for each mode of polarisation when one considers the detailed balance relations.

For continuum states, the statistical weights of each level to be unity.

The contribution to the absorption coefficient from states differing in momentum by the momentum of an emitted photon is:

$$\begin{aligned}
d\alpha_\nu^{(i)} &= [N_1 B_{12}^{(i)} - N_2 B_{21}^{(i)}]h\nu = [N_1 - N_2]B_{21}^{(i)}h\nu \\
&= [\Delta N(\boldsymbol{p} - \hbar\boldsymbol{k}) - \Delta N(\boldsymbol{p})]\frac{c^2}{h\nu^3}h\nu A_{21}^{(i)} \\
&= [\Delta N(\boldsymbol{p} - \hbar\boldsymbol{k}) - \Delta N(\boldsymbol{p})]\frac{c^2}{\nu^2}A_{21}^{(i)}
\end{aligned}$$

where the (i) refers to the two modes of polarisation of the emitted transverse waves. In this equation

$$\Delta N(\boldsymbol{p} - \hbar\boldsymbol{k}) = f(\boldsymbol{p} - \hbar\boldsymbol{k}) d^3 p_1 \qquad \Delta N(\boldsymbol{p}) = f(\boldsymbol{p}) d^3 p_2$$

Relation between the respective volumes of momentum space

How do we relate $d^3 p_1$ and $d^3 p_2$?

Remember that we are considering momentum states that are related by

$$\boldsymbol{p}_2 = \boldsymbol{p}_1 + \hbar\boldsymbol{k}$$

This can be thought of as a mapping between different regions of momentum space with the elementary volumes related by the Jacobean of the transformation which we can write out in full as

$$
\begin{aligned}
p_{x,2} &= p_{x,1} + \hbar k_x \\
p_{y,2} &= p_{y,1} + \hbar k_y \\
p_{z,2} &= p_{z,1} + \hbar k_x
\end{aligned}
$$

The Jacobean of this transformation is just 1. Hence

$$
d^3 p_2 = d^3 p_1
$$

Therefore, we can put for the populations in the different states

$$N_1 = \Delta N(\boldsymbol{p} - \hbar\boldsymbol{k}) = f(\boldsymbol{p} - \hbar\boldsymbol{k})d^3p$$

$$N_2 = \Delta N(\boldsymbol{p}) = f(\boldsymbol{p})d^3p$$

where

$$d^3p = d^3p_1 = d^3p_2$$

The contribution to the absorption coefficient from particles in this region of momentum space is therefore:

$$d\alpha_\nu^{(i)} = [f(\boldsymbol{p} - \hbar\boldsymbol{k}) - f(\boldsymbol{p})]\frac{c^2}{\nu^2}A_{21}^{(i)}d^3\boldsymbol{p}$$

We assume that the momentum of the emitted photon is much less than that of the emitting particle.

When $\hbar k = h\nu / c \ll p$, we can expand the distribution function to first order:

$$f(\boldsymbol{p} - \hbar\boldsymbol{k}) = f(\boldsymbol{p}) - \hbar\boldsymbol{k} \cdot \nabla f(\boldsymbol{p})$$

For an isotropic distribution of electrons

$$f(\boldsymbol{p}) = f(p)$$

$$\Rightarrow \nabla f(\boldsymbol{p}) = \frac{\partial}{\partial p} f(p)\hat{\boldsymbol{p}} = \frac{df(p)}{dp}\hat{\boldsymbol{p}}$$

$$\Rightarrow \hbar\boldsymbol{k} \cdot \nabla f(\boldsymbol{p}) = (\hbar\boldsymbol{k} \cdot \hat{\boldsymbol{p}})\frac{df(p)}{dp}$$

Specific case of synchrotron emission

We now utilise one of the features of synchrotron emission, namely that the photon is emitted in the direction of the particle to within an angle of γ^{-1} radians, i.e. $\hbar \boldsymbol{k} \propto \boldsymbol{p}$. Hence,

$$\hbar \boldsymbol{k} \cdot \hat{\boldsymbol{p}} \approx \hbar k = \frac{h\nu}{c}$$

The difference in the phase-space distribution functions at the two different momenta is:

$$f(\boldsymbol{p} - \hbar \boldsymbol{k}) - f(\boldsymbol{p}) \approx -\frac{h\nu}{c}\frac{df(p)}{dp}$$

Since,

$$N(E) = \frac{4\pi}{c^3}E^2 f(p) \qquad \text{and} \qquad \frac{d}{dp} = c\frac{d}{dE}$$

then

$$f(\boldsymbol{p} - \hbar\boldsymbol{k}) - f(\boldsymbol{p}) = -\frac{h\nu c^3}{4\pi}\frac{d}{dE}\left[\frac{N(E)}{E^2}\right]$$

Contribution to the absorption coefficient

The contribution to the absorption coefficient from this volume of momentum space is therefore:

$$d\alpha_\nu^{(i)} = -\frac{h\nu c^3}{4\pi}\frac{d}{dE}\left[\frac{N(E)}{E^2}\right]\times\frac{c^2}{\nu^2}A_{21}^{(i)}d^3p$$

The total absorption coefficient is therefore:

$$\alpha_\nu^{(i)} = -\int\frac{h\nu c^3}{4\pi}\frac{d}{dE}\left[\frac{N(E)}{E^2}\right]\frac{c^2}{\nu^2}A_{21}^{(i)}p^2dp\,d\Omega_p$$

where the integral is over momentum and solid angle Ω_p in momentum space. In order to calculate the absorption coefficient, all we have to do now is evaluate $A_{21}^{(i)}$.

Emissivity and A_{21}

Recall the definition of the Einstein coefficient:

$$A_{21}^{(i)} d\nu d\Omega = \begin{array}{c}\text{Probability per unit time for spontaneous emission} \\ \text{of a photon in mode } i \text{ in the ranges } d\nu \text{ and } d\Omega\end{array}$$

Therefore, the coefficient $A_{21}^{(i)}$ is related to the single electron emissivity $\dfrac{dP_\nu^{(i)}}{d\Omega}$ through

$$\nu A_{21}^{(i)} d\nu d\Omega = \frac{dP_\nu^{(i)}}{d\Omega} d\nu d\Omega$$

The quantity

$$\frac{dP_\nu^{(i)}}{d\Omega} = \begin{array}{c}\text{Power radiated per unit time}\\ \text{per unit frequency per unit solid angle}\end{array}$$

In ***Synchrotron Radiation I***, we calculated the single electron power emitted per unit circular frequency

$$P^{(i)}(\omega) = \frac{\sqrt{3}}{16\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}[F(x)\pm G(x)]$$

with the + sign for the perpendicular component and the - sign for the parallel component.

