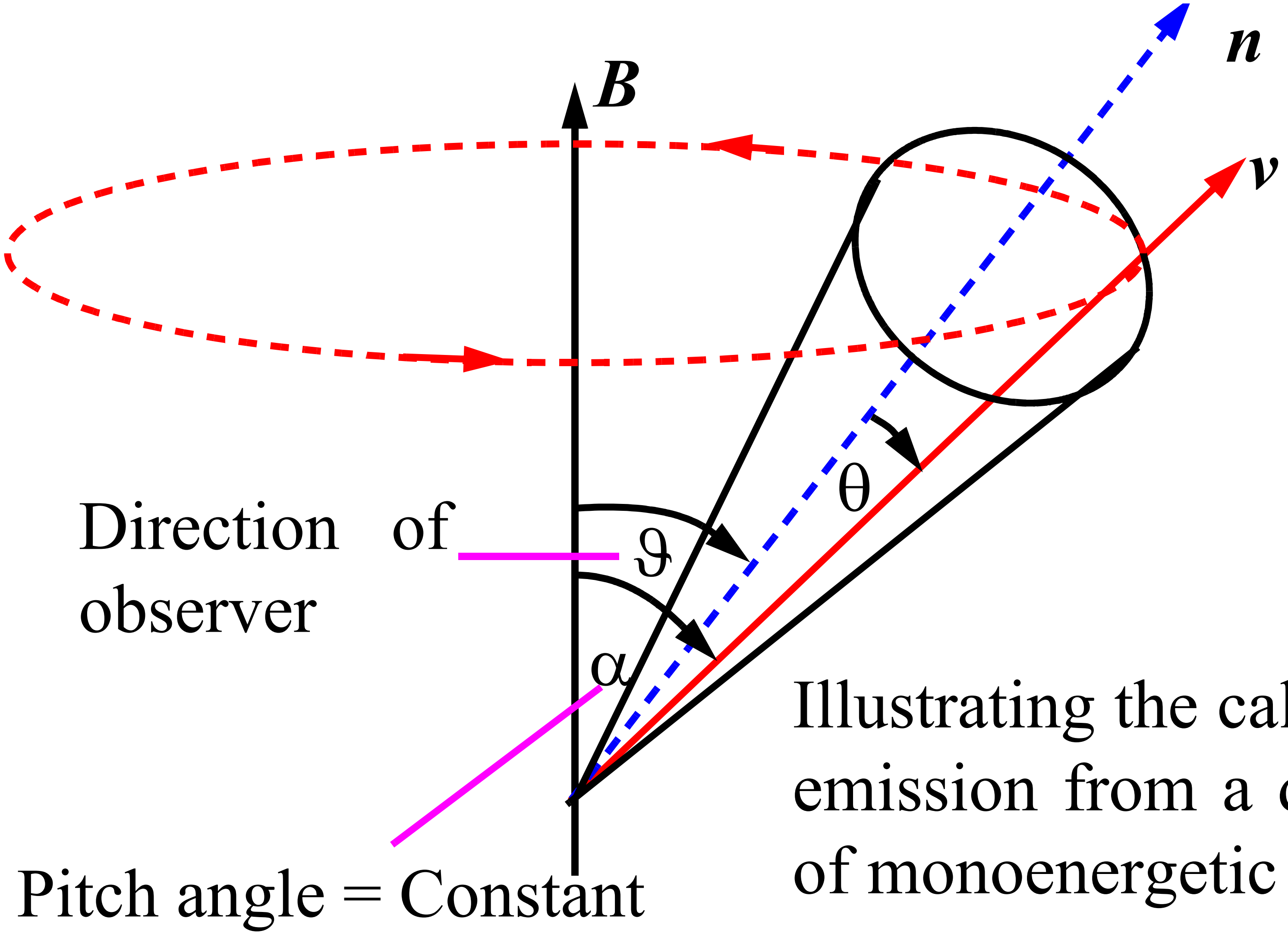
Synchrotron Radiation II

1 Synchrotron radiation from Astrophysical Sources.

1.1 Distributions of electrons

In this chapter we shall deal with synchrotron radiation from two different types of distribution of electrons. It is instructive to deal first with emission from a monoenergetic distribution of electrons. We use a similar diagram as we used for the calculation of the total emission from an electron.



Illustrating the calculation of emission from a distribution of monoenergetic electrons.

In this case however, the direction $\boldsymbol{n}$ is fixed but we are calculating the integrated emission from a number of different electrons with velocities which point near the direction of $\boldsymbol{n}$ but which are nevertheless different.

