Compton Scattering II

1 Introduction

In the previous chapter we considered the ***total*** power produced by a single electron from inverse Compton scattering. This is useful but limited information. Here we calculate the resulting spectra.

1.1 Importance

This emission mechanism is frequently encountered in a number of areas, for example:

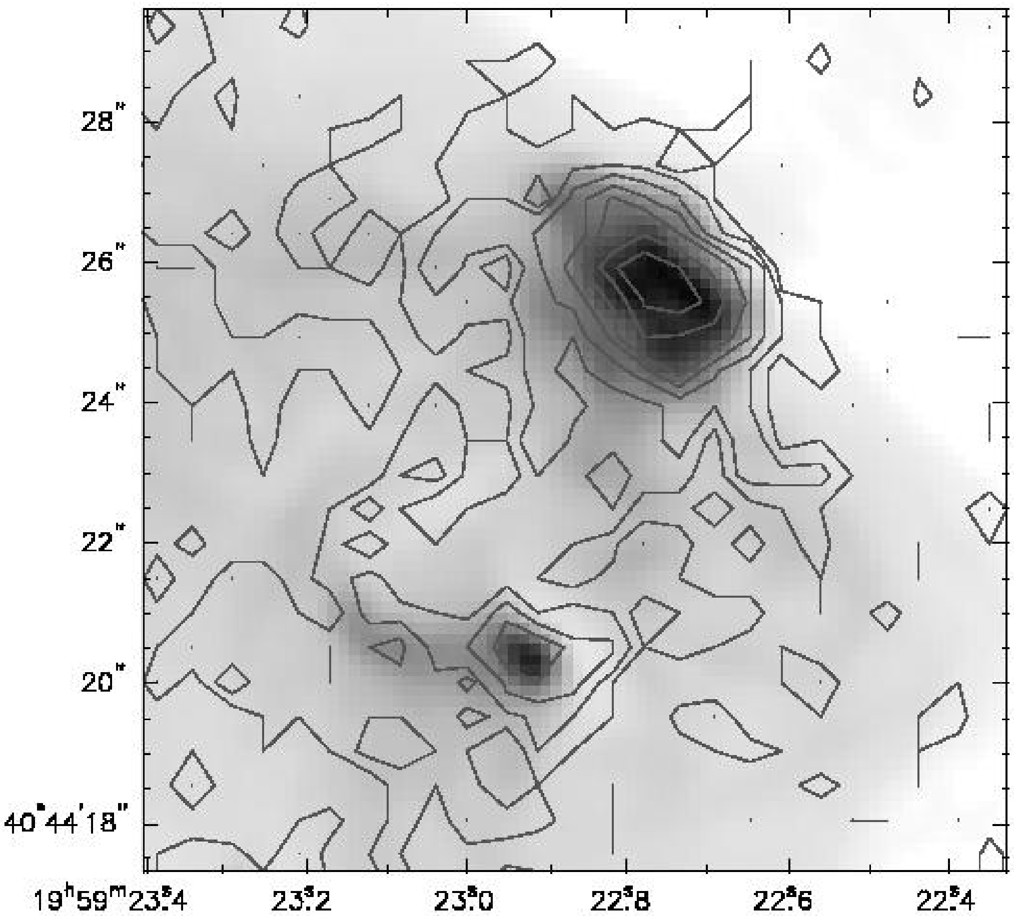
- The X-ray emission from the lobes of radio galaxies can be used, along with the synchrotron emission to estimate the magnetic field and the energy density of particles and magnetic field. The additional information available from X-ray observations unlocks the degeneracy between particle estimates and fields involved in the expression for, say, the surface brightness of synchrotron emission. Such estimates can then be compared with the minimum energy estimates obtained from synchrotron theory.

- Blazars emit X-ray synchrotron and GeV or TeV γ-rays. The simultaneous measurement of these spectra enable us to estimate the properties of blazar jets, even though they are unresolved.

It is well to keep in mind two possible scenarios:

1. The “soft” photons which are scattered originate from an external source, e.g. starlight or the microwave background. This is known as External Inverse Compton (EIC) emission.
2. The soft photons originate from within the source itself and are scattered by the same population of relativistic electrons. This is known as Synchrotron Self Compton (SSC) emission.

1.2 Example



The western hot spots in Cygnus A. The radio image is shown as a grey-scale; the X-ray image is shown as contours overlaid on the radio image. From Wilson et al. (2000) astro-ph 0009308

2 Inverse compton spectrum from a single relativistic electron

2.1 Relative importance of SSC and EIC emission

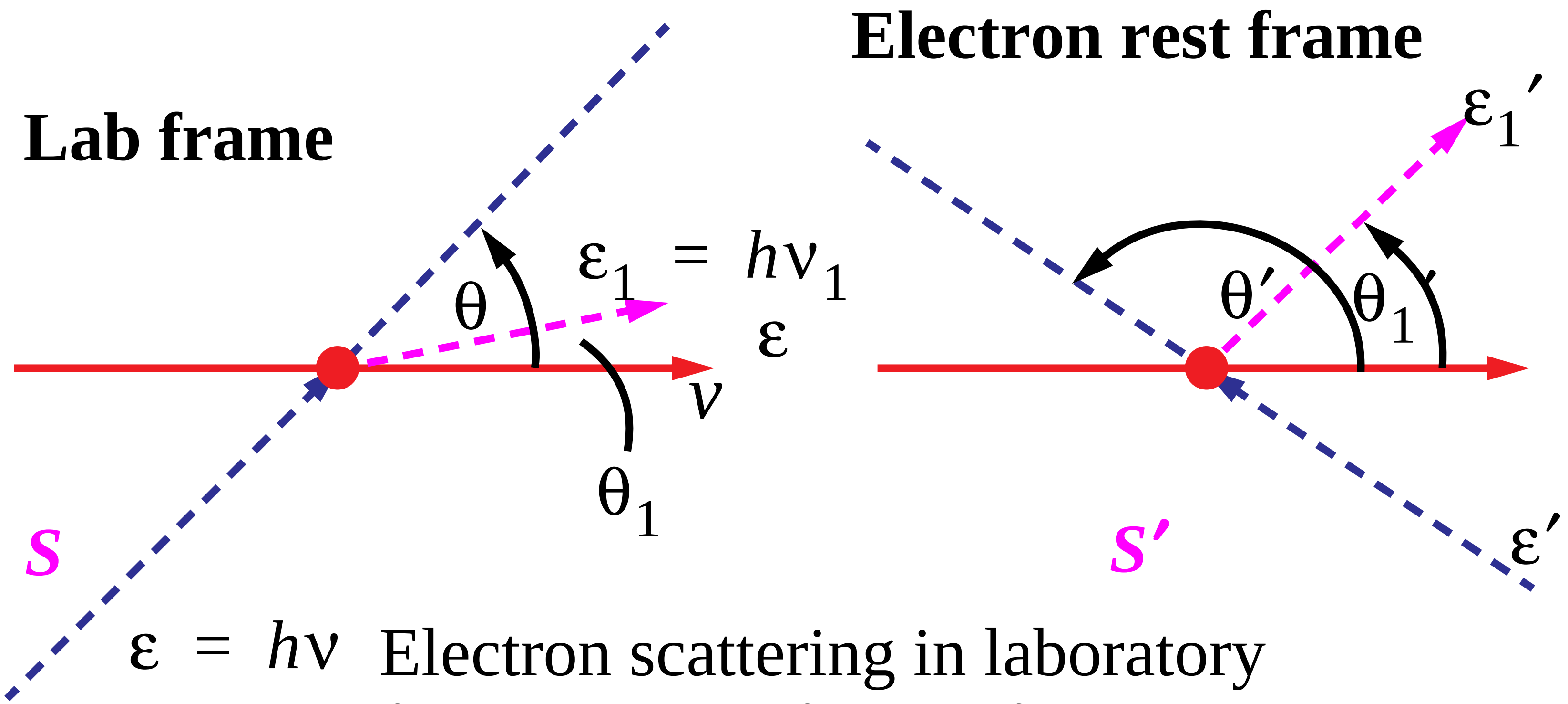
One can readily ascertain the relative importance of SSC and EIC emission from the expression for the Compton power:

$$
\begin{aligned}
P_{\mathrm{Compton}} &= \frac{4}{3}\sigma_T c\gamma^2\beta^2 U_{\mathrm{ph}} \\
&= \frac{4}{3}\sigma_T c\gamma^2 U_{\mathrm{ph}} \quad \text{for relativistic particles}
\end{aligned}
\qquad (1)
$$

Since the power is proportional to the radiation energy density, SSC is going to be more important than EIC when

$$\frac{\text{Radiation energy density}}{\text{of the synchrotron emission}} > \frac{\text{Radiation energy density}}{\text{of background emission}} \qquad (2)$$