Now recall the above expression for the absorption coefficient

$$
\begin{aligned}
\alpha_\nu^{(i)} &= -\int \frac{h\nu c^3}{4\pi} \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] \frac{c^2}{\nu^2} A_{21}^{(i)} p^2 dp d\Omega_p \\
&= -\frac{c^3}{4\pi\nu^2} \int \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] h\nu A_{21}^{(i)} p^2 dp d\Omega_p \\
&= -\frac{c^3}{4\pi\nu^2} \int \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] \frac{dP_\nu^{(i)}}{d\Omega} p^2 dp d\Omega_p
\end{aligned}
$$

In this integral we take $p^2 dp \rightarrow \frac{1}{c^3} E^2 dE$ and

$$\alpha_\nu^{(i)} = -\frac{1}{4\pi\nu^2} \int_E \frac{d}{dE}\left[\frac{N(E)}{E^2}\right] E^2 dE \int_{\Omega_p} \frac{dP_\nu^{(i)}}{d\Omega} d\Omega_p$$

Integrating over solid angle:

$$\int \frac{dP_\nu^{(i)}}{d\Omega} d\Omega_p = P_\nu^{(i)} = 2\pi P^{(i)}(\omega)$$

Since we use $N(\gamma)$ rather than $N(E)$ we also change the integral over E into one over γ, remembering that

$$N(E) = \frac{1}{m_e c^2} N(\gamma)$$

This gives

$$\alpha_\nu^{(i)} = -\frac{\sqrt{3}}{32\pi^2}\left(\frac{1}{\nu^2}\right)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)$$

$$\times \int_{\gamma_1}^{\gamma_2} \gamma^2 \frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right][F(x) \pm G(x)]\, d\gamma$$

We shall use the following below:

$$\frac{\alpha_\nu^\perp + \alpha_\nu^\parallel}{2} = -\frac{\sqrt{3}}{32\pi^2}\left(\frac{1}{\nu^2}\right)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)\int_{\gamma_1}^{\gamma_2}\gamma^2\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right]F(x)d\gamma$$

$$\frac{\alpha_\nu^\perp - \alpha_\nu^\parallel}{2} = -\frac{\sqrt{3}}{32\pi^2}\left(\frac{1}{\nu^2}\right)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)\int_{\gamma_1}^{\gamma_2}\gamma^2\frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right]G(x)d\gamma$$

The first quantity is often referred to as the mean absorption coefficient. The ratio

$$\frac{\alpha_\nu^\perp - \alpha_\nu^\parallel}{\alpha_\nu^\perp + \alpha_\nu^\parallel} = \frac{\int_{\gamma_1}^{\gamma_2} \gamma^2 \frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right] G(x)\, d\gamma}{\int_{\gamma_1}^{\gamma_2} \gamma^2 \frac{d}{d\gamma}\left[\frac{N(\gamma)}{\gamma^2}\right] F(x)\, d\gamma}$$

7.3 Absorption coefficients for a infinite power-law distribution

For a power-law distribution:

$$N(\gamma) = K\gamma^{-a} \Rightarrow \gamma^2 \frac{d}{d\gamma}\left(\frac{N(\gamma)}{\gamma^2}\right) = -(a+2)K\gamma^{-(a+1)}$$

and

$$\alpha_\nu^{(i)} = \frac{\sqrt{3}}{32\pi^2}(a+2)\left(\frac{1}{\nu^2}\right)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)K$$

$$\times \int_{\gamma_1}^{\gamma_2} \gamma^{-(a+1)}[F(x) \pm G(x)]d\gamma$$

As with computing the emission coefficient, we change the variable of integration to x, using:

$$\gamma = \left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\theta}\right)^{1/2} x^{-1/2} = \left(\frac{4\pi}{3}\frac{\nu}{\Omega_0 \sin\theta}\right)^{1/2} x^{-1/2}$$

$$d\gamma = -\frac{1}{2}\left(\frac{4\pi}{3}\frac{\nu}{\Omega_0 \sin\theta}\right)^{1/2} x^{-3/2} dx$$

so that

$$\alpha_\nu^{(i)} = \frac{\sqrt{3}}{64\pi^2}\left(\frac{3}{4\pi}\right)^{a/2}(a+2)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)^{\frac{(a+2)}{2}} K\nu^{-\frac{(a+4)}{2}}$$

$$\times \int_{x_2}^{x_1} x^{\frac{(a-2)}{2}}[F(x) \pm G(x)]dx$$

That is,

$$\alpha_\nu^{\perp} = \frac{\sqrt{3}}{64\pi^2}\left(\frac{3}{4\pi}\right)^{a/2}(a+2)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)^{\frac{(a+2)}{2}} K\nu^{-\frac{(a+4)}{2}}$$

$$\times \int_{x_2}^{x_1} x^{\frac{(a-2)}{2}} [F(x) + G(x)]dx$$

$$\alpha_\nu^{\parallel} = \frac{\sqrt{3}}{64\pi^2}\left(\frac{3}{4\pi}\right)^{a/2}(a+2)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)^{\frac{(a+2)}{2}} K\nu^{-\frac{(a+4)}{2}}$$

$$\times \int_{x_2}^{x_1} x^{\frac{(a-2)}{2}} [F(x) - G(x)]dx$$

Consequently,

$$\frac{\alpha_\nu^\perp + \alpha_\nu^\parallel}{2} = \frac{\sqrt{3}}{64\pi^2}\left(\frac{3}{4\pi}\right)^{a/2}(a+2)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(\Omega_0 \sin\theta)^{\frac{(a+2)}{2}}$$

$$\times K\nu^{-\frac{(a+4)}{2}}\int_{x_2}^{x_1} x^{\frac{(a-2)}{2}} F(x)\,dx$$

and

$$\frac{\alpha_\nu^\perp - \alpha_\nu^\parallel}{\alpha_\nu^\perp + \alpha_\nu^\parallel} = \frac{\int_{x_2}^{x_1} x^{\frac{(a-2)}{2}} G(x)\,dx}{\int_{x_2}^{x_1} x^{\frac{(a-2)}{2}} F(x)\,dx}$$

As before, when the frequency ν is well inside the limits determined by the upper and lower cutoff Lorentz factors, the limits of the integral may be taken to be zero and infinity, respectively. Using the expressions for the integrals of a power times $F(x)$ and $G(x)$, we have,

$$\int_0^\infty x^{\frac{(a-2)}{2}} F(x)dx = \frac{2^{\frac{a+2}{2}}}{a+2}\Gamma\left(\frac{a}{4}+\frac{11}{6}\right)\Gamma\left(\frac{a}{4}+\frac{1}{6}\right)$$

$$\int_0^\infty x^{\frac{(a-2)}{2}} G(x)dx = 2^{\frac{(a-2)}{2}}\Gamma\left(\frac{a}{4}+\frac{5}{6}\right)\Gamma\left(\frac{a}{4}+\frac{1}{6}\right)$$

Therefore,

$$\frac{\alpha_\nu^\perp - \alpha_\nu^\parallel}{\alpha_\nu^\perp + \alpha_\nu^\parallel} = \frac{2^{\frac{(a-2)}{2}}\Gamma\left(\frac{a}{4}+\frac{5}{6}\right)\Gamma\left(\frac{a}{4}+\frac{1}{6}\right)}{\frac{2^{\frac{a+2}{2}}}{a+2}\Gamma\left(\frac{a}{4}+\frac{11}{6}\right)\Gamma\left(\frac{a}{4}+\frac{1}{6}\right)} = \frac{a+2}{a+\frac{10}{3}}$$