1.2 Pitch angle distribution

Although the particles are monoenergetic they may have different pitch angles, α. We take α and ϕ as the polar angles describing the direction of the velocity vector of an electron. We know that

$$\alpha = \text{constant} \qquad \phi = \Omega_B t + \phi_0$$

The corresponding elementary solid angle is

$$d\Omega_p = \sin\alpha \, d\alpha \, d\phi$$

Note that this is different from the solid angle we have been using for the radiation field.

Particle distribution in solid angle

The distribution of particles in direction, wrt the magnetic field is described by:

$$N(\boldsymbol{k}) d\Omega_p = N(\boldsymbol{k}) \sin\alpha \, d\alpha \, d\phi$$

= No density of particles with velocities pointing within solid angle $d\Omega_p$ about the direction $\boldsymbol{k}$

If the distribution of electrons is ***isotropic***, then $N(\boldsymbol{k})$ is constant and the number density is

$$\int_{\Omega} N(\boldsymbol{k}) d\Omega = 4\pi N(\boldsymbol{k}).$$

Hence

$$N(\boldsymbol{k}) = \frac{n}{4\pi}$$

$$\text{No of particles within ranges } d\alpha \text{ and } d\phi = \frac{n}{4\pi} \sin\alpha\, d\alpha\, d\phi$$

1.3 Calculation of emissivity

Outline

1. We wish to calculate the emission into a solid angle around the direction $\boldsymbol{n}$. This involves contributions from electrons of different pitch angles as illustrated. Electrons which have a pitch angle close to the direction of $\boldsymbol{n}$ will contribute more to the emission about $\boldsymbol{n}$ than electrons whose velocity points further away from $\boldsymbol{n}$.

2. The number *density* of electrons determines the number of electrons passing through the relevant direction per unit time, thence the number of pulses per unit time emitted by electrons in the direction of $\boldsymbol{n}$, thence the emissivity.

3. The radiation from different electrons moving with different pitch angles is integrated over pitch angle to determine the total emission into that solid angle.

No of pulses per unit time

In calculating the energy in a pulse emitted by a relativistic electron, we used as a parameter, θ, the minimum angle between the velocity vector and the direction to the observer, $\boldsymbol{n}$. Each particle has angular velocity

$$\frac{d\phi}{dt} = \Omega_B$$

where

$$\Omega_B = \frac{eB}{\gamma m_e} = \gamma^{-1}\Omega_0 \qquad \Omega_0 = \frac{eB}{m_e} = \text{Non-relativistic gyrofrequency}$$

Let us estimate the number of particles passing $\boldsymbol{n}$ within the range $\alpha \to \alpha + d\alpha$ in time dt. The range of ϕ corresponding to particles that are about to pass the direction $\boldsymbol{n}$ in time dt is

$$d\phi = \frac{d\phi}{dt}dt$$

Hence the number of particles passing $\boldsymbol{n}$ in time dt is

$$\frac{n}{4\pi}\sin\alpha d\alpha d\phi = \frac{n}{4\pi}\sin\alpha d\alpha\frac{d\phi}{dt}dt = \frac{n}{4\pi}\Omega_B\sin\alpha d\alpha dt$$

This is the number of pulses of radiation that are emitted from within this range of pitch angles in time dt and from the range of pitch angles $d\alpha$. Hence,

$$\text{No of pulses per unit time} = \frac{n}{4\pi}\Omega_B\sin\alpha d\alpha$$

Contribution of particles of a given pitch angle

The contribution to the emissivity from particles within the range $\alpha \to \alpha + d\alpha$ is:

$$dj_{\alpha\beta} = \frac{\text{Energy per unit frequency per}}{\text{unit solid angle per pulse}} \times \frac{\text{No of pulses}}{\text{per unit time}}$$

$$dj_{\alpha\beta} = \frac{n}{4\pi}\Omega_B \sin\alpha \frac{dW_{\alpha\beta}(\theta)}{d\Omega d\omega} d\alpha$$

We know that the important range of θ is $\sim 1/\gamma$ and that γ is usually large. Hence we put

$$\alpha = \vartheta + \theta \approx \vartheta \qquad d\alpha = d\theta$$

The emissivity is

$$j_{\alpha\beta} = \frac{n}{4\pi}\Omega_B\int_0^{\pi}\sin\alpha\frac{dW_{\alpha\beta}(\theta)}{d\Omega d\omega}d\alpha$$

$$\approx \frac{n}{4\pi}(\Omega_B\sin\vartheta)\int_{-\infty}^{\infty}\frac{dW_{\alpha\beta}(\theta)}{d\Omega d\omega}d\theta$$

The parallel and perpendicular emissivities are very similar to what we calculated before for the total single electron emission.