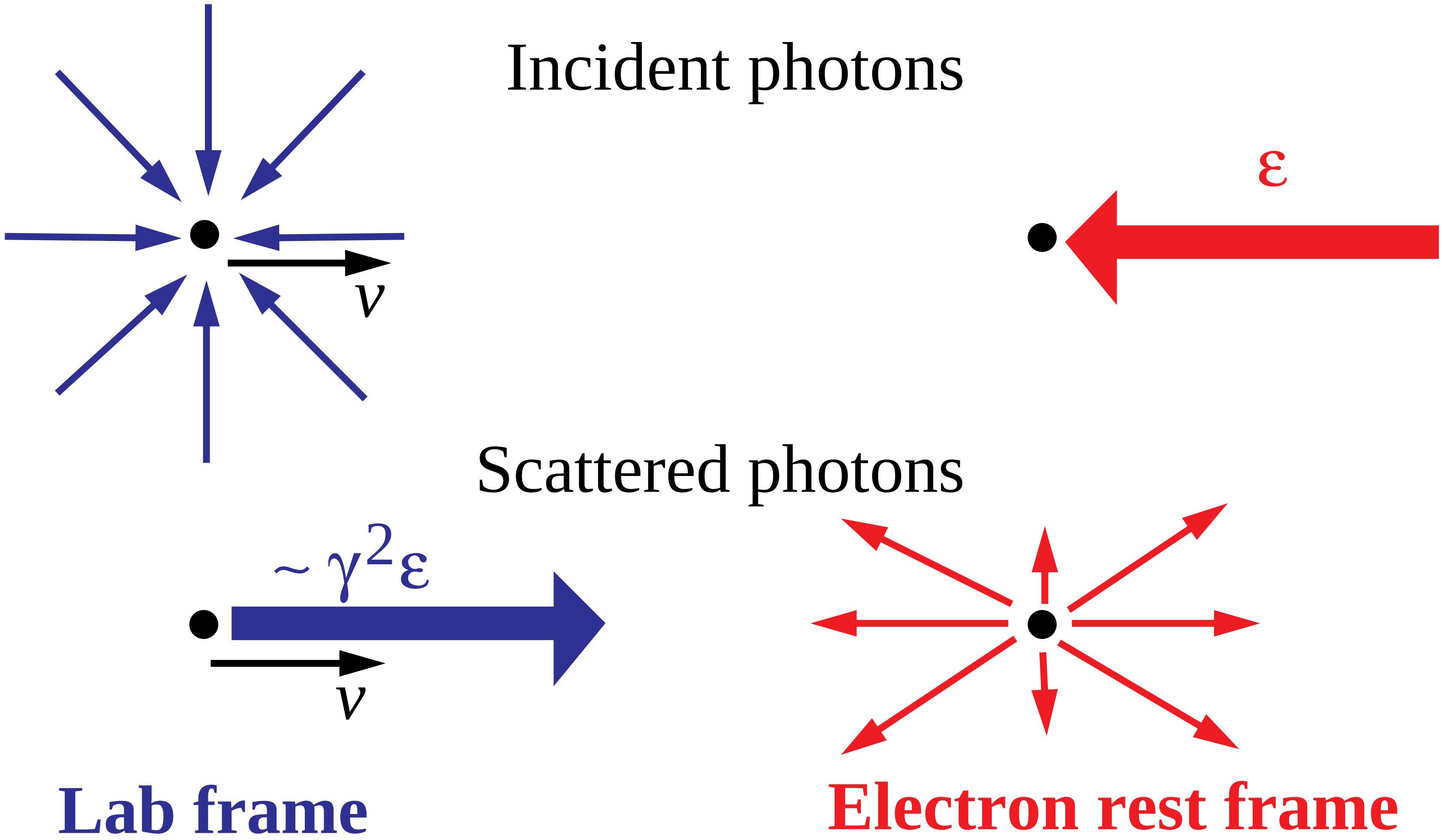
2.2 Calculation of the emitted spectrum



Electron scattering in laboratory frame and rest frame of electron

Note that all angles are measured clockwise from the positive *x*-axis as defined by the electron velocity.

Outline of what happens in practice



Photon-electron collisions in the rest and lab frames

One of the simplifications of inverse compton emission from relativistic electrons is due to the aberration of the photons in the rest frame of the electron.

Let $\mu' = \cos\theta'$ for the incoming photon in the rest frame and $\mu = \cos\theta$ for the incoming photon in the lab frame. Then the relativistic aberration formula gives:

$$\mu' = \frac{\mu - \beta}{1 - \beta\mu} \qquad (3)$$

For $\beta \approx 1$ and most values of μ:

$$\mu' \approx \frac{\mu - 1}{1 - \mu} \approx -1 \tag{4}$$

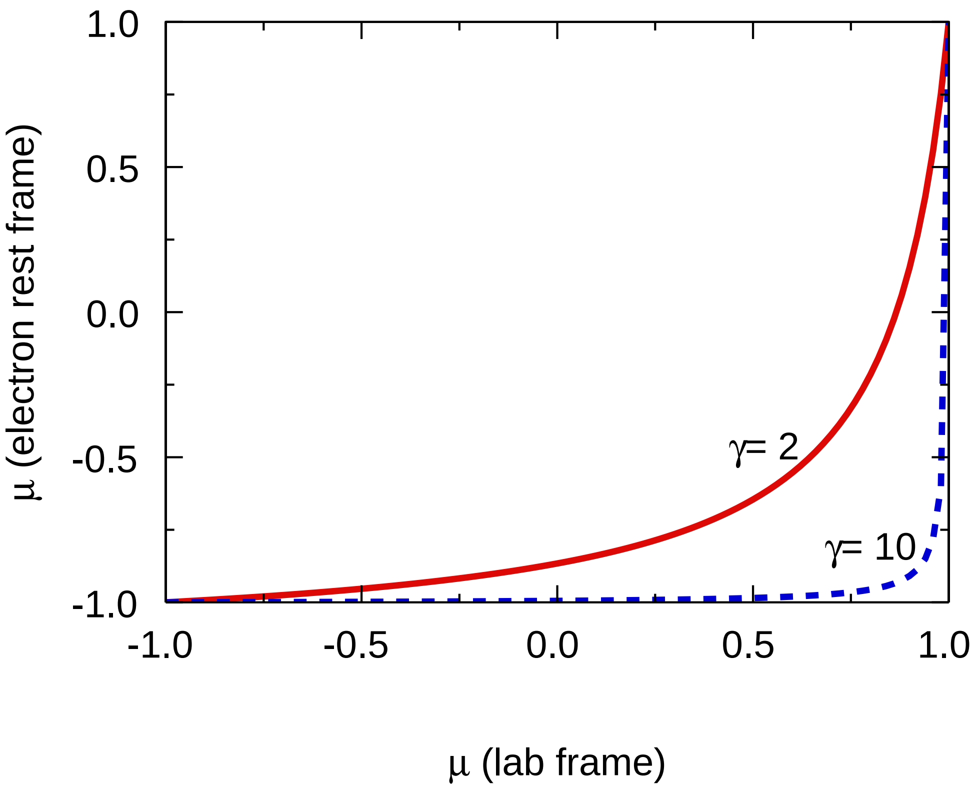
The only exception is when

$$\mu - \beta \approx 1 - \beta\mu \Rightarrow \mu \approx 1 \tag{5}$$

and then

$$\mu' \approx 1 \tag{6}$$

Thus the values of μ' cluster around $\mu' = -1$ for high Lorentz factors. This corresponds to virtually head on collisions between electrons and photons in the rest frame. The only exception is for collisions which are close to overtaking.



The plot at the left shows the result of the aberration formula for $\mu = \cos\theta$, for 2 different values of γ.

It is well to remember that the lab frame distribution is isotropic so that $\mu' \approx -1$ is going to be the most common result.

Electron energy

On the other hand, the ratio of the photon energy in the rest frame to that in the lab frame is given by

$$\frac{\text{Rest frame photon energy}}{\text{Lab frame photon energy}} = \frac{\varepsilon'}{\varepsilon} = \gamma(1-\beta\mu) \qquad (7)$$

and varies linearly with μ. There is no specific value of $\frac{\varepsilon'}{\varepsilon}$ which is favoured.

Minimum and maximum values of rest frame photon energy

The minimum value is for

$$\mu = 1 \Rightarrow \varepsilon' = \gamma(1-\beta)\varepsilon \approx \gamma\left(1 - 1 + \frac{1}{2\gamma^2}\right)\varepsilon = \frac{\varepsilon}{2\gamma} \qquad (8)$$

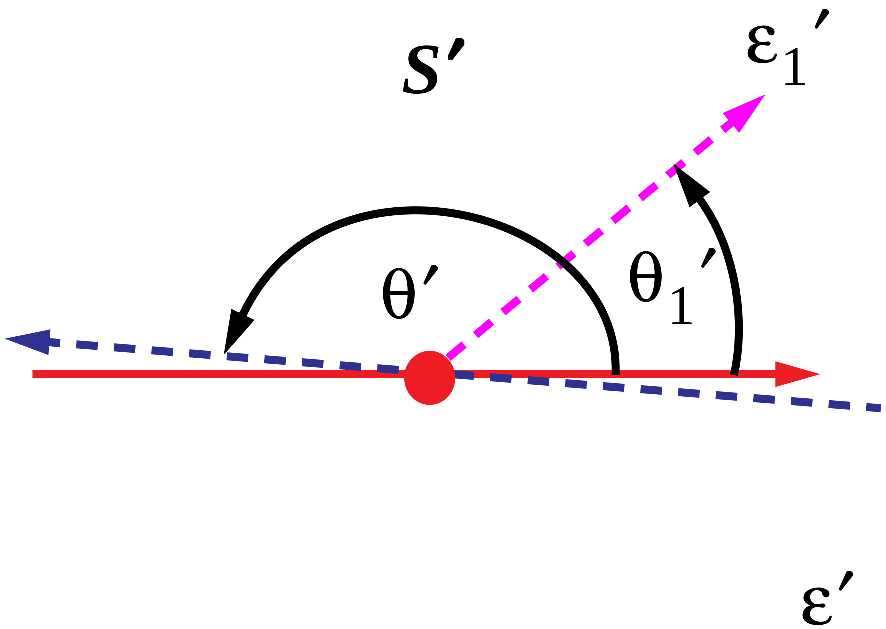
The maximum value is at

$$\mu = -1 \Rightarrow \varepsilon' = \gamma(1+\beta)\varepsilon \approx 2\gamma\varepsilon \qquad (9)$$

That is,

$$\frac{\varepsilon}{2\gamma} < \varepsilon' < 2\gamma\varepsilon \qquad (10)$$

2.3 Evaluation of rate of scattering



We consider the situation first in the rest frame. The complexity of the scattering is considerably simplified by the fact that the photons are effectively incident on the electron at one angle $\theta' = \pi$. This means that we only have to consider one scattering angle for the scattered photons.

In the rest frame:

$$\frac{dN'}{dt'd\Omega_1'} = \begin{array}{c}\text{No of photons scattered}\\ \text{per unit time per unit solid angle}\\ \text{in the electron rest frame}\end{array} \qquad (11)$$

$$= \int_{\boldsymbol{p}'} \frac{d\sigma_T}{d\Omega_1'} c f'(\boldsymbol{p}') d^3\boldsymbol{p}'$$

The differential cross section is given by Thomson scattering:

$$\frac{d\sigma_T}{d\Omega_1'} = \frac{r_0^2}{2}(1 + \mu_1'^2) \qquad (12)$$

We can write the cross-section this way because we are approximating the photons as all coming in along the x-axis.

Relation to distribution in lab frame

As we did for the total power calculation, we transform the integral over the rest frame momentum space into one over the lab frame momentum space. Note that

$$\begin{aligned} d^3p &= p^2 \sin\theta \, dp \, d\theta \, d\phi = -p^2 dp \, d(\cos\theta) \, d\phi \\ &= -p^2 dp \, d\mu \, d\phi \end{aligned} \qquad (13)$$

so that integrations over θ become integrations over μ. Also

$$\int_0^{\pi} \sin\theta \, d\theta \rightarrow -\int_1^{-1} d\mu \rightarrow \int_{-1}^{1} d\mu \qquad (14)$$

Hence,

$$d^3p \rightarrow p^2 dp d\mu d\phi \tag{15}$$

in integrations over momentum space.