This gives for the ratio of absorption coefficients:

$$\frac{\alpha_\nu^\parallel}{\alpha_\nu^\perp} = \frac{2}{3a+8}$$

The mean absorption coefficient is:

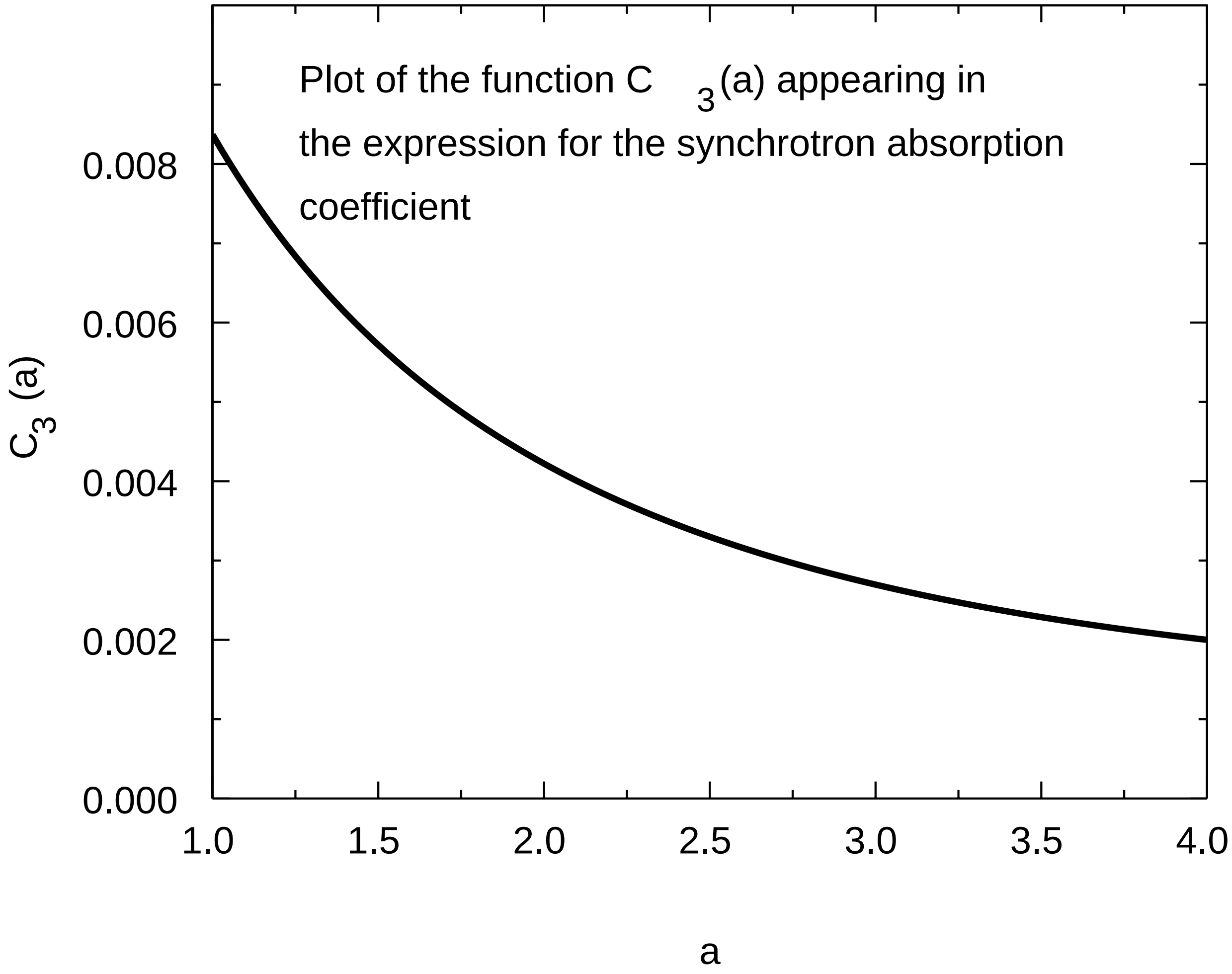
$$\frac{\alpha_\nu^{\perp} + \alpha_\nu^{\parallel}}{2} = C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right) K(\Omega_0 \sin\theta)^{\frac{a+2}{2}} \nu^{-\frac{(a+4)}{2}}$$

where

$$C_3(a) = 3^{\frac{(a+1)}{2}} 2^{-\frac{(a+10)}{2}} \pi^{-\left(\frac{a+4}{2}\right)} \Gamma\left(\frac{a}{4} + \frac{11}{6}\right) \Gamma\left(\frac{a}{4} + \frac{1}{6}\right)$$

As with the expression for the emission coefficient, this expression is separated into a numerical coefficient which depends upon a, a factor involving physical constants, a part involving the non-relativistic gyrofrequency, a factor involving the parameter for the electron density and a factor involving a power of the frequency. This is often the absorption coefficient which is quoted in text books. However, this coefficient alone does not convey the whole story since synchrotron absorption unlike many other absorption processes is polarised.

7.4 Numerical value of $C_3(a)$



At the left is a plot of the function $C_3(a)$ appearing in the above expression for the synchrotron absorption coefficient.

8 The transfer of polarised synchrotron radiation in an optically thick region

The transfer equations for the two modes of polarisation are:

$$\frac{dI_\nu^\perp}{ds} = j_\nu^\perp - \alpha_\nu^\perp I_\nu^\perp$$

$$\frac{dI_\nu^\parallel}{ds} = j_\nu^\parallel - \alpha_\nu^\parallel I_\nu^\parallel$$

Recall that the perpendicular and parallel components of intensity are related to the Stokes parameters by:

$$I_\nu^\perp = \frac{1}{2}(I_\nu + Q_\nu)$$

$$I_\nu^\parallel = \frac{1}{2}(I_\nu - Q_\nu)$$

The reverse transformation is:

$$I_\nu = I_\nu^\perp + I_\nu^\parallel$$

$$Q_\nu = I_\nu^\perp - I_\nu^\parallel$$

We write the transfer equations in terms of source functions.

$$\frac{dI_\nu^\perp}{ds} = j_\nu^\perp - \alpha_\nu^\perp I_\nu^\perp \Rightarrow \frac{dI_\nu^\perp}{d\tau_\nu^\perp} = \frac{j_\nu^\perp}{\alpha_\nu^\perp} - I_\nu^\perp = S_\nu^\perp - I_\nu^\perp$$

$$\frac{dI_\nu^{||}}{ds} = j_\nu^{||} - \alpha_\nu^{||} I_\nu^{||} \Rightarrow \frac{dI_\nu^{||}}{d\tau_\nu^{||}} = \frac{j_\nu^{||}}{\alpha_\nu^{||}} - I_\nu^{||} = S_\nu^{||} - I_\nu^{||}$$