Since

$$\frac{dW_{\parallel}}{d\Omega d\omega} = \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2 (\theta_\gamma \gamma\theta)^2 K_{1/3}^2(\eta)$$

$$\frac{dW_{\perp}}{d\Omega d\omega} = \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2 \theta_\gamma^4 K_{2/3}^2(\eta)$$

then

$$j_{\|}(\omega) = \frac{n}{4\pi}(\Omega_B \sin\vartheta) \times \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2$$

$$\times \int_{-\infty}^{\infty} (\theta_\gamma \gamma\theta)^2 K_{1/3}^2(\eta) d\theta$$

$$j_{\perp}(\omega) = \frac{n}{4\pi}(\Omega_B \sin\vartheta) \times \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2$$

$$\times \int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta) d\theta$$

We have given the results for the integrals already:

$$\int_{-\infty}^{\infty} (\gamma\theta)^2 \theta_\gamma^2 K_{1/3}^2(\eta) d(\gamma\theta) = \frac{\pi}{\sqrt{3}} x^{-2} [F(x) - G(x)]$$

$$\int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta) d(\gamma\theta) = \frac{\pi}{\sqrt{3}} x^{-2} [F(x) + G(x)]$$

$$x = \frac{\omega}{\omega_c}$$

Collecting terms and factors, using

$$a = \frac{c}{\Omega_B \sin\alpha} = \frac{c\gamma}{\Omega_0 \sin\alpha} \qquad \text{where} \qquad \Omega_0 = \frac{eB}{m_e}$$

$$\omega_c = \frac{3}{2}\Omega_0\gamma^2 \sin\alpha$$

we obtain the result:

$$j_{\perp,\,\|}(\omega) = \frac{\sqrt{3}ne^2}{64\pi^3\varepsilon_0 c}(\Omega_0 \sin\vartheta)[F(x) \pm G(x)]$$

2 Polarisation of monoenergetic electron emissivity

2.1 Stokes parameters

To a good approximation the integrated circular polarisation of synchrotron emission is zero because of the approximately equal contributions from $\theta < 0$ and $\theta > 0$.

Remember that (see chapter on radiation field):

$$j_I(\omega) = j_\perp(\omega) + j_\parallel(\omega)$$

$$j_Q(\omega) = j_\perp(\omega) - j_\parallel(\omega)$$

Hence

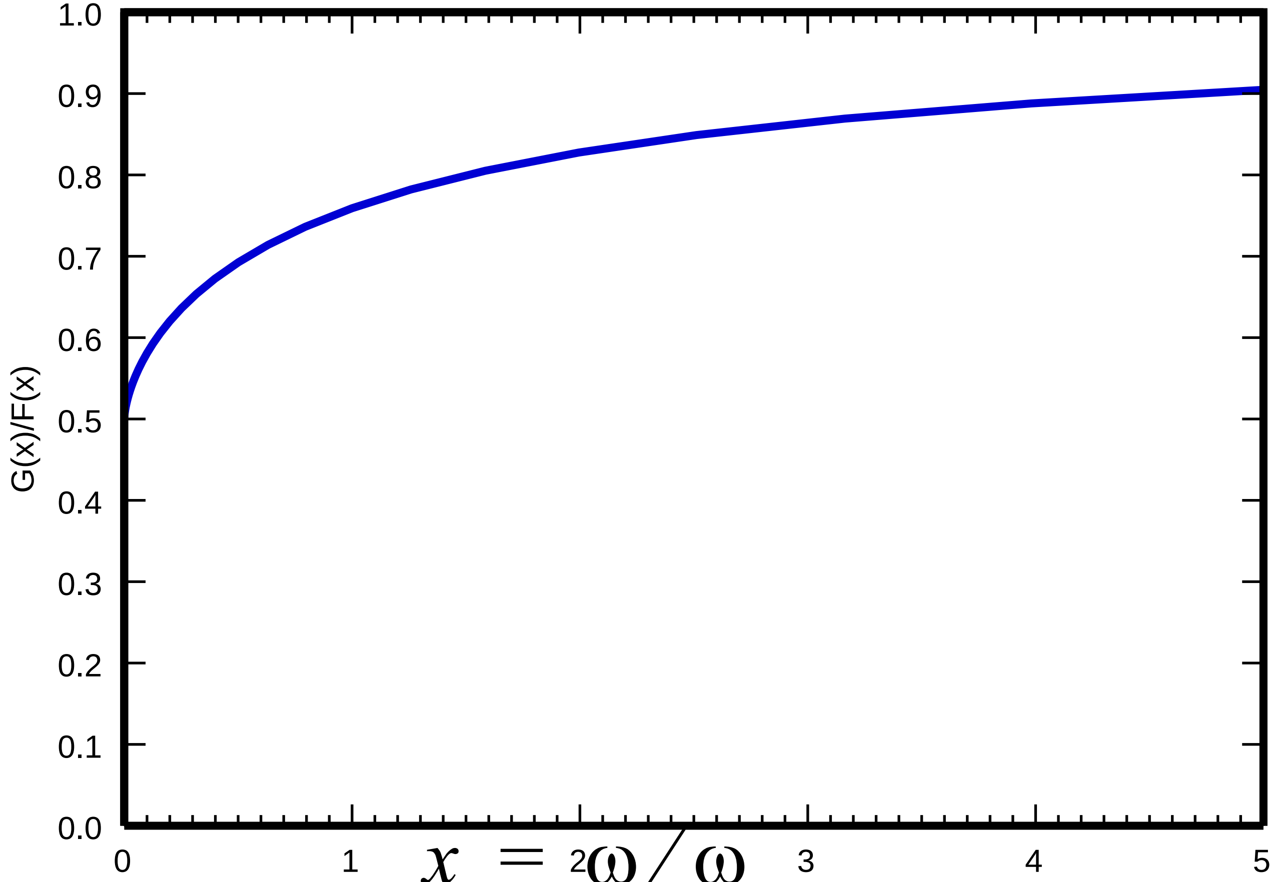
$$j_I(\omega) = \frac{\sqrt{3}ne^2}{32\pi^3\varepsilon_0 c}(\Omega_0 \sin\vartheta)F(x)$$