In *Compton Scattering I*, we found that:

$$d^3p' = \gamma(1-\beta\mu)d^3\boldsymbol{p} \tag{16}$$

Hence,

$$f'(\boldsymbol{p}')d^3\boldsymbol{p}' = f(\boldsymbol{p})\gamma(1-\beta\mu)d^3\boldsymbol{p} \tag{17}$$

We also have

$$\varepsilon' = \gamma(1 - \beta\mu)\varepsilon \tag{18}$$

Since the distribution in the lab frame is isotropic. Therefore,

$$4\pi f(p) p^2 dp = n(\varepsilon) d\varepsilon \tag{19}$$

And

$$f'(\boldsymbol{p}')d^3\boldsymbol{p}' = \left(\frac{\varepsilon'}{\varepsilon}\right) f(\boldsymbol{p}) d^3\boldsymbol{p} = \left(\frac{\varepsilon'}{\varepsilon}\right) f(p) p^2 dp d\mu d\phi \tag{20}$$

$$= \frac{1}{4\pi}\left(\frac{\varepsilon'}{\varepsilon}\right) n(\varepsilon) d\varepsilon d\mu d\phi \tag{21}$$

The integral over ϕ is easy:

$$\begin{aligned}\frac{dN'}{dt'd\Omega_1'} &= \frac{1}{4\pi}\int_{\boldsymbol{p}'}\frac{d\sigma_T}{d\Omega_1'}c\left(\frac{\varepsilon'}{\varepsilon}\right)n(\varepsilon)d\varepsilon d\mu d\phi \\ &= \frac{1}{2}\int_{(\mu,\,\varepsilon)}\frac{d\sigma_T}{d\Omega_1'}c\left(\frac{\varepsilon'}{\varepsilon}\right)n(\varepsilon)d\varepsilon d\mu\end{aligned} \qquad (22)$$

Change of variables from (ε, μ) *to* $(\varepsilon, \varepsilon')$

We now make a change of variables from (ε, μ) to $(\varepsilon, \varepsilon')$ utilising the transformation

$$\begin{aligned} \varepsilon &= \varepsilon \\ \varepsilon' &= \gamma\varepsilon(1 - \beta\mu) \end{aligned} \qquad (23)$$

That is we are replacing μ by the photon energy (ε') seen by the electron in the rest frame.

Our reason for doing this is that, later on, we relate ε' to the energy of the scattered photon and thus derive the differential no of photons scattered per unit energy of the final scattered energy – that is, the energy we receive at the telescope.

Remember:

$$\begin{aligned} \varepsilon &= \text{Energy of incident photon in lab frame} \\ \varepsilon' &= \text{Energy of incident photon in rest frame} \end{aligned} \qquad (24)$$

The Jacobean

$$\frac{\partial(\varepsilon, \varepsilon')}{\partial(\varepsilon, \mu)} = \begin{vmatrix} 1 & 0 \\ \gamma(1-\beta\mu) & -\gamma\beta\varepsilon \end{vmatrix} = \gamma\beta\varepsilon$$

$$\Rightarrow d\varepsilon d\mu = \frac{1}{\gamma\beta\varepsilon} d\varepsilon d\varepsilon' \approx \frac{1}{\gamma\varepsilon} d\varepsilon d\varepsilon' \qquad (25)$$

Hence,

$$\frac{dN'}{dt'd\Omega_1'} = \frac{1}{2}\int_{(\mu,\,\varepsilon)} \frac{d\sigma_T}{d\Omega_1'} c\left(\frac{\varepsilon'}{\varepsilon}\right) n(\varepsilon)\, d\varepsilon\, d\mu \tag{26}$$

$$= \int_{(\varepsilon,\,\varepsilon')} \frac{d\sigma_T}{d\Omega_1'} c\left(\frac{\varepsilon'}{2\gamma\varepsilon^2}\right) n(\varepsilon)\, d\varepsilon\, d\varepsilon' \tag{27}$$

We now write this in the form:

$$\frac{dN'}{dt'd\Omega_1'd\varepsilon d\varepsilon'} = c\frac{d\sigma_T}{d\Omega_1'}\left(\frac{\varepsilon'}{2\gamma\varepsilon^2}\right) n(\varepsilon) \tag{28}$$

Scattering in the Thomson limit

Now use the Thomson limit approximation for the energy of the scattered photon:

$$\epsilon_1' = \epsilon' \qquad (29)$$

As we saw in ***Compton Scattering I***, the condition for this is that

$$\gamma\epsilon \ll m_e c^2 \qquad (30)$$

Using $\varepsilon_1' = \varepsilon'$ gives:

$$\frac{dN'}{dt'd\Omega_1'd\varepsilon d\varepsilon_1'} = c\frac{d\sigma_T}{d\Omega_1'}\left(\frac{\varepsilon_1'}{2\gamma\varepsilon^2}\right)n(\varepsilon)$$

$$= \frac{cr_0^2}{2}(1+\mu_1'^2)\left(\frac{\varepsilon_1'}{2\gamma\varepsilon^2}\right)n(\varepsilon) \qquad (31)$$

Relationship to the scattering rate in the lab frame

The next step is to relate this differential expression to quantities in the lab frame. The two main variables to transform are t' and ε_1'. We use the Lorentz transformation for time

$$t = \gamma\left(t' - \frac{\beta x'}{c}\right) \qquad (32)$$

We also use the Lorentz transformation relating photon energy in the lab frame to photon energy in the rest frame given that the scattered photon has a direction given by $\mu_1' = \cos\theta_1'$.

This is:

$$\varepsilon_1 = \gamma\varepsilon_1'(1+\beta\mu_1') \qquad (33)$$

These transformations imply that:

$$dt = \gamma dt' \qquad d\varepsilon_1 = \gamma(1+\beta\mu_1')d\varepsilon_1'$$

$$dt' = \gamma^{-1}dt \qquad d\varepsilon_1' = \frac{d\varepsilon_1}{\gamma(1+\beta\mu_1')} \qquad (34)$$

Also, since dN' is a number it is a Lorentz invariant and

$$\frac{dN}{dt\,d\Omega_1'\,d\varepsilon\,d\varepsilon_1} = \frac{dN'}{dt'\,d\Omega_1'\,d\varepsilon\,d\varepsilon_1'} \times \frac{dt'}{dt} \times \frac{d\varepsilon_1'}{d\varepsilon_1}$$

$$= \frac{cr_0^2}{\gamma^2}\frac{1+\mu_1'^2}{(1+\beta\mu_1')^2}\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)\frac{n(\varepsilon)}{\varepsilon} \qquad (35)$$

Rate of scattering in all directions

We now calculate the rate of scattering of photons into all directions. We integrate over solid angle in the rest frame, with $d\Omega_1' \rightarrow d\mu_1' d\phi'$

$$\frac{dN}{t\,d\varepsilon\,d\varepsilon_1} = c\frac{r_0^2}{\gamma^2}\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)\frac{n(\varepsilon)}{\varepsilon}\int_0^{2\pi}\left[\int_{\mu'_{1,\,\min}}^{\mu'_{1,\,\max}}\frac{(1+\mu_1'^2)}{(1+\mu_1')^2}d\mu_1'\right]d\phi' \qquad (36)$$

Limits on scattered energy in lab frame

In order to complete the above integral, we have to determine the limits on μ_1'. We first calculate the limits on ε_1.

We have the relation between rest frame incident soft photon energy ε' and lab frame scattered energy ε_1:

$$\varepsilon_1 = \gamma\varepsilon'(1+\beta\mu_1') = \gamma\varepsilon'(1+\mu_1') \qquad (37)$$

and the previously derived limits on ε':

$$\frac{\varepsilon}{2\gamma} < \varepsilon' < 2\gamma\varepsilon \qquad (38)$$

For a given ε' the upper limit on ε_1 is obtained for $\mu_1' = 1$.

This gives $\varepsilon_1 \approx 2\gamma\varepsilon'$

Therefore, from the upper limit on ε'

$$\varepsilon_1 < 4\gamma^2\varepsilon \qquad (39)$$

That is, the kinematics of Compton scattering establishes a maximum energy, $4\gamma^2\varepsilon$ of the scattered photons.

Later we make use of the variable:

$$x = \frac{\text{Final photon energy}}{\text{Maximum photon energy}} = \frac{\varepsilon_1}{4\gamma^2\varepsilon} \qquad (40)$$

Limits on μ_1' *for a fixed* ε_1 *and* ε

Since

$$\varepsilon_1 = \gamma\varepsilon'(1+\mu_1') < 2\gamma^2\varepsilon(1+\mu_1') \qquad (41)$$

then, for a given ε, ε_1

$$1 + \mu_1' > \frac{\varepsilon_1}{2\gamma^2\varepsilon} \qquad (42)$$

This provides the lower limit on the integral over μ_1'.

We also have from the lower limit on ε':

$$\varepsilon_1 = \gamma\varepsilon'(1 + \mu_1') > \frac{\gamma\varepsilon}{2\gamma}(1 + \mu_1') = \frac{\varepsilon}{2}(1 + \mu_1') \qquad (43)$$

Therefore,

$$(1 + \mu_1') < 2\frac{\varepsilon_1}{\varepsilon} \qquad (44)$$

We also have the obvious constraint on μ_1', namely $\mu_1' < 1$ so that

$$(1 + \mu_1') < 2 \qquad (45)$$

Since we are normally interested in $\varepsilon_1 / \varepsilon \gg 1$ then the latter limit is usually the most relevant.

Summary of limits

$$\frac{\varepsilon_1}{2\gamma^2\varepsilon} < (1+\mu_1') < 2 \qquad (46)$$

Evaluation of integral

The indefinite integral

$$\int \frac{(1+\mu_1'^2)}{(1+\mu_1')^2} d\mu_1' = \left(2\frac{(1+\mu_1')}{2} - 2\ln\left(\frac{1+\mu_1'}{2}\right) - \left(\frac{1+\mu_1'}{2}\right)^{-1}\right) \qquad (47)$$

Evaluation between the lower and upper limits gives:

$$1 - 2\frac{\varepsilon_1}{4\gamma^2\varepsilon} + 2\ln\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right) + \left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)^{-1} \qquad (48)$$

Putting it all together:

$$\frac{dN}{dt\,d\varepsilon\,d\varepsilon_1} = c\frac{r_0^2}{\gamma^2}\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)\frac{n(\varepsilon)}{\varepsilon}\int_0^{2\pi}\left[\int_{\mu_{1,\,\min}'}^{\mu_{1,\,\max}'}\frac{(1+\mu_1'^2)}{(1+\mu_1')^2}d\mu_1'\right]d\phi$$

$$= \frac{2\pi c r_0^2}{\gamma^2}\frac{n(\varepsilon)}{\varepsilon} \qquad (49)$$