In a slab geometry with all parameters constant, the solutions are:

$$I_\nu^\perp = S_\nu^\perp (1 - \exp - \tau_\nu^\perp)$$

$$I_\nu^{||} = S_\nu^{||} (1 - \exp - \tau_\nu^{||})$$

For small optical depths $\tau_\nu^\perp, \tau_\nu^\parallel \ll 1$, these relations become the standard ones for optically thin emission. In the opposite limit of infinite optical depth,

$$I_\nu^\perp = S_\nu^\perp = \frac{j_\nu^\perp}{\alpha_\nu^\perp}$$

$$I_\nu^\parallel = S_\nu^\parallel = \frac{j_\nu^\parallel}{\alpha_\nu^\parallel}$$

The ratio

$$\frac{Q}{I} = \frac{I_\nu^{\perp} - I_\nu^{\parallel}}{I_\nu^{\perp} + I_\nu^{\parallel}} = \frac{\dfrac{j_\nu^{\perp}}{\alpha_\nu^{\perp}} - \dfrac{j_\nu^{\parallel}}{\alpha_\nu^{\parallel}}}{\dfrac{j_\nu^{\perp}}{\alpha_\nu^{\perp}} + \dfrac{j_\nu^{\parallel}}{\alpha_\nu^{\parallel}}} = \frac{\dfrac{\alpha_\nu^{\parallel}}{\alpha_\nu^{\perp}} - \dfrac{j_\nu^{\parallel}}{j_\nu^{\perp}}}{\dfrac{\alpha_\nu^{\parallel}}{\alpha_\nu^{\perp}} + \dfrac{j_\nu^{\parallel}}{j_\nu^{\parallel}}}$$

We have:

$$\frac{\alpha_\nu^{\parallel}}{\alpha_\nu^{\perp}} = \frac{2}{3a+8} \qquad \frac{j_\nu^{\parallel}}{j_\nu^{\perp}} = \frac{2}{3a+5}$$

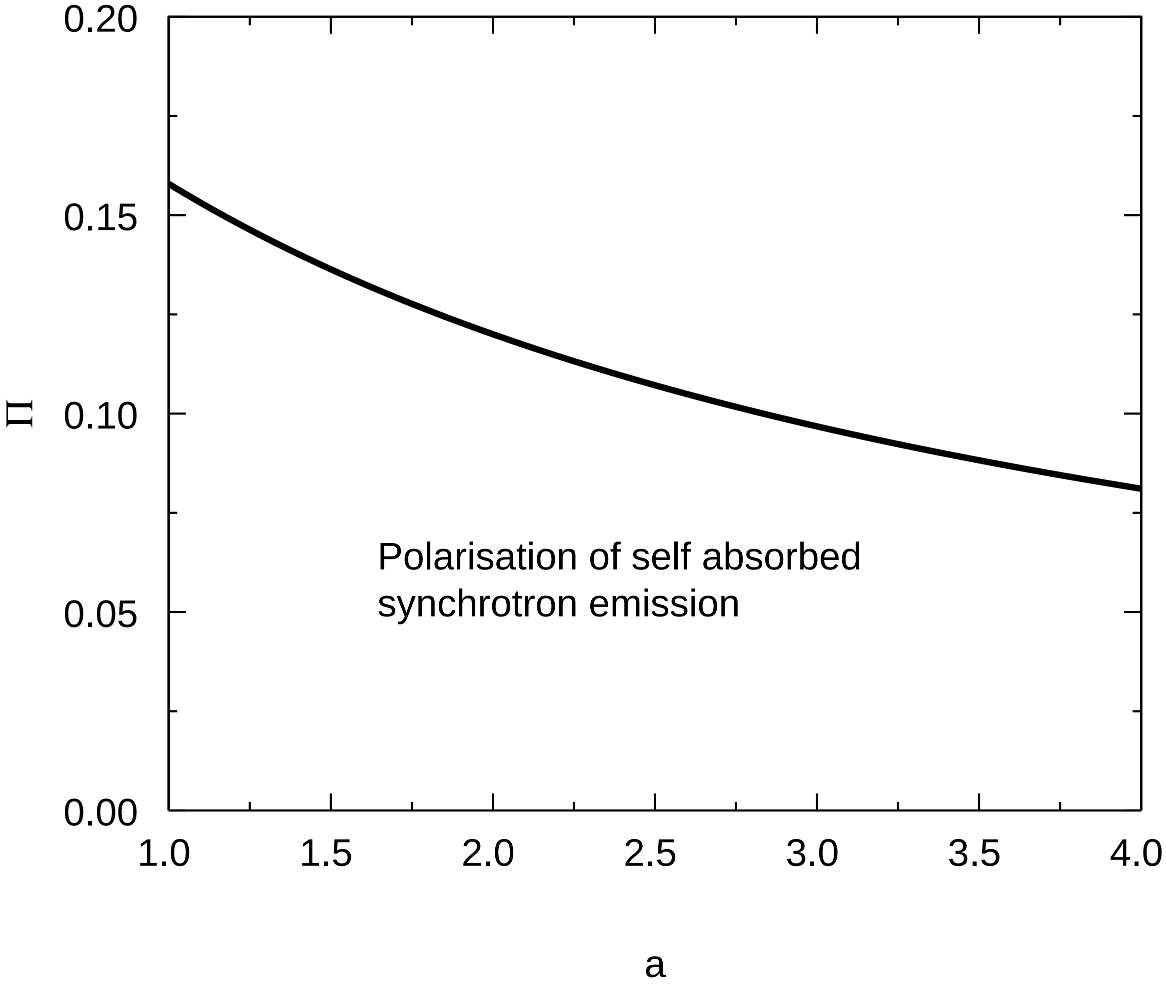
Using the previously determined ratios for these quantities

$$\frac{Q_\nu}{I_\nu} = -\frac{3}{6a + 13}$$

An important result here is that the ratio is negative signifying that the parallel component of the intensity is the larger. This means that the major axis of the polarisation ellipse is parallel to the magnetic field. As one can see from the following plot the fractional polarisation

$$\Pi = \frac{3}{6a + 13}$$

is lower for optically thick emission.



The fractional polarisation of a self absorbed synchrotron source.

9 The spectral slope for optically thick emission

For optically thick emission, we have:

$$I_\nu^{\perp} = S_\nu^{\perp} = \frac{j_\nu^{\perp}}{\alpha_\nu^{\perp}} \qquad I_\nu^{\|} = S_\nu^{\|} = \frac{j_\nu^{\|}}{\alpha_\nu^{\|}}$$

$$\Rightarrow I_\nu = S_\nu^{\perp} + S_\nu^{\|} = \left[\frac{j_\nu^{\perp}}{\alpha_\nu^{\perp}} + \frac{j_\nu^{\|}}{\alpha_\nu^{\|}}\right]$$

Write this in terms of the total emissivity and the mean absorption coefficient in the following way:

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left[\frac{j_\nu^{\perp}/j_\nu}{\alpha_\nu^{\perp}/\alpha_\nu} + \frac{j_\nu^{\|}/j_\nu}{\alpha_\nu^{\|}/\alpha_\nu}\right]$$

where $\alpha_\nu = \dfrac{\alpha_\nu^{\perp} + \alpha_\nu^{\|}}{2}$ is the mean absorption coefficient.