$$j_Q(\omega) = \frac{\sqrt{3}ne^2}{32\pi^3\varepsilon_0 c}(\Omega_0 \sin\vartheta)G(x)$$

It is readily shown, under the assumptions that we have introduced that $j_U(\omega) \approx 0$ (exercise).

The fractional polarisation of the emissivity (the intrinsic polarisation) is

$$r(\omega) = \frac{j_Q(\omega)}{j_I(\omega)} = \frac{G(x)}{F(x)}$$



$x = \omega / \omega_c$

The function $(G(x))/(F(x))$ is plotted at left. Note that the polarisation is very high (greater than 50%) and approaches unity as $x \to \infty$.

2.2 Direction of polarisation

Since

$$j_V(\omega) = j_U(\omega) = 0$$

the radiation is linearly polarised with the major axis of the polarisation ellipse in the direction of $\boldsymbol{e}_{\perp}$, i.e. in the direction perpendicular to the projection of the magnetic field onto the plane of propagation.

We shall revisit this topic later after discussing emission from a power-law distribution of electrons.

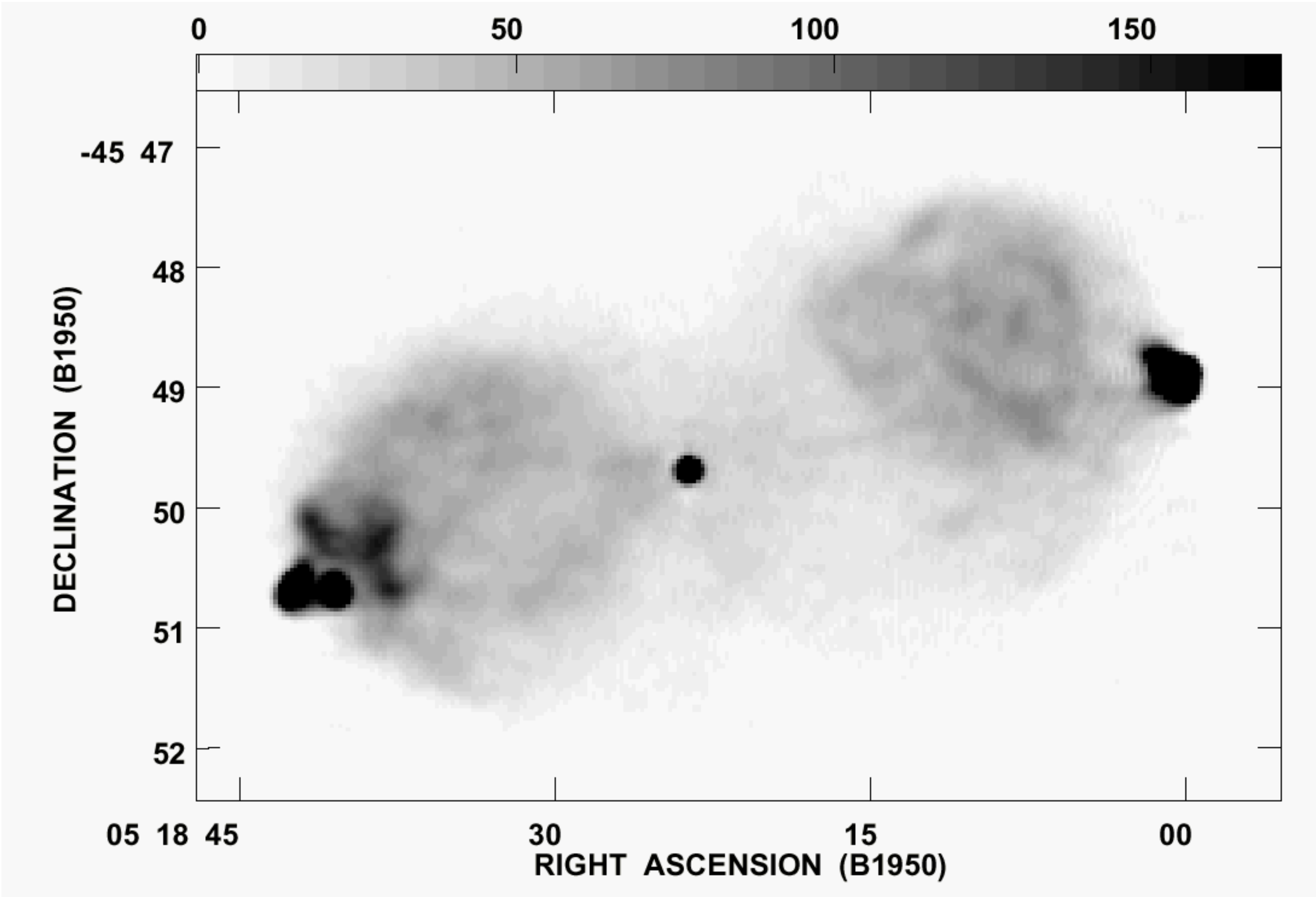
3 Emission from a power-law distribution of electrons

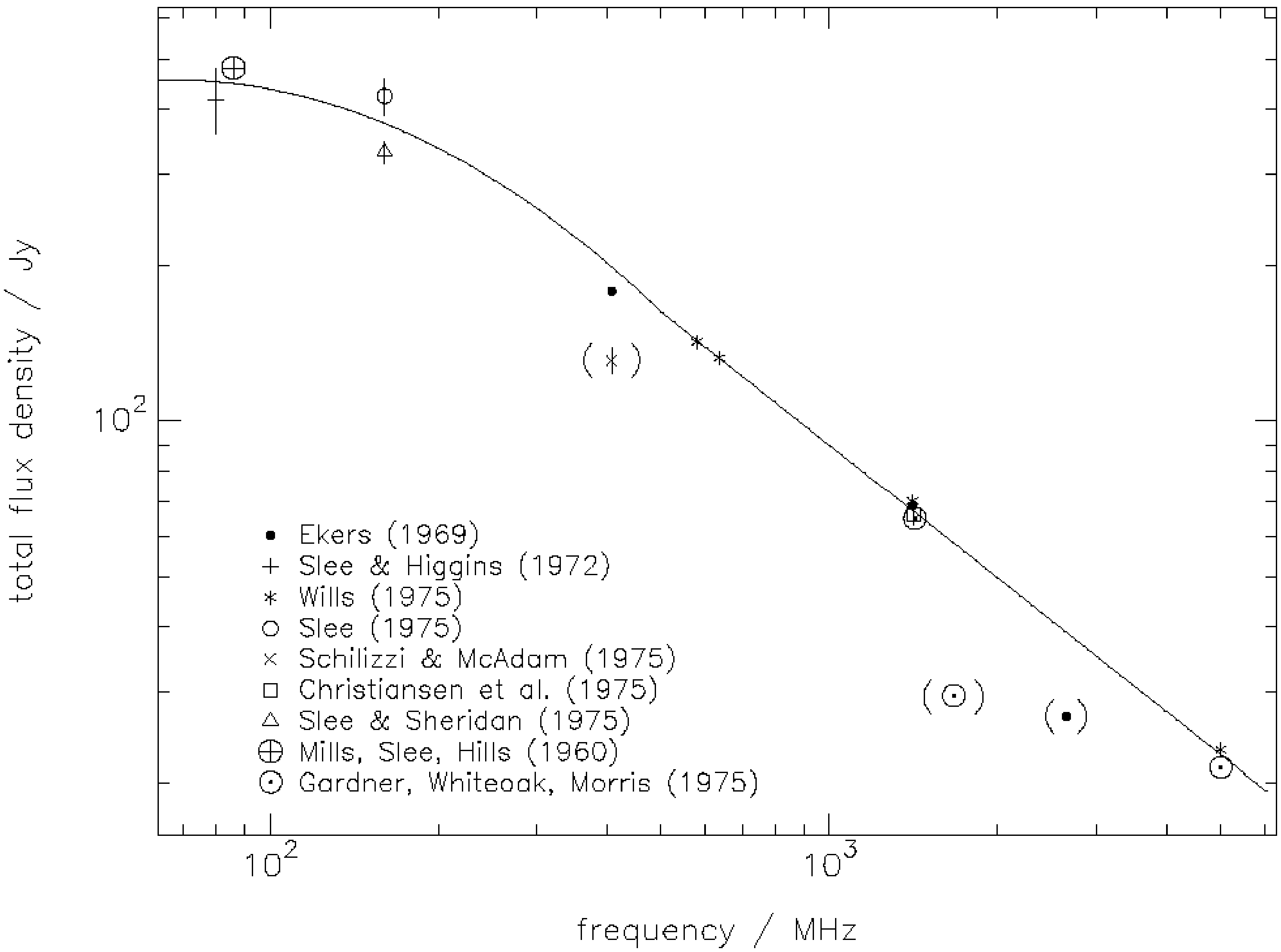
In many astrophysical environments, the spectrum of synchrotron radiation is a power-law over a large region in frequency, i.e. the flux density is well approximated by