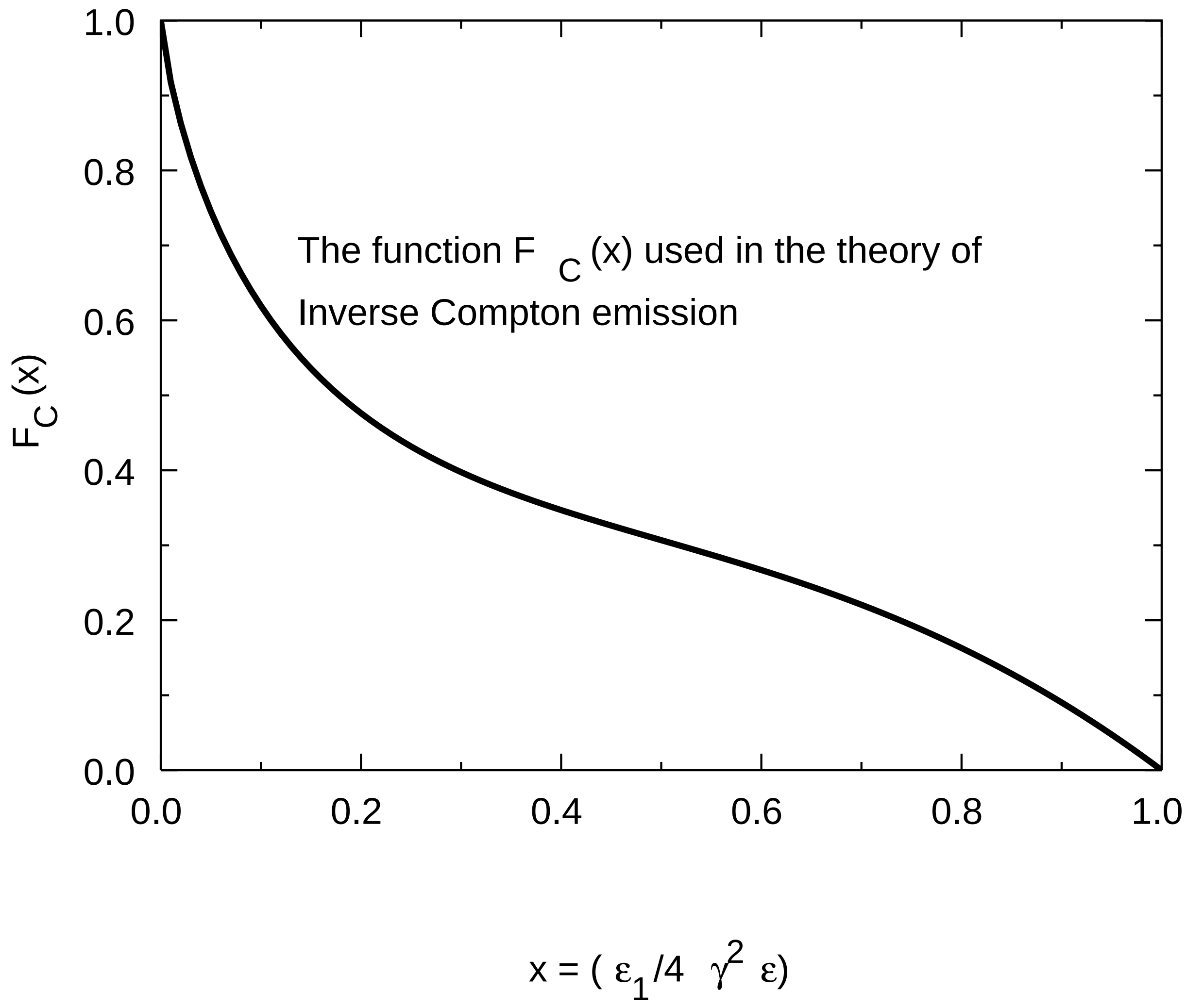
$$\times\left[1+\frac{\varepsilon_1}{4\gamma^2\varepsilon}+2\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)\ln\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)-2\left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right)^2\right]$$

We put

$$x = \left(\frac{\varepsilon_1}{4\gamma^2\varepsilon}\right) \qquad \pi r_0^2 = \frac{3}{8}\sigma_T \tag{50}$$

to obtain

$$\frac{dN}{dt\,d\varepsilon\,d\varepsilon_1} = \frac{3}{4}\frac{(c\sigma_T)}{\gamma^2}\frac{n(\varepsilon)}{\varepsilon}F_C(x)$$
$$F_C(x) = 2x\ln x + x + 1 - 2x^2 \tag{51}$$

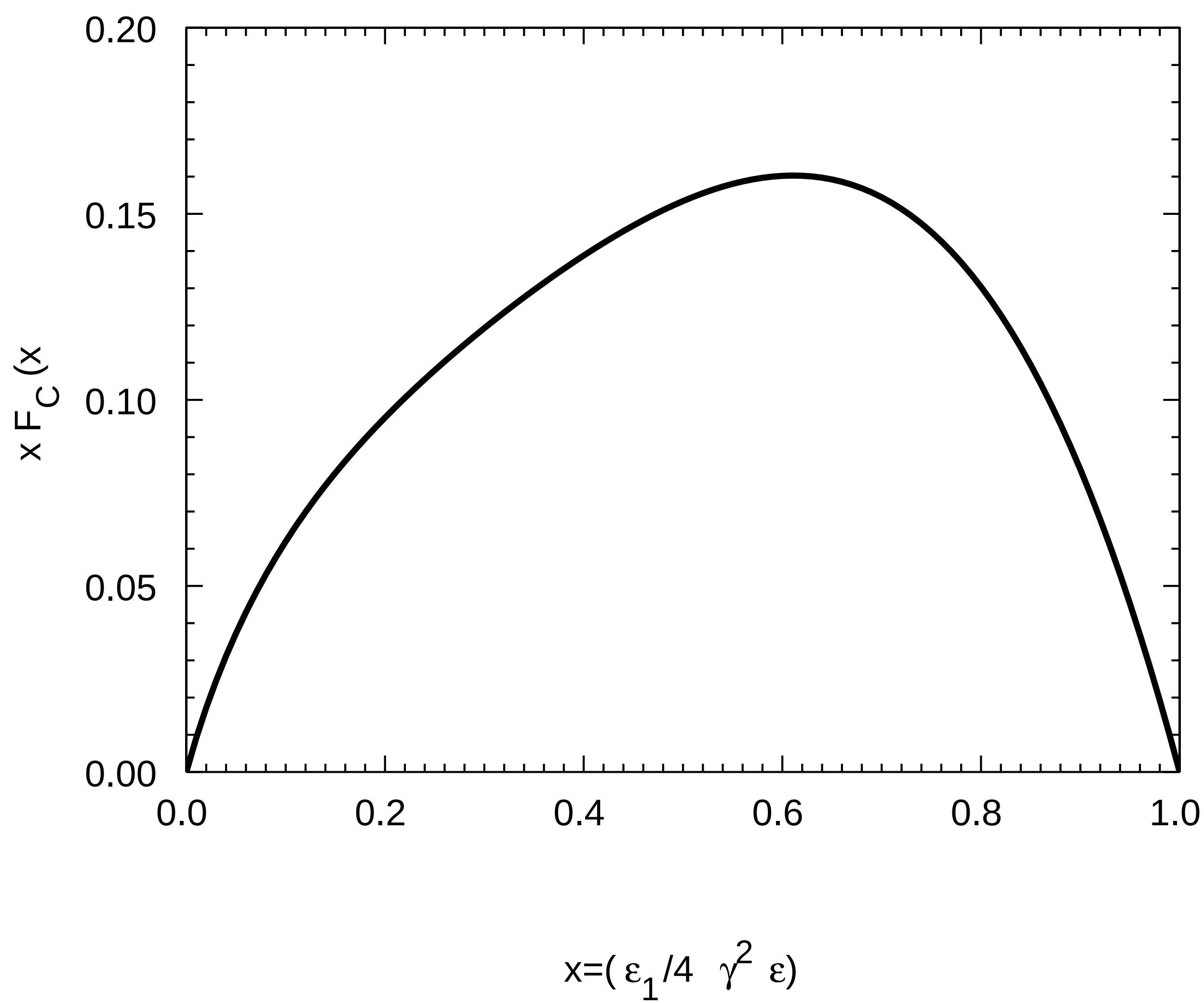


A plot of the function $F_C(x)$ is shown at the left. This shows that there is a broad distribution of electron energies resulting from Compton scattering.

Rate of energy of scattered photons

The above expression is more informative if we compute the rate of *energy* scattered per unit final photon energy per unit incident energy. This is

$$
\begin{aligned}
\epsilon_1 \frac{dN}{dt\,d\epsilon_1\,d\epsilon} &= \epsilon_1 \frac{3}{4}\frac{(c\sigma_T)}{\gamma^2}\frac{n(\epsilon)}{\epsilon}F_C(x) \\
&= 3(c\sigma_T)n(\epsilon)xF_C(x)
\end{aligned}
\tag{52}
$$



The function $xF_C(x)$ is plotted at the left. This function gives a better idea of the rate of photon energy scattered to high energies as a result of Compton scattering.

As you can see this function indicates a peak rate of scattered energy at

$$\epsilon_1 \approx 0.6 \times 4\gamma^2\epsilon \approx 2.4\gamma^2\epsilon \tag{53}$$

Check on calculation

As a check on this expression, one can evaluate the total energy radiated against the previous expression by integrating with respect to ϵ_1.

Since $x = \varepsilon_1/(4\gamma^2\varepsilon)$, then

$$\int \varepsilon_1 \frac{dN}{dt\,d\varepsilon_1\,d\varepsilon} d\varepsilon_1 = 12\gamma^2(c\sigma_T)\varepsilon n(\varepsilon)\int_0^1 x F_C(x)\,dx \qquad (54)$$

The value of the integral is $1/9$ and

$$\frac{\text{Energy emitted per unit}}{\text{soft photon energy}} = \frac{4}{3}(c\sigma_T)\varepsilon n(\varepsilon)\gamma^2 \qquad (55)$$

The total power emitted comes from integrating over ε and is

$$P = \frac{4}{3}(c\sigma_T)\gamma^2\int \varepsilon n(\varepsilon)\,d\varepsilon \qquad (56)$$

in agreement with the ultrarelativistic limit derived earlier.

3 Inverse Compton spectrum from a distribution of electrons

3.1 Direction of emission

The goal of the following is to determine the emissivity of a distribution of electrons that are emitting photons via the inverse Compton process.

This calculation is simplified by the fact that, in the lab frame, the emitted photons are scattered in the direction of motion of the electron.