We have already determined the ratios

$$s = \frac{j_\nu^{\|}}{j_\nu^{\perp}} = \frac{2}{3a+5} \qquad r = \frac{\alpha_\nu^{\|}}{\alpha_\nu^{\perp}} = \frac{2}{3a+8}$$

Therefore,

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left[\frac{1/(1+s)}{2/(1+r)} + \frac{s/(1+s)}{2r/(1+r)}\right]$$

$$= \frac{j_\nu}{\alpha_\nu}\frac{(6a+13)(3a+10)}{2(3a+8)(3a+7)}$$

Now use

$$j_\nu = C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right)K(\Omega_0\sin\theta)^{\frac{a+1}{2}}\nu^{-\frac{(a-1)}{2}}$$

$$\alpha_\nu = C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)K(\Omega_0\sin\theta)^{\frac{a+2}{2}}\nu^{-\frac{(a+4)}{2}}$$

The intensity is then

$$I_\nu = \frac{(6a+13)(3a+10)}{2(3a+8)(3a+7)}\frac{C_1(a)}{C_3(a)}m_e(\Omega_0\sin\theta)^{-1/2}\nu^{5/2}$$

$$= C_5(a)m_e(\Omega_0\sin\theta)^{-1/2}\nu^{5/2}$$

where

$$C_5(a) = \frac{(6a+13)(3a+10)}{2(3a+8)(3a+7)} \frac{C_1(a)}{C_3(a)}.$$

The important result here is that $I_\nu \propto \nu^{5/2}$.

The entire spectrum from optically thick to optically thin regimes is as indicated in the following plot for the case of $a = 2.1$

Plots of perpendicular and parallel components of intensity

Write

$$S_\nu^{\|} = \frac{j_\nu^{\|}}{\alpha_\nu^{\|}} = \left[\frac{\overline{\alpha_\nu}}{\alpha_\nu^{\|}}\frac{j_\nu^{\|}}{j_\nu}\right] \times \frac{j_\nu}{\overline{\alpha_\nu}} = \left[\frac{\overline{\alpha_\nu}}{\alpha_\nu^{\|}}\frac{j_\nu^{\|}}{j_\nu}\right]\overline{S_\nu}$$

$$S_\nu^{\perp} = \frac{j_\nu^{\perp}}{\alpha_\nu^{\perp}} = \left[\frac{\overline{\alpha_\nu}}{\alpha_\nu^{\perp}}\frac{j_\nu^{\perp}}{j_\nu}\right] \times \frac{j_\nu}{\overline{\alpha_\nu}} = \left[\frac{\overline{\alpha_\nu}}{\alpha_\nu^{\perp}}\frac{j_\nu^{\perp}}{j_\nu}\right]\overline{S_\nu}$$

The mean source function $\overline{S_\nu}$ is given by:

$$\overline{S_\nu} = \frac{j_\nu}{\overline{\overline{\alpha_\nu}}} = \frac{C_1(a)}{C_3(a)} m_e (\Omega_0 \sin\theta)^{-1/2} \nu^{5/2}$$

From the above

$$S_\nu^\perp = \frac{(3a+5)(3a+10)}{2(3a+8)(3a+7)} \overline{S_\nu} \qquad S_\nu^\parallel = \frac{(3a+10)}{2(3a+7)} \overline{S_\nu}$$

We define a frequency ν_0 at which the mean optical depth is unity by

$$\overline{\alpha_\nu} L = C_3(a) \left(\frac{e^2}{\varepsilon_0 m_e c} \right) (KL) (\Omega_0 \sin\theta)^{\frac{a+1}{2}} \nu_0^{-\frac{(a+4)}{2}}$$

and then put

$$\overline{S_\nu} = A\left(\frac{\nu}{\nu_0}\right)^{5/2}$$

Then the perpendicular component is

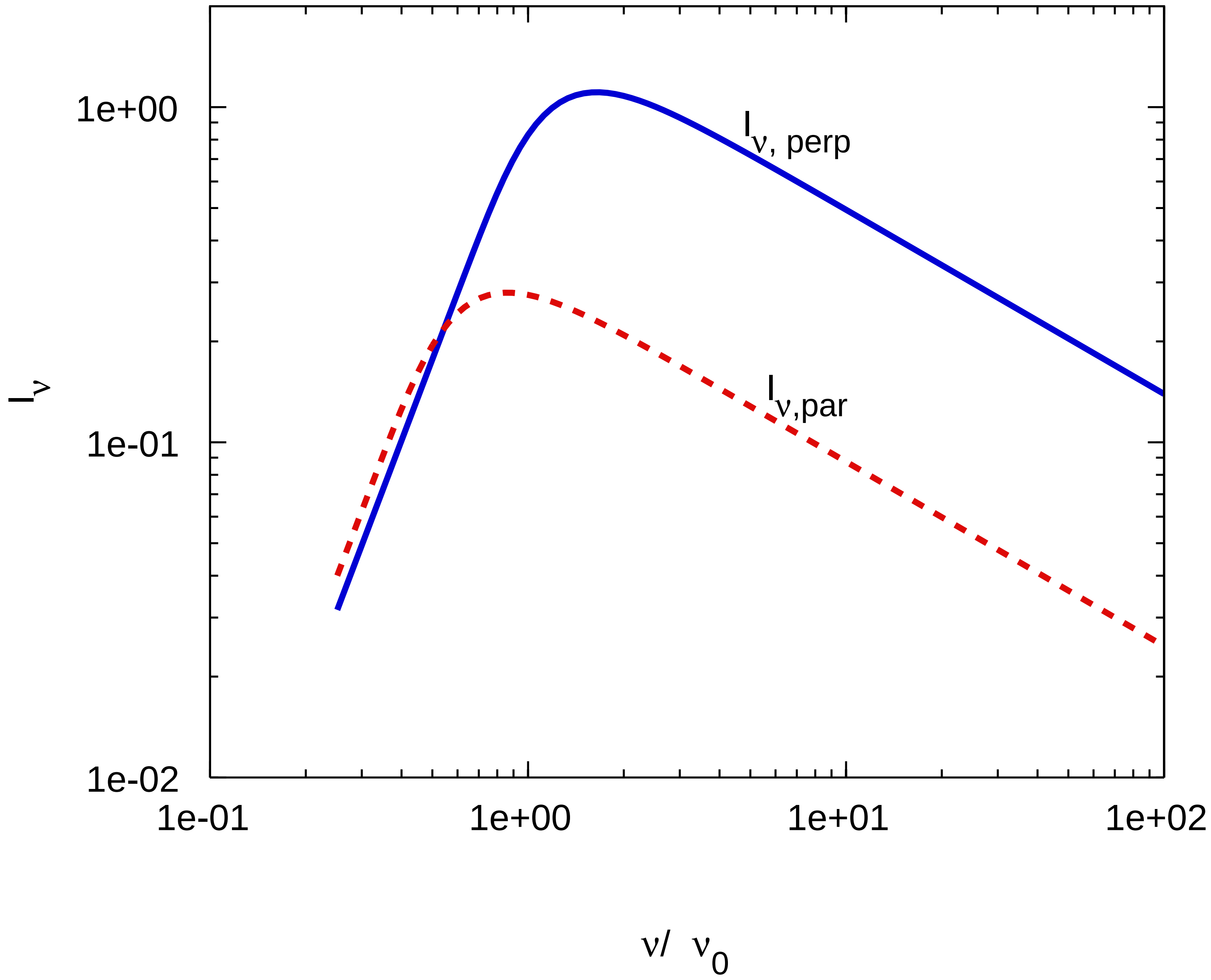
$$I_\nu^{\perp} = \frac{(3a+5)(3a+10)}{2(3a+8)(3a+7)}A\left(\frac{\nu}{\nu_0}\right)^{5/2}$$

$$\times\left[1-\exp-\frac{2(3a+8)}{3a+10}\left(\frac{\nu}{\nu_0}\right)^{-\frac{(a+4)}{2}}\right]$$