$$F_\nu \propto \nu^{-\alpha}$$

where α is called the spectral index. (Often the opposite convention $F_\nu \propto \nu^{\alpha}$ is used. The reason for the above convention is that most optically thin nonthermal spectra have $\alpha > 0$ when α is defined using the first convention.)

Example: Integrated flux density of Pictor A





total flux density / Jy
10^2
frequency / MHz
10^2
10^3
Ekers (1969)
Slee & Higgins (1972)
Wills (1975)
Slee (1975)
Schilizzi & McAdam (1975)
Christiansen et al. (1975)
Slee & Sheridan (1975)
Mills, Slee, Hills (1960)
Gardner, Whiteoak, Morris (1975)

Interpretation

The power-law spectra of many sources is conventionally interpreted as synchrotron radiation from an ensemble of electrons with a power-law distribution in energy. Sometimes this is represented by

$$N(E) = CE^{-a} \qquad E_1 < E < E_2$$

where a is the electron spectral index and $N(E)dE$ represents the number density of electrons with energies between E and $E + dE$. The energies E_1 and E_2 are the lower and upper cut-off energies.

Because of the direct connection between Lorentz factor and frequency of emission, we shall use

$$N(\gamma) = K\gamma^{-a} \qquad \gamma_1 < \gamma < \gamma_2$$

where K has the dimensions of number density. $N(\gamma)d\gamma$ is the number density of electrons with Lorentz factors between γ and $\gamma + d\gamma$ and γ_1 and γ_2 are the lower and upper cutoff Lorentz factors. Sometimes however, we wish to relate the two and putting

$$N(E)dE = N(\gamma)d\gamma \Rightarrow N(E) = N(\gamma)\frac{d\gamma}{dE} = \frac{1}{mc^2}N(\gamma)$$

Usually $m = m_e$, i.e. we normally only consider synchrotron radiation from relativistic electrons. However, all of the formulae are relevant for radiation from charged particles of any mass. For a power-law distribution of particles:

$$N(E) = \frac{K}{mc^2}\left(\frac{E}{mc^2}\right)^{-a} = K(mc^2)^{a-1}E^{-a}$$

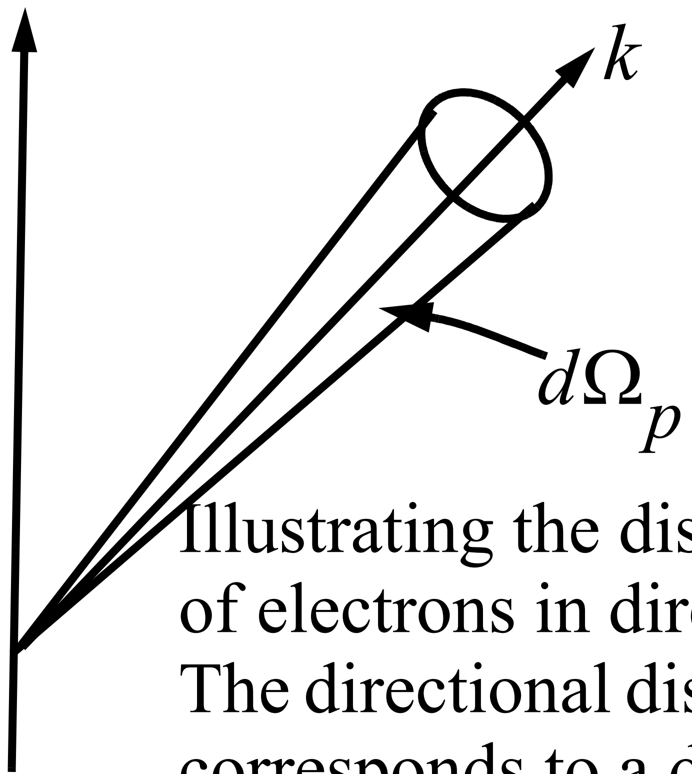
$$\Rightarrow C = K(mc^2)^{a-1}$$

with the cutoff energies and Lorentz factors related by:

$$E_1 = \gamma_1 mc^2 \qquad E_2 = \gamma_2 mc^2$$

Such power-law distributions are *nonthermal*. The distribution of particle energies is not that described by say a Maxwellian distribution described by a fixed temperature. This is still the case for a relativistic Maxwellian distribution.

3.1 Isotropic distributions



Illustrating the distribution of electrons in direction. The directional distribution corresponds to a distribution in pitch angle.

We generally assume that the electron distribution is isotropic. We take

$$N(\gamma, \boldsymbol{k})\, d\gamma\, d\Omega$$

as the number density of electrons within $d\gamma$ about γ and within solid angle $d\Omega_p$ about the direction $\boldsymbol{k}$. Then the distribution is isotropic if $N(\gamma, \boldsymbol{k})$ is independent of $\boldsymbol{k}$. In this case,

$$N(\gamma, \boldsymbol{k}) = \frac{1}{4\pi} N(\gamma)$$

since the integral of $N(\gamma, \boldsymbol{k})$ over all solid angle gives $N(\gamma)$.

$$\int_{4\pi} N(\gamma, \boldsymbol{k}) d\Omega = \frac{1}{4\pi} \int_{4\pi} N(\gamma) d\Omega = N(\gamma)$$

Monoenergetic -> Distribution

Given the way that we defined a monoenergetic distribution, then the transition from monoenergetic to a distribution in energy is effected by:

$$N(\boldsymbol{k}) \rightarrow N(\gamma, \boldsymbol{k}) d\gamma$$

3.2 Integral quantities

Number density

The total particle number density is determined from:

$$n = \int_{\gamma_1}^{\gamma_2} N(\gamma) d\gamma = K \int_{\gamma_1}^{\gamma_2} \gamma^{-a} d\gamma$$

where γ_1 and γ_2 are respectively the lower and upper limits to the electron distribution. In many sources we do not really know the value of γ_1 and we only have an approximate idea of γ_2. However, many properties of the radiation do not depend upon these quantities.

The above integral is easy:

$$n = \frac{K}{(a-1)}[\gamma_1^{-(a-1)} - \gamma_2^{-(a-1)}]$$

$$= \frac{K}{a-1}\gamma_1^{-(a-1)}\left[1 - \left(\frac{\gamma_2}{\gamma_1}\right)^{-(a-1)}\right]$$

$$\approx \frac{K}{a-1}\gamma_1^{-(a-1)}$$

The form of the integral we have taken here is for $a > 1$. Usually (but not always; the Galactic Centre is an exception) $a > 2$. For $a > 1$ the number density is dominated by low energy particles. For $\gamma_2 \gg \gamma_1$, the high energy part of the expression can be neglected.