To see this consider a photon that is scattered at an angle $\cos^{-1}\mu_1'$ in the rest frame. In the lab frame

$$\mu_1 = \cos\theta_1 = \frac{\mu_1' + \beta}{1 + \beta\mu_1'} \tag{57}$$

When $\beta \approx 1$, for most angles θ_1',

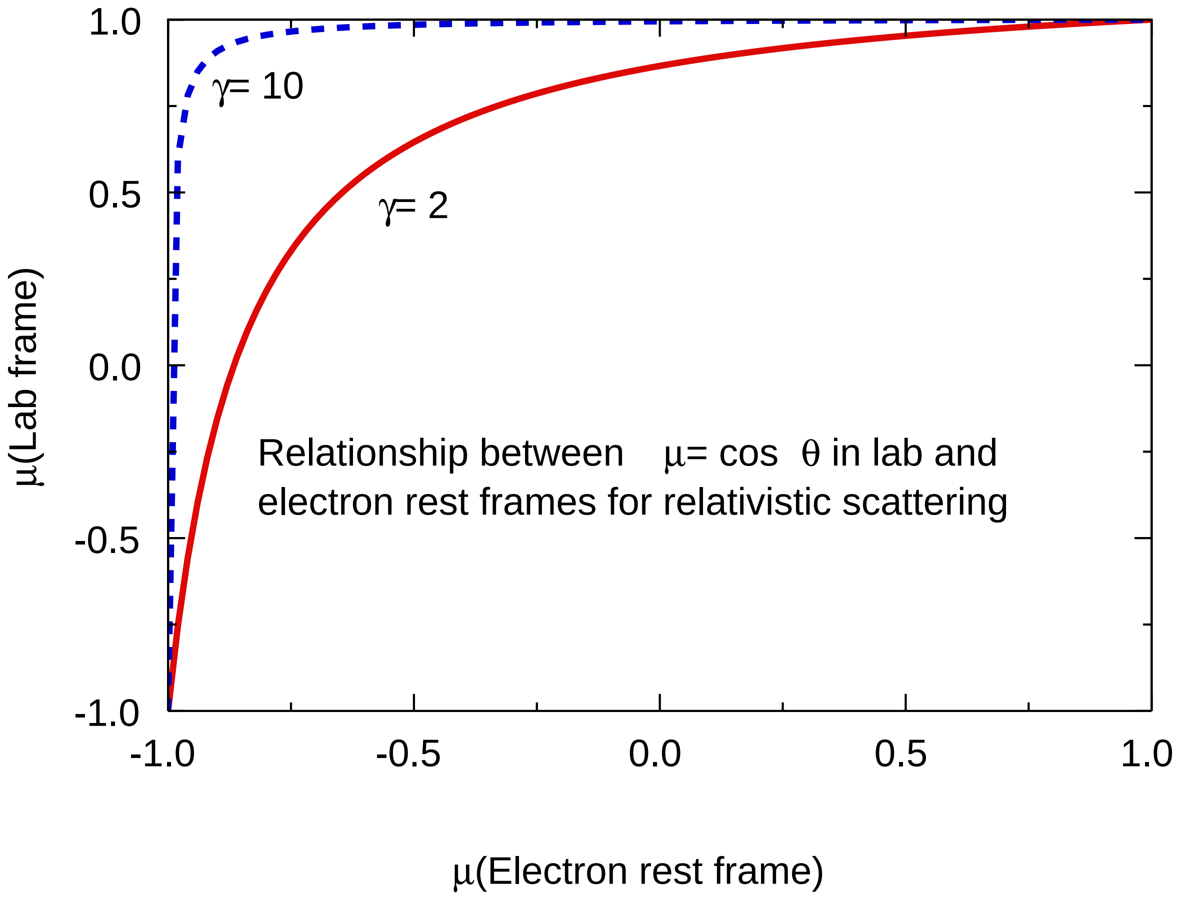
$$\mu_1 \approx 1 \tag{58}$$

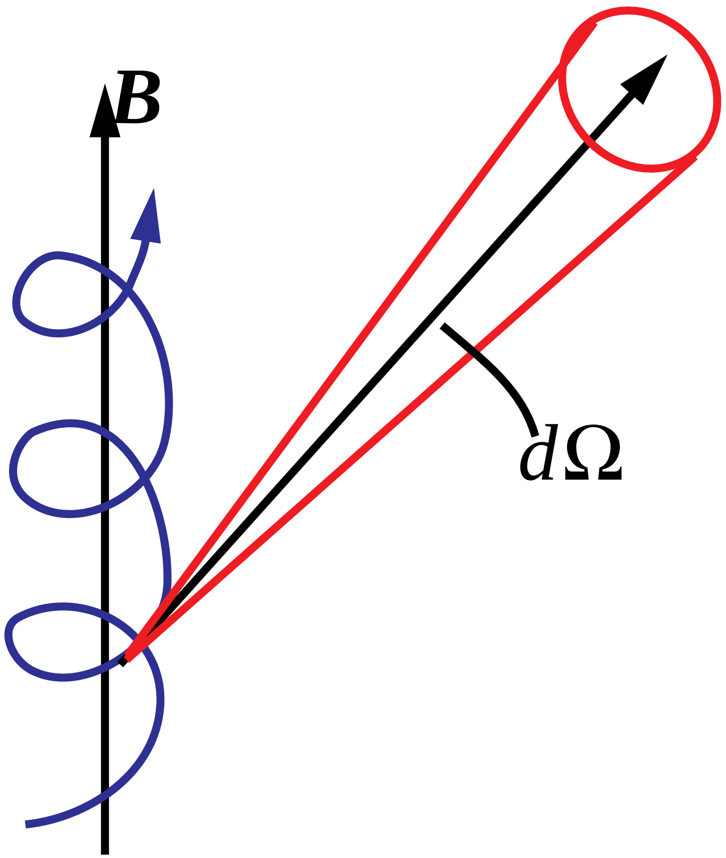
The only exception is when $\mu_1' \to -1$, then

$$\mu_1 \to -1 \tag{59}$$

Since Thomson scattering in the rest frame is quasi-isotropic, most photons will be scattered in the forward direction given by $\mu_1 = 1, \theta = 0$.

This plot effectively illustrates this point. The consequence of this is that most of the photons are emitted in the direction of motion of the electron. This is similar to synchrotron emission.





As before let $N(\gamma)$ be the number of photons per unit Lorentz factor. Then the number of electrons with Lorentz factors in the range $d\gamma$ pointing in the direction of the observer is:

$$\frac{N(\gamma)}{4\pi} d\gamma d\Omega \qquad (60)$$

Hence the rate of energy per unit time per unit scattered photon energy radiated into this solid angle, produced by soft photons in the range $d\epsilon$ is

$$\epsilon_1 \frac{dN}{dt\,d\epsilon\,d\epsilon_1} \frac{N(\gamma)}{4\pi} d\gamma\, d\epsilon\, d\epsilon_1\, d\Omega \tag{61}$$

(Note that we have multiplied by ϵ_1 to determine the energy associated with the number of photons.)

Hence, the emissivity of inverse Compton scattered radiation is

$$
\begin{aligned}
j_{\varepsilon_1} &= \int_{\varepsilon}\left[\int_{\gamma}\frac{1}{4\pi}\varepsilon_1\frac{dN}{dt\,d\varepsilon\,d\varepsilon_1}N(\gamma)d\gamma\right]d\varepsilon \\
&= \frac{3(c\sigma_T)\varepsilon_1}{16\pi}\int_{\varepsilon}\left[\frac{n(\varepsilon)}{\varepsilon}\int_{\gamma}F_C(x)\frac{N(\gamma)}{\gamma^2}d\gamma\right]d\varepsilon
\end{aligned}
\qquad (62)
$$

The emissivity therefore involves a double integration over the particle distribution and the input photon distribution.

Integration over Lorentz factor

We can do this part first since we generally have a good idea of what the particle distribution is.

We assume, as before, that the particle distribution is given by

$$N(\gamma) = K_e \gamma^{-a} \qquad \gamma_{\min} < \gamma < \gamma_{\max} \tag{63}$$

Remember that the energy of the incident and scattered photons satisfy:

$$\frac{1}{2\gamma^2} < \frac{\varepsilon_1}{\varepsilon} < 4\gamma^2 \tag{64}$$

so that if we are considering values of $\varepsilon_1 / \varepsilon$ that approach either $1/(2\gamma_{min}^2)$ or $4\gamma_{max}^2$ then there will be "end effects" associated with the integral. We assume that this is not the case and estimate the inner integral

$$\int_\gamma F_C(x) \frac{N(\gamma)}{\gamma^2} d\gamma \tag{65}$$

We calculate the integral by changing the integration variable to

$$x = \varepsilon_1 / (4\gamma^2 \varepsilon). \tag{66}$$

Then

$$\gamma = \left(\frac{\varepsilon_1}{4\varepsilon}\right)^{1/2} x^{-1/2} \qquad d\gamma = -\frac{1}{2}\left(\frac{\varepsilon_1}{4\varepsilon}\right)^{1/2} x^{-3/2} dx$$

$$\gamma^{-2} N(\gamma) = K\left(\frac{\varepsilon_1}{4\varepsilon}\right)^{-\frac{(a+2)}{2}} x^{\frac{a+2}{2}} \qquad (67)$$

The initial limits on x are defined by γ_{min} and γ_{max} implying that

$$\frac{\varepsilon_1}{4\gamma_{\mathrm{max}}^2 \varepsilon} < x < \frac{\varepsilon_1}{4\gamma_{\mathrm{min}}^2 \varepsilon} \qquad (68)$$

For the upper limit however, we also have the requirement $x < 1$, so that

$$x < \left(\frac{\varepsilon_1 / 4\varepsilon}{\gamma_{\min}^2}, 1 \right) \qquad (69)$$

If $\dfrac{\varepsilon_1}{\varepsilon} > 4\gamma_{\min}^2$, as is usually the case, (that is we are considering very high energies ε_1) then we have $x < 1$.

Making the above substitutions and using the derived limits:

$$\int_{\gamma_{\min}}^{\gamma_{\max}} f(x)\frac{N(\gamma)}{\gamma^2}d\gamma = \frac{K_e}{2}\left(\frac{\varepsilon_1}{4\varepsilon}\right)^{\frac{-(a+1)}{2}}\int_0^1 x^{\frac{(a-1)}{2}} F_C(x)dx \tag{70}$$

$$= K_e\frac{2(a^2+4a+11)}{(a+1)(a+3)^2(a+5)}\left(\frac{\varepsilon_1}{4\varepsilon}\right)^{\frac{-(a+1)}{2}}$$

Then,

$$j_{\varepsilon_1} = \frac{3(c\sigma_T)\varepsilon_1}{16\pi}\int_{\varepsilon}\left[\frac{n(\varepsilon)}{\varepsilon}\int_{\gamma}F_C(x)\frac{N(\gamma)}{\gamma^2}d\gamma\right]d\varepsilon$$

$$= \frac{3}{\pi}2^{(a-2)}\frac{(a^2+4a+11)}{(a+1)(a+3)^2(a+5)}(c\sigma_T)K_e\varepsilon_1^{-\frac{(a-1)}{2}} \quad (71)$$

$$\times\int_{\varepsilon_{\min}}^{\varepsilon_{\max}}\varepsilon^{\frac{(a-1)}{2}}n(\varepsilon)d\varepsilon \qquad \mathrm{W\ m^{-3}\ J^{-1}}$$

We write this more compactly as:

$$j_{\varepsilon_1} = f(a)(c\sigma_T)K_e \varepsilon_1^{-\frac{(a-1)}{2}} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^{\frac{(a-1)}{2}} n(\varepsilon)\, d\varepsilon$$

$$f(a) = \frac{3}{\pi} 2^{(a-2)} \frac{(a^2 + 4a + 11)}{(a+1)(a+3)^2(a+5)} \qquad (72)$$

3.2 Emissivity in terms of photon number

High energy observers count photons and photon-counting statistics come into the estimation of errors. Hence, the results of X-ray or γ-ray observations are often represented in terms of a photon flux: number of photons per second per unit area.

Accordingly, the emissivity can also be represented in terms of number. Let

$$\frac{dN}{dt\,dV\,d\varepsilon\,d\Omega} = \begin{array}{c}\text{Rate of emission of photons per unit time} \\ \text{per unit volume per unit energy per Steradian}\end{array} \qquad (73)$$

Then

$$\frac{dN}{dt\,dV\,d\varepsilon\,d\Omega} = \varepsilon^{-1} j_{\varepsilon_1}$$

$$= f(a)(c\sigma_T)K_e \varepsilon_1^{-\frac{(a+1)}{2}} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^{\frac{(a-1)}{2}} n(\varepsilon)\,d\varepsilon \qquad (74)$$

An important feature is a difference of 1 between the ***photon index*** and the spectral index.