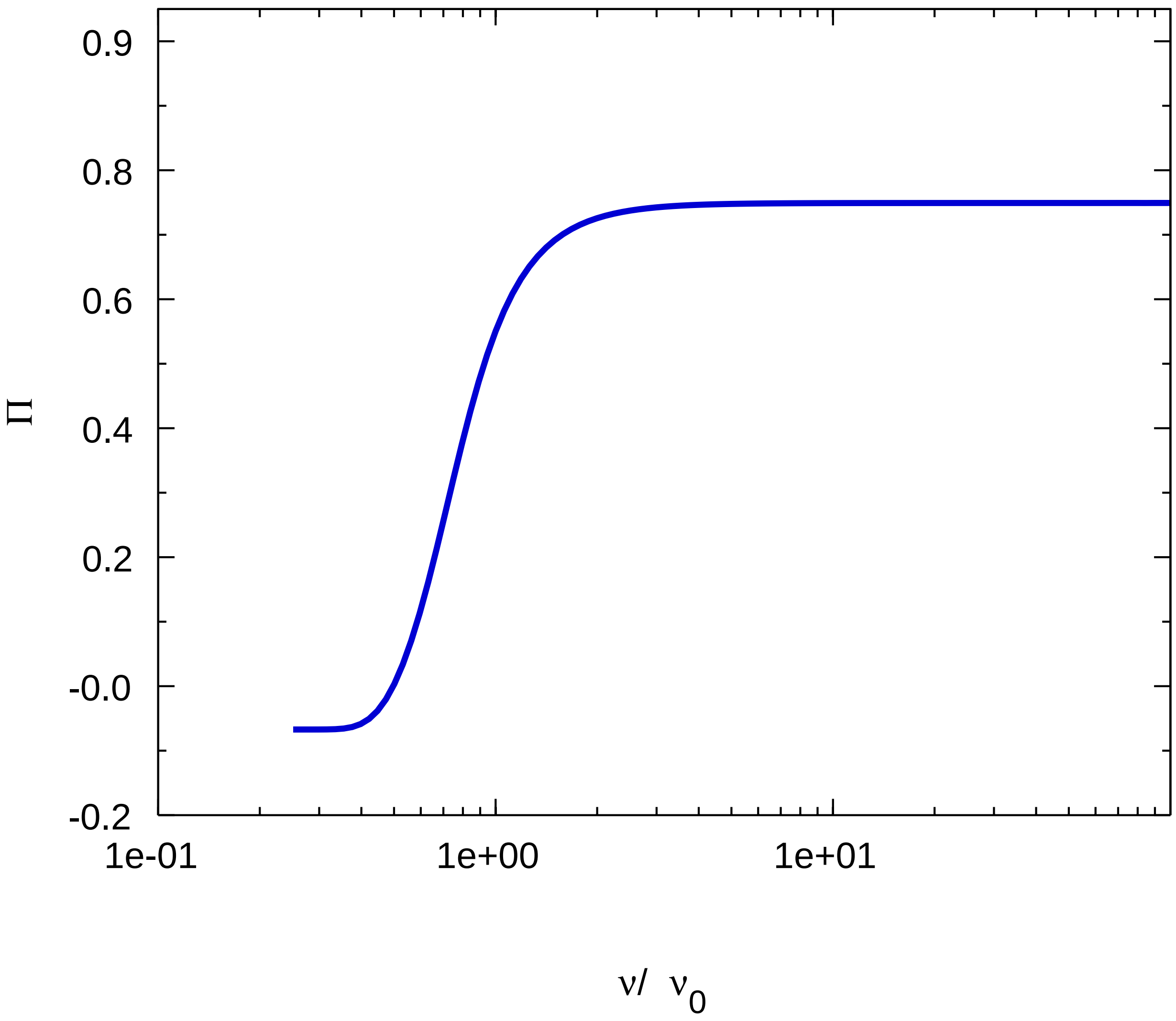
and the parallel component is:

$$I_{\nu}^{\parallel} = \frac{(3a+10)}{2(3a+7)} A \left(\frac{\nu}{\nu_0}\right)^{5/2} \times \left[1 - \exp - \frac{4}{3a+10} \left(\frac{\nu}{\nu_0}\right)^{-\frac{(a+4)}{2}} \right]$$

These equations have used to produce the plots below.


1e+00
1e-01
1e-02
1e-01
1e+00
1e+01
1e+02
I_ν
$I_{\nu,\ perp}$
$I_{\nu,par}$
$\nu/\ \nu_0$

Polarisation



The above curve shows the polarisation defined by

$$\Pi = \frac{Q_\nu}{I_\nu}$$

varying between the limits of $\frac{-3}{6a+13}$ (for the optically thick case) and $\frac{a+1}{a+7/3}$ for the optically thin regime.

Synchrotron self-absorption is just one of the processes that can lead to a low frequency cutoff in the spectrum of a radio source. Others include free-free absorption (due to foreground ionised matter) and a process known as induced Compton scattering. A low energy cut-off in the low frequency spectrum can also lead to a low frequency cutoff.

10 When is synchrotron self absorption important?

Consider the optical depth based upon the mean absorption coefficient:

$$\bar{\tau}_\nu = \int_{\mathrm{slab}} \alpha_\nu ds \approx \alpha_\nu L$$

We also consider optically thin emission and consider the transition to the optically thick regime. Therefore, the surface brightness is

$$I_\nu = \int_{\mathrm{slab}} j_\nu ds \approx j_\nu L$$

We can therefore relate the optical depth to the surface brightness via:

$$\frac{\bar{\tau}_\nu}{I_\nu} \approx \frac{\alpha_\nu L}{j_\nu L} = \frac{\alpha_\nu}{j_\nu}$$

$$\Rightarrow \bar{\tau}_\nu = \frac{\alpha_\nu}{j_\nu} I_\nu = \frac{1}{\bar{S}_\nu} \times I_\nu$$

where $\overline{S_\nu}$ is the "mean" source function.