Energy density

$$\varepsilon = \text{Energy density} = K\int_{\gamma_1}^{\gamma_2} \gamma^{-a} \times \gamma m c^2 d\gamma$$

$$= Kmc^2 \int_{\gamma_1}^{\gamma_2} \gamma^{1-a} d\gamma = \frac{Kmc^2}{a-2}[\gamma_1^{-(a-2)} - \gamma_2^{-(a-2)}]$$

$$= \frac{Kmc^2}{a-2}\gamma_1^{-(a-2)}\left[1 - \left(\frac{\gamma_2}{\gamma_1}\right)^{-(a-2)}\right] \qquad \text{for } a > 2$$

$$= Kmc^2 \ln\left(\frac{\gamma_2}{\gamma_1}\right) \qquad \text{for } a = 2$$

For $a > 2$ (the usual case), the energy density is also dominated by the lowest energy particles. When $a \le 2$, the energy diverges as $\gamma_2 \to \infty$ so that a cutoff at upper energies is required, in this case. Theoretically, for $a > 2$, the distribution of energies can extend to infinity. However, it seldom does because of radiative losses considered later.

The tip of the iceberg

In principle, nonthermal distributions can extend down to $\gamma \sim 1 - 10$, whereas what we observe via radio astronomy (or higher energy bands) corresponds to $\gamma > 10^4$ or thereabouts. Hence radio astronomy samples the "tip of the iceberg".

Pressure

The pressure of a relativistic gas satisfies

$$p = \frac{1}{3}\varepsilon$$

4 Synchrotron emission from a power-law

4.1 Some general points

The integrated emission from a power-law distribution of electrons follows straightforwardly from the expressions derived for a monoenergetic electron distribution.

We use the correspondence:

$$\frac{dn}{d\Omega_p} \to N(\gamma, \boldsymbol{k})d\gamma$$

$$n \to N(\gamma)d\gamma$$

4.2 Isotropic distribution of electrons

For an isotropic distribution, $N(\gamma, \boldsymbol{k}) = \frac{1}{4\pi}N(\gamma)$ and we use the expression for a monoenergetic distribution

$$j_{\perp, \parallel}(\omega) = \frac{\sqrt{3}ne^2}{64\pi^3\varepsilon_0 c}(\Omega_0 \sin\vartheta)[F(x) \pm G(x)]$$

with $n \rightarrow N d\gamma$ and integrating over Lorentz factor.

This gives:

$$j_{\perp,\,\parallel}(\omega) = \frac{\sqrt{3}}{64\pi^3\varepsilon_0 c}\frac{e^2(\Omega_0 \sin\vartheta)}{m_e}\int_{\gamma_1}^{\gamma_2}[F(x) \pm G(x)]N(\gamma)d\gamma$$

We now change the variable of the integration from γ to $x = \omega/\omega_c$.

Recall our expression for the critical frequency:

$$\omega_c = \frac{3}{2}(\Omega_0 \sin\vartheta)\gamma^2$$

$$\Rightarrow \gamma^2 = \frac{2}{3}(\Omega_0 \sin\vartheta)^{-1}\omega_c = \frac{2}{3}\left(\frac{\omega}{\Omega_0 \sin\vartheta}\right)\left(\frac{\omega}{\omega_c}\right)^{-1}$$

$$= \left(\frac{2\omega}{3\Omega_0 \sin\vartheta}\right)x^{-1}$$

Therefore,

$$\gamma = \left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{1/2} x^{-1/2}$$

$$\Rightarrow d\gamma = -\frac{1}{2}\left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{1/2} x^{-3/2} dx$$

Hence,

$$j_{\perp,\,\parallel}(\omega) = \frac{\sqrt{3}}{128\pi^3}\left(\frac{e^2}{\varepsilon_0 c}\right)(\Omega_0 \sin\vartheta)\left(\frac{2\omega}{3\Omega_0 \sin\vartheta}\right)^{1/2}$$

$$\times \int_{x_2}^{x_1} x^{-3/2}[F(x) \pm G(x)]N[\gamma(x)]dx$$

Collecting terms:

$$j_{\perp,\parallel}(\omega) = \left(\frac{2^{1/2}}{128\pi^3}\right)\left(\frac{e^2}{\varepsilon_0 c}\right)(\Omega_0 \sin\vartheta)\left(\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{1/2}$$

$$\times \int_{x_2}^{x_1} x^{-3/2}[F(x) \pm G(x)]N[\gamma(x)]dx$$

The limits of integration are determined by the values of x corresponding to the upper and lower Lorentz factors:

$$x_{1,2} = \left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\theta}\right)\gamma_{1,2}^{-2}$$

Note that $x_2 < x_1$ because of the dependence of x on the inverse square.

This expression is valid for any isotropic distribution of electrons. We now take a power-law distribution:

$$N(\gamma) = K\gamma^{-a} = K\left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{-a/2} x^{a/2}$$

so that

$$j_{\perp,\parallel}(\omega) = \frac{\sqrt{3}}{128\pi^3}\left(\frac{e^2}{\varepsilon_