4 Features

This expression for the emissivity has some important features:

- The expression for the emissivity is proportional to the Thomson cross-section. When one calculates the surface brightness one multiplies by the length, L, through the source to obtain the Thomson optical depth $\sigma_T L$. This always figures in applications involving electron scattering.
- The emissivity is a power-law with exactly the same spectral index as the synchrotron emission. This is a tell-tale signature of inverse Compton emission.
- The dependence on the soft photon spectrum enters

through the factor $\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^{(a-1)/2} n(\varepsilon) d\varepsilon$. The energy dependence of the inverse Compton emission is independent of the type of input photon spectrum.

5 Applications

There are 2 traditional main applications for this formula.

- When the incident photon spectrum is a black body, e.g. the microwave background or approximating the emission from starlight by a black body
- When the synchrotron emission from a source produces a quasi-isotropic spectrum in the emitting volume

5.1 Evaluation of the soft photon integral

We require

$$\int_0^\infty \varepsilon^{(a-1)/2} n(\varepsilon) d\varepsilon \qquad (75)$$

where $n(\varepsilon)$ is the number density of photons per unit energy.

The ***energy density*** of photons per unit energy is

$$u(\varepsilon) = \varepsilon n(\varepsilon) \Rightarrow n(\varepsilon) = \varepsilon^{-1} u(\varepsilon) \qquad (76)$$

We had previously defined the energy density per unit frequency by

$$u_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\Omega \tag{77}$$

For the isotropic distribution we have used here,

$$u_\nu = \frac{4\pi}{c} I_\nu \tag{78}$$

Moreover

$$u_\nu d\nu = u(\varepsilon)d\varepsilon \Rightarrow u(\varepsilon) = u_\nu \frac{d\nu}{d\varepsilon} = h^{-1}u_\nu \tag{79}$$

Therefore

$$n(\varepsilon) = \varepsilon^{-1} u(\varepsilon) = \frac{4\pi}{h^2 c} \nu^{-1} u_\nu \qquad (80)$$

5.2 Blackbody distribution

When the soft photon field is an isotropic black body distribution such as the microwave background, then the specific intensity is just given by the black body value:

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} \qquad (81)$$

The energy density per unit energy is then:

$$n(\varepsilon) = \frac{8\pi}{hc^3}\nu^2\left[e^{\frac{h\nu}{kT}} - 1\right]^{-1} \qquad (82)$$

Hence,

$$\begin{aligned}&\int_0^\infty \varepsilon^{(a-1)/2} n(\varepsilon) d\varepsilon \\ &= \frac{8\pi}{h^3c^3}(kT)^{(a+5)/2}\int_0^\infty x^{(a+3)/2}[e^x - 1]^{-1} dx\end{aligned} \qquad (83)$$

The integral

$$\int_0^\infty x^{(a+3)/2}[e^x - 1]^{-1}dx = \Gamma\left(\frac{a+5}{2}\right)\zeta\left(\frac{a+5}{2}\right) \qquad (84)$$

Hence,

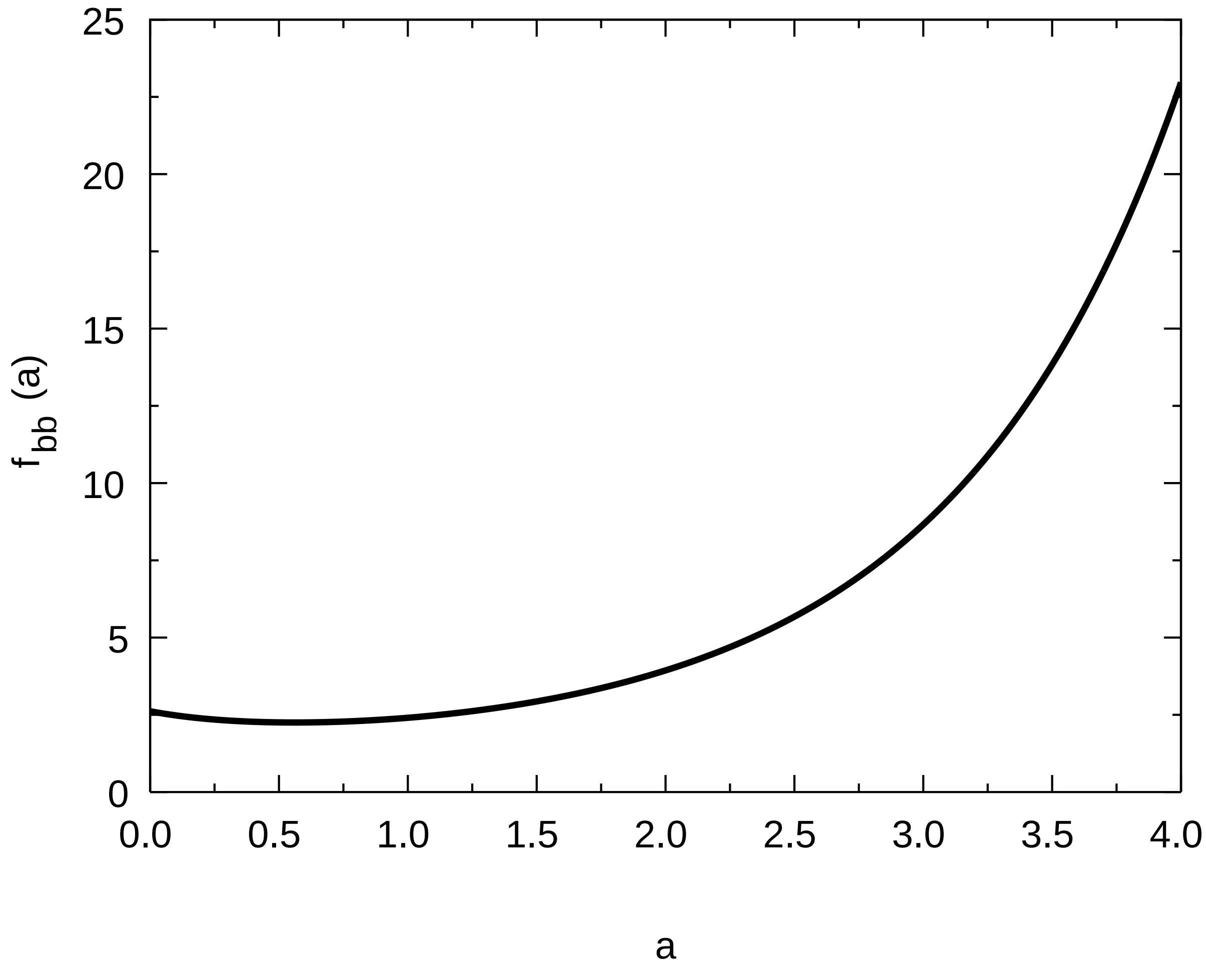
$$j_{\varepsilon_1} = f(a)\frac{8\pi\sigma_T}{h^3c^2}\Gamma\left(\frac{a+5}{2}\right)\zeta\left(\frac{a+5}{2}\right)(kT)^{\frac{(a+5)}{2}}K_e\varepsilon_1^{-\frac{(a-1)}{2}}$$

Group the numerical and a-dependent terms into,

$$f_{bb}(a) = \frac{3.2^{a+1}(a^2+4a+11)}{(a+1)(a+3)^2(a+5)} \times \Gamma\left(\frac{a+5}{2}\right)\zeta\left(\frac{a+5}{2}\right) \quad (85)$$

The emissivity from a black body soft photon field and a relativistic electron distribution is therefore:

$$j_{\varepsilon_1} = \frac{\sigma_T}{h^3c^2} f_{bb}(a)(kT)^{\frac{a+5}{2}} K_e \varepsilon_1^{-\frac{(a-1)}{2}} \quad (86)$$



The function $f_{bb}(a)$ is plotted at the left.

This can be used in estimates of the inverse Compton emission produced by the microwave background, for example.

(87)

5.3 Synchrotron Self-Compton emission

In this case, the soft photons are synchrotron photons emitted from the same volume by the same electrons. Therefore, the power-law index of the soft photon input is $(a-1)/2$ and we can take $u(\varepsilon) \propto \varepsilon^{-(a-1)/2}$. We suppose that the power-law holds between soft photon frequencies of ε_l and ε_u (l for lower and u for upper).

Thus,

$$n(\varepsilon) = \varepsilon^{-1}u(\varepsilon) = [\varepsilon_0^{-1}u(\varepsilon_0)]\left(\frac{\varepsilon}{\varepsilon_0}\right)^{-(a+1)/2} \qquad (88)$$

where ε_0 is some fiducial soft photon energy.

The integral,

$$\int_{\varepsilon_l}^{\varepsilon_u} \varepsilon^{(a-1)/2} n(\varepsilon) d\varepsilon = \varepsilon_0^{(a-1)/2} u(\varepsilon_0) \int_{\varepsilon_l}^{\varepsilon_u} \left(\frac{\varepsilon}{\varepsilon_0}\right)^{-1} \frac{d\varepsilon}{\varepsilon_0} \quad (89)$$

$$= \varepsilon_0^{(a-1)/2} u(\varepsilon_0) \ln(\varepsilon_u / \varepsilon_l)$$

Notes:

- Because of the assumed power-law, the parameter $\varepsilon_0^{(a-1)/2} u(\varepsilon_0)$ is independent of ε_0.
- The parameter $\ln(\varepsilon_u / \varepsilon_l)$ is known as the ***Compton loga-***

rithm. The upper energy can be determined from observation. The lower frequency may not be as well determined, although synchrotron self-absorption (to be treated later) sets a lower limit on ε_l. Fortunately, the evaluation of the emissivity is not sensitive to either the upper or lower cutoff since they appear in a logarithmic term.

Final form in terms of energy

In the case of SSC emission, then, the emissivity is:

$$\begin{aligned} j_{\varepsilon_1} &= f(a)(c\sigma_T)K_e \varepsilon_1^{-\frac{(a-1)}{2}} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^{\frac{(a-1)}{2}} n(\varepsilon)\, d\varepsilon \\ &= f(a)(c\sigma_T)K_e (\varepsilon_1/\varepsilon_0)^{-(a-1)/2} u(\varepsilon_0) \ln\left(\frac{\varepsilon_u}{\varepsilon_l}\right) \end{aligned} \qquad (90)$$

When we compare this with the expression for synchrotron emission from, say the isotropic version resulting from a tangled field

$$j_\nu^{\mathrm{syn}} = C_2(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K_e \Omega_0^{(a+1)/2} \nu^{-(a-1)/2} \qquad (91)$$

then we can see that the ratio of SSC to synchrotron emissivity can give us an estimate of the magnetic field since the factor of K_e occurs in both expressions and therefore cancels out.