Using again the relations for emissivity and mean absorption coefficient:

$$j_\nu = C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right)K(\Omega_0 \sin\theta)^{\frac{a+1}{2}}\nu^{-\frac{(a-1)}{2}}$$

$$\alpha_\nu = C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)K(\Omega_0 \sin\theta)^{\frac{a+2}{2}}\nu^{-\frac{(a+4)}{2}}$$

we obtain

$$\frac{\alpha_\nu}{j_\nu} = \frac{C_3(a)}{C_1(a)}(\Omega_0 \sin\theta)^{1/2}\nu^{-5/2}$$

The mean optical depth is then given in terms of the optically thin surface brightness by:

$$\overline{\tau_\nu} = \frac{C_3(a)}{C_1(a)} m_e^{-1} (\Omega_0 \sin\theta)^{1/2} \times (I_\nu \nu^{-5/2})$$

Note that this expression depends upon the magnetic field only and not on the parameter K. Now in the optically thin regime, we can write:

$$I_\nu = I_{\nu_0}\left(\frac{\nu}{\nu_0}\right)^{-\alpha}$$

$$\Rightarrow \overline{\tau_\nu} = \frac{C_3(a)}{C_1(a)} m_e^{-1}(\Omega_0 \sin\theta)^{1/2} \times \left[I_{\nu_0}\nu_0^{-5/2}\left(\frac{\nu}{\nu_0}\right)^{-(\alpha+5/2)}\right]$$

$$= \frac{C_3(a)}{C_1(a)}\left[\frac{e^{1/2}}{m_e^{3/2}}\right][B\sin\theta]^{1/2}\left[I_{\nu_0}\nu_0^{-5/2}\left(\frac{\nu}{\nu_0}\right)^{-(\alpha+5/2)}\right]$$

Now let us put in some fiducial values:

$$B = 1\mathrm{G} = 10^{-4}\ \mathrm{T}$$

$$I_{\nu_0} = \frac{1\ \mathrm{Jy}}{\mathrm{sq.\ arcsec}} = 4.3\times 10^{-16}\ \mathrm{W\ m^{-2}\ Hz^{-1}\ Sr^{-1}}$$

$$\nu_0 = 1\ \mathrm{GHz}$$

The result is:

$$\overline{\tau_\nu} = 6.26\times 10^{-5}\frac{C_3(a)}{C_1(a)}\left[\frac{B\sin\theta}{\mathrm{G}}\right]^{1/2}\left[\frac{I_{\nu_0}}{\mathrm{Jy\ arcsec^{-2}}}\right]\left(\frac{\nu}{\mathrm{GHz}}\right)^{-(\alpha+5/2)}$$

For typical kpc scale regions of radio galaxies, supernova remnants etc., we can see that there is no chance of plasma becoming optically thin, except at very low frequencies. Typical values would be

$$B \sim 10^{-5}\ \mathrm{G} \qquad I_{\nu_0} \sim 10\ \mathrm{mJy\ arcsec^{-2}}$$

However, in the cores of radio galaxies and quasars, we can have

$$I_{\nu_0} \sim 1\ \mathrm{Jy\ mas^{-2}} \qquad B \sim 10^{-2}\ \mathrm{G}$$

In which case the optical depth at 1 GHz would be

$$\overline{\tau_\nu} = 6.3\frac{C_3(a)}{C_1(a)}\left[\frac{B\sin\theta}{0.01\mathrm{G}}\right]^{1/2}\left[\frac{I_{1\mathrm{GHz}}}{\mathrm{Jy\ mas^{-2}}}\right]\left(\frac{\nu}{\mathrm{Ghz}}\right)^{-(\alpha+5/2)}$$

For a spectral index $\alpha = 0.6$, $a = 2.2$, $C_3/C_1 \approx 29.8$ and

$$\overline{\tau_\nu} = 186\left[\frac{B\sin\theta}{0.01\,\mathrm{G}}\right]^{1/2}\left[\frac{I_{1\mathrm{GHz}}}{\mathrm{Jy\ mas}^{-2}}\right]\left(\frac{\nu}{\mathrm{Ghz}}\right)^{-(\alpha+5/2)}$$

and the plasma can clearly be optically thick at frequencies much higher than a GHz. This is a characteristic feature of quasar spectra.

11 Brightness temperature

11.1 Definition

A concept which commonly enters in to radio astronomy and which is tied in with the notion of self absorption is brightness temperature. Consider the blackbody spectrum:

$$I_\nu = \frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1}$$

In the classical limit $h\nu \ll kT$, this becomes the Rayleigh-Jeans law:

$$I_\nu = \frac{2kT}{c^2}\nu^2$$

This is the emission at low frequencies (long wavelengths) from a blackbody (optically thick thermal emitter). This spectrum is the result of the balance of emission and absorption in the emitting region.

Astronomers use the Rayleigh-Jeans law to ascribe a brightness temperature to a source:

$$T_b = \frac{c^2}{2k}\nu^{-2}I_\nu$$

For a nonthermal source, the brightness temperature is a strong function of frequency:

$$T_b = \frac{c^2}{2k}I_{\nu_0}\nu_0^{-2}\left(\frac{\nu}{\nu_0}\right)^{-(\alpha+2)}$$

and increases rapidly at low frequency. Eventually, at low enough frequency, the brightness temperature exceeds the kinetic temperature of the emitting electrons.

The characteristic of a nonthermal source is that the particles have not come into an equilibrium distribution (i.e. a relativistic Maxwellian of a single temperature) because the collision times are too long to achieve this. However, we can ascribe a temperature to particles of a given energy. Moreover, in a self-absorbed source, the photons which are absorbed are similar in energy to the photons which are emitted since the absorption process is the reverse of the emission process. Therefore, at each energy, we expect something similar to a blackbody equilibrium. A characteristic of such an equilibrium is that the brightness temperature of the radiation cannot exceed the equivalent temperature of the electrons. This gives a simple explanation of the frequency dependence of the surface brightness of a self-absorbed source.

11.2 Equivalent temperature

For a gas in which the ratio of specific heats is χ and the number density of particles is the energy density is given by

$$u = \frac{NkT}{\chi - 1} = \frac{nKT}{4/3 - 1} = 3NkT$$

for a relativistic gas in which $\chi = 4/3$. Hence, for electrons of Lorentz factor γ we have an equivalent temperature defined by:

$$N\gamma m_e c^2 = 3NkT$$

$$\Rightarrow T = \frac{1}{3}\gamma\frac{m_e c^2}{k}$$

We know from the theory of synchrotron emission that the circular frequency of emission is given by:

$$\omega \sim \frac{3\Omega_0}{2}\gamma^2 \Rightarrow \nu = \left(\frac{3\Omega_0}{4\pi}\right)\gamma^2$$

$$\Rightarrow \gamma \sim \left(\frac{4\pi\nu}{3\Omega_0}\right)^{1/2}$$

Hence,

$$T \sim \frac{1}{3}\frac{m_e c^2}{k}\left(\frac{4\pi\nu}{3\Omega_0}\right)^{1/2}$$

11.3 Condition for optically thick radiation - a thermodynamic argument

As stated above, the brightness temperature of the radiation cannot exceed the kinetic temperature of the particles producing it. Hence, a synchrotron source becomes optically thick when

$$T \sim T_b$$

$$\Rightarrow \frac{1}{3}\frac{m_e c^2}{k}\left(\frac{4\pi\nu}{3\Omega_0}\right)^{1/2} \sim \frac{c^2}{2k}\nu^{-2} I_\nu$$

$$\Rightarrow \nu^{-5/2} I_\nu \sim \frac{2}{3}\left(\frac{4\pi}{3}\right)^{1/2} m_e \Omega_0^{-1/2}$$

This defines a critical frequency at which the source becomes self-absorbed. For lower frequencies the processes of emission and absorption must be balanced to guarantee the equilibration between brightness temperature and kinetic temperature, so that

$$I_\nu \sim \frac{2}{3}(2\pi)^{1/2} m_e \Omega_0^{-1/2} \nu^{5/2}$$

This is close to the relation for optically thick sources that we derived above from the theory of synchrotron absorption. This derivation also gives a physical explanation for the $\nu^{5/2}$ dependence of a synchrotron self-absorbed source as opposed to the ν^2 dependence for a thermal source. The extra factor of $\nu^{1/2}$ arises from the frequency dependence of the kinetic temperature.

12 Preliminary estimate of the magnetic field and particle energy density in a quasar

We can use the above theory to estimate both the magnetic energy density and magnetic field in a self-absorbed source. This excursion into the estimation of observational parameters is only preliminary since relativistic effects mislead us in our estimation of the rest frame flux density. However, it is instructive to go through the exercise to see what deductions we can make.

When a source is self-absorbed, we have two constraints on the number density and magnetic field, derived from the surface brightness and the optical depth.

The emissivity and absorption coefficient are given by:

$$j_\nu = C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right)K(\Omega_0 \sin\theta)^{\frac{a+1}{2}}\nu^{-\frac{(a-1)}{2}}$$

$$\alpha_\nu = C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)K(\Omega_0 \sin\theta)^{\frac{a+2}{2}}\nu^{-\frac{(a+4)}{2}}$$

Let the path length through the source be L, so that at the peak of the spectrum

$$I_\nu \approx j_\nu L = C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right)(KL)(\Omega_0 \sin\theta)^{\frac{a+1}{2}}\nu^{-\frac{(a-1)}{2}}$$

$$1 \approx \alpha_\nu L = C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(KL)(\Omega_0 \sin\theta)^{\frac{a+2}{2}}\nu^{-\frac{(a+4)}{2}}$$

These are our two constraints on the parameters KL and $\Omega_0 \sin\theta$.

We usually write the surface brightness as

$$I_\nu = \frac{F_\nu}{\psi^2}$$

where F_ν is the flux density from a region of angular size ψ. We also use the relationship $a = 2\alpha + 1$ in the above so that our equations read:

$$C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right)(KL)(\Omega_0 \sin\theta)^{1+\alpha}\nu^{-\alpha} = I_\nu$$

$$C_3(a)\left(\frac{e^2}{\varepsilon_0 m_e c}\right)(KL)(\Omega_0 \sin\theta)^{\alpha+3/2}\nu^{-(\alpha+5/2)} = 1$$

These equations can be solved (using Maple or brute force) to give the solutions for Ω_0 and KL:

$$\Omega \sin\theta = \left(\frac{C_1}{C_3}\right)^2 m_e^2 I_\nu^{-2} \nu^5 = \left(\frac{C_1}{C_3}\right)^2 m_e^2 F_\nu^{-2} \psi^4 \nu^5$$

$$KL = \left(\frac{\varepsilon_0 c}{e^2 m_e^2}\right)\left(\frac{C_3^2}{C_1^3}\right) I_\nu^3 \nu^{-5} \left[m_e^{-1} \frac{C_3}{C_1} I_\nu \nu^{-2}\right]^{2\alpha}$$

$$= \left(\frac{\varepsilon_0 c}{e^2 m_e^2}\right)\left(\frac{C_3^2}{C_1^3}\right) F_\nu^3 \psi^6 \nu^{-5} \left[m_e^{-1} \frac{C_3}{C_1} F_\nu \psi^{-2} \nu^{-2}\right]^{2\alpha}$$

Numerically,

$$\Omega \sin\theta = 4.6\times10^{3}\left(\frac{C_1}{C_3}\right)^2\left(\frac{F_\nu}{\mathrm{Jy}}\right)^{-2}\left(\frac{\psi}{\mathrm{mas}}\right)^4\nu_9^5$$

$$\Rightarrow B\sin\theta = 1.2\times10^{-4}\left(\frac{C_1}{C_3}\right)^2\left(\frac{F_\nu}{\mathrm{Jy}}\right)^{-2}\left(\frac{\psi}{\mathrm{mas}}\right)^4\nu_9^5$$

$$KL = 9.62\times10^{21}\left(\frac{C_3^2}{C_1^3}\right)\left(\frac{F_\nu}{\mathrm{Jy}}\right)^3\left(\frac{\theta}{\mathrm{mas}}\right)^{-6}\nu_9^{-5}(2\alpha)$$

$$\times\left[467\left(\frac{F_\nu}{\mathrm{Jy}}\right)\left(\frac{\psi}{\mathrm{mas}}\right)^{-2}\nu_9^{-2}\right]$$

Units:

$$KL = \mathrm{m}^{-3} \times \mathrm{m} = \mathrm{m}^{-2}$$

$$B = \text{Tesla (T)}$$

Let us estimate the parameters for a typical quasar with

$F_\nu = 1\ \mathrm{Jy}$ $\qquad \psi = 1\ \mathrm{mas}$ $\qquad \nu_{\mathrm{peak}} = 1\ \mathrm{GHz}$ $\qquad \alpha = 0.6$

For $a = 2.2$, $C_1 = 4.44\times10^{-3}$ and $C_3 = 0.132$, so that

$$B\sin\theta = 1.4\times10^{-7}\ \mathrm{T} \qquad KL = 3.1\times10^{30}\,\mathrm{m}^{-2}$$

and for a typical path length through the source, $L = 1\ \mathrm{pc} = 3.1\times10^{16}\ \mathrm{m}$, $K = 9.8\times10^{13}\ \mathrm{m}^{-3}$.

The energy density in particles for a lower cutoff Lorentz factor γ_1 is

$$\varepsilon = \frac{Km_e c^2}{a-2}\gamma_1^{-(a-2)} = 40\gamma_1^{-0.2}\ \mathrm{J\ m^{-3}}$$

For $\gamma_1 = 10$ this is approximately $25\ \mathrm{J\ m^{-3}}$ and for $\gamma_1 = 100$ $\varepsilon \approx 16\ \mathrm{J\ m^{-3}}$.

For comparison, the energy density in the magnetic field is:

$$\frac{B^2}{2\mu_0} \sim 7.3\times 10^{-9}\ \mathrm{J\ m^{-3}}$$

approximately nine orders of magnitude lower and it would appear that the plasma is well removed from equipartition.

Note however the sensitive dependence of the magnetic field and particle energy on the flux density:

$$B \propto F_\nu^{-2} \qquad KL \propto F_\nu^{3 + 2\alpha}$$

so that the ratio of particle energy density to magnetic energy density is proportional to $F_\nu^{7 + 2\alpha} \approx F_\nu^{8.2}$. It is this single factor which is most affected by relativistic beaming from the moving plasma.