Emissivity in terms of frequency

Note that, for the inverse Compton emissivity,

$$j_{\nu_1} = h j_{\varepsilon_1} \qquad (92)$$

and Planck's constant combines with the factor $u(\varepsilon_0)$ as follows:

$$h u(\varepsilon_0) = u_{\nu_0} \qquad (93)$$

Hence, a useful expression for SSC emissivity, when comparing different frequencies is

$$j_{\nu_1}^{\mathrm{SSC}} = f(a)(c\sigma_T)K_e(\nu_1/\nu_0)^{-(a-1)/2}u_{\nu_0}\ln\left(\frac{\varepsilon_u}{\varepsilon_l}\right) \quad (94)$$

5.4 Using synchrotron and self-Compton fluxes to estimate the magnetic field

The flux density from a stationary source at a luminosity distance, D_L, is:

$$F_\nu = \frac{1}{D_L^2} \int_V j_\nu dV \qquad (95)$$

We assume a spherical source with constant electron density and uniform synchrotron emissivity. The flux density from the source at frequency ν_1 is then

$$F_{\nu_1} = \frac{1}{D_L^2} f(a)(c\sigma_T)K_e(\nu_1/\nu_0)^{-(a-1)/2}\ln(\varepsilon_u/\varepsilon_l) \times \int_V u_{\nu_0}\, dV \tag{96}$$

We could simply assume at this point that the photon energy density is constant throughout the volume. However, one of the exercises shows that:

$$u_\nu(r) = u_\nu(0)\left[\frac{1}{2} + \frac{1}{4}(\xi^{-1} - \xi)\ln\left(\frac{1+\xi}{1-\xi}\right)\right] \qquad (97)$$

$$\xi = \frac{r}{R}$$

where R is the radius of the volume.

Hence,

$$\int_V u_\nu dV = 4\pi u_\nu(0)\int_0^R \left[\frac{1}{2} + \frac{1}{4}(\xi^{-1} - \xi)\ln\left(\frac{1+\xi}{1-\xi}\right)\right] r^2 dr$$

$$= 4\pi u_\nu(0) R^3 \int_0^1 \left[\frac{1}{2} + \frac{1}{4}(\xi^{-1} - \xi)\ln\left(\frac{1+\xi}{1-\xi}\right)\right] \xi^2 d\xi \qquad (98)$$

$$= \pi R^3 u_{\nu_0}(0) = \frac{3}{4} u_{\nu_0}(0) V$$

The SSC flux, therefore is:

$$F_{\nu_1}^{\mathrm{SSC}} = \frac{3}{4}\frac{1}{D_L^2} f(a)(c\sigma_T)K_e(\nu_1/\nu_0)^{-(a-1)/2} \times \ln(\varepsilon_u/\varepsilon_l)u_{\nu_0}V \tag{99}$$

By comparison, the synchrotron flux is:

$$F_{\nu}^{\mathrm{syn}} = \frac{C_2(a)}{D^2}\left(\frac{e^2}{\varepsilon_0 c}\right)K_e\Omega_0^{\frac{a+1}{2}}\nu^{-\frac{(a-1)}{2}}V \tag{100}$$

6 Models of synchrotron and SSC emission

As above let us consider a spherical model in which the radius of the sphere is R. In the expression for the SSC flux

$$F_{\nu_1}^{\mathrm{SSC}} = \frac{3}{4}\frac{1}{D^2} f(a)(c\sigma_T)K_e\left(\frac{\nu_1}{\nu_0}\right)^{-\frac{(a-1)}{2}} \qquad (101)$$

$$\times \ln(\varepsilon_u/\varepsilon_l)u_{\nu_0}(0)V$$

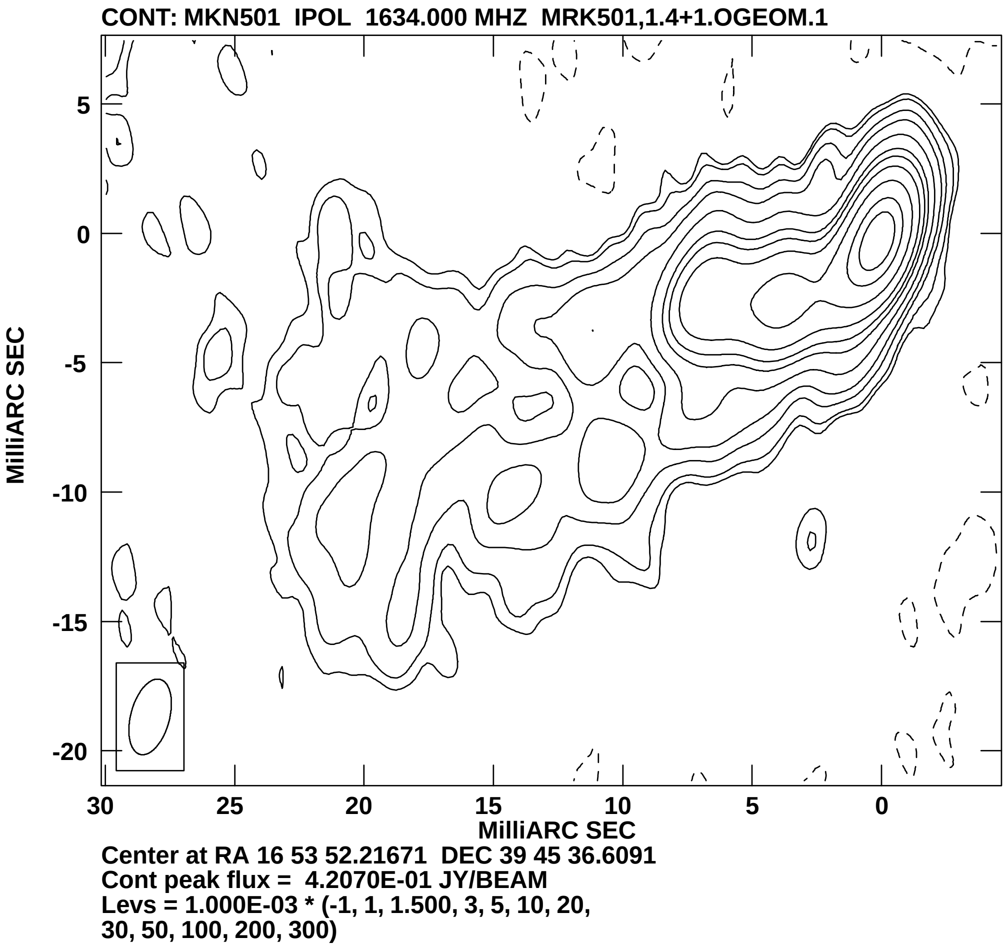
we have the factor

$$u_{\nu_0} = \frac{2\pi}{c} I_\nu^{\mathrm{peak}} \qquad (102)$$

and

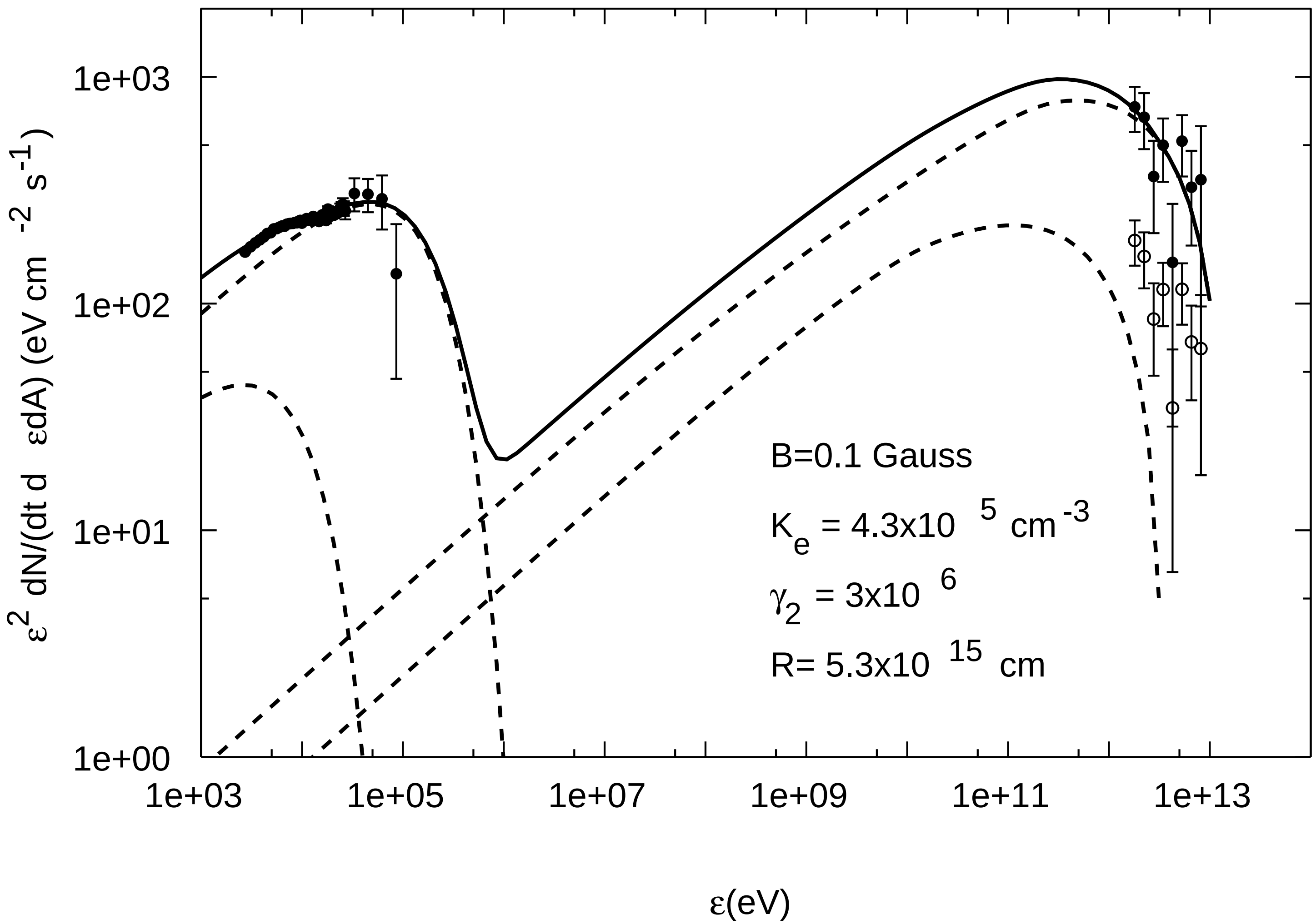
$$I_\nu^{\rm peak} = 2 j_\nu^{\rm syn} R \qquad (103)$$

Therefore, a model of the combined synchrotron and SSC spectrum involves the synchrotron parameters as well. When constructing models for the entire spectrum, we take this into account. An example is the following spectrum of Markarian 501. In this case, we have also had to allow for the fact that the Klein Nishina cross-section had to be used and the motion of the emitting region is relativistic. Also, the end points of the distribution in γ have to be included so that this example is quite a comprehensive one.



VLBI Image of Markarian 501

The X-ray and γ-ray emission originate from a region about an order of magnitude smaller than the smallest resolvable scale in this image.



X and TeV γ-ray spectrum of Markarian 501

7 Inferences from SSC emission on resolvable scales

In general, we model synchrotron plus SSC spectra as we showed for MKN 501. However, when the emission is resolved it is possible to use both sets of data to estimate the magnetic field. Using the expressions for the synchrotron and SSC fluxes:

$$F_{\nu}^{\mathrm{syn}} = \frac{C_2(a)}{D^2}\left(\frac{e^2}{\varepsilon_0 c}\right) K_e \Omega_0^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}} V$$

$$F_{\nu_1}^{\mathrm{SSC}} = \frac{0.75 f(a)}{D^2}(c\sigma_T) K_e \left(\frac{\nu_1}{\nu_0}\right)^{-\frac{(a-1)}{2}} \ln(\varepsilon_u / \varepsilon_l) u_{\nu_0}(0) V \tag{104}$$

On dividing one equation by the other, a number of factors cancel (K_e, D^2, V) and $\dfrac{e^2}{\varepsilon_0 c^2 \sigma_T} = \dfrac{3}{2}\dfrac{m_e}{r_0}$. This gives:

$$\Omega_0^{\frac{a+1}{2}} = \left[\frac{f(a)}{2C_2(a)}\right]\left(\frac{r_0}{m_e}\right)\left(\frac{\nu_1}{\nu_0}\right)^{-\frac{(a-1)}{2}} \frac{\nu^{(a-1)}}{2} \ln\left(\frac{\varepsilon_u}{\varepsilon_l}\right) u_{\nu_0}(0) \times \frac{F_\nu^{\mathrm{syn}}}{F_{\nu_1}^{SSC}} \tag{105}$$

and, of course, $B = (m_e / e)\Omega_0$.

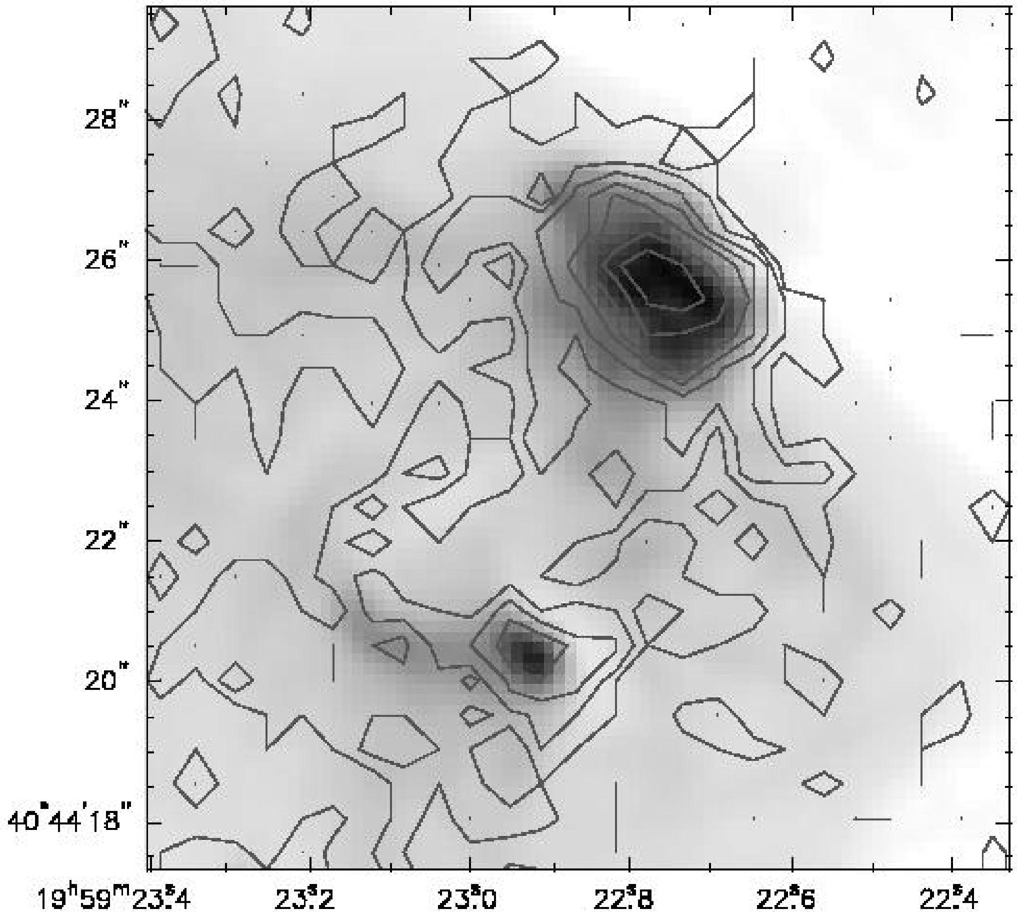
All of the parameters in the right hand side of the above equation are observable, except for ε_l. We shall see later how we can put constraints on ε_l.

The parameter

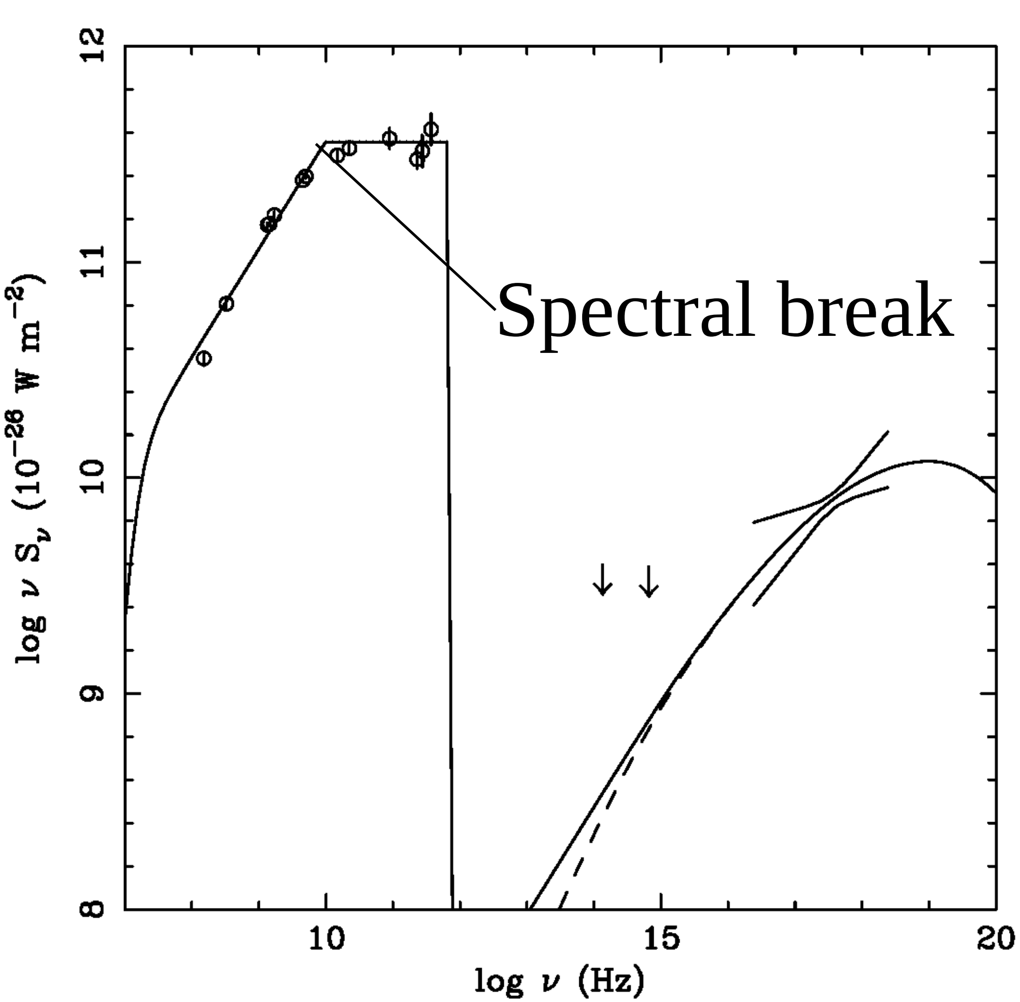
$$u_{\nu_0} = \frac{2\pi}{c} I_\nu^{\mathrm{peak}} \tag{106}$$

is also directly estimable from the peak surface brightness of the emitting region.

8 Example on Cygnus A hot spots



X-ray image (contours) of Cygnus A overlaid on the radio image (greyscale) (Wilson, Young and Shopbell, 2000). The bright X-ray regions are coincident with the radio hot spots.



The radio and X-ray spectrum (in the form of fluence) of the brightest hot spot in Cygnus A. The "bow-tie" represents the region of allowable X-ray slopes.

The solid lines show model fits to the data. However, we can estimate the magnetic field using the data from this plot.

For example take the point at

$$\nu = 10^{8.3}\ \mathrm{Hz} \qquad \nu F_\nu^{\mathrm{syn}} = 10^{10.8} \times 10^{-26}\ \mathrm{W\ m^{-2}}$$

$$\nu_u = 10^{10}\ \mathrm{Hz}$$

$$\nu_1 = 10^{17.5}\ \mathrm{Hz} \qquad \nu_1 F_{\nu_1} = 10^{9.9}\ \mathrm{Hz} \tag{107}$$

$$\alpha = 0.55$$

The following information on the radio source gives us the value of the energy density of the radio emission in the hot spot. This is obtained from the paper *The Structure and Polarisation of Cygnus A at λ 3.6 cm*, R.A. Perley and C.L. Carilli, in *Cygnus A – Study of a Radio Galaxy*, ed. C.L. Carilli & D.E. Harris.

$$\begin{aligned} \text{Peak flux per beam} &= 90\ \text{mJy} \\ \text{Beam size} &= 0.35'' \times 0.35'' \\ \nu_0 &= 5\times 10^9\ \text{Hz} \end{aligned} \tag{108}$$

The value of the magnetic field, evaluated using equation (105) is

$$B = 1.4\times10^{-4}\ \text{Gauss} \tag{109}$$

Note that the slope of the spectrum implied by the “bowtie” is not the same as the slope of the radio spectrum. In the model shown in the figure, this is the result of the X-ray emission being affected by the “end effects” referred to earlier. Nevertheless estimates such as these are quite useful for the starting points of detailed models.

By adopting an angular size for the hot spot we can estimate the minimum energy magnetic field. From the image, one obtains

$$D_{\text{hot spot}} \approx 3.5'' \tag{110}$$

The peak surface brightness of the hot spot can be evaluated from the flux per beam and the beam size [see equation (108)] and these parameters can be used to evaluate the minimum energy magnetic field.

One obtains for the minimum energy magnetic field in the hot spot

$$B_{\text{min}} = 9.2\times10^{-5}\ \text{Gauss} \tag{111}$$

This is within a factor of 2 of the inverse Compton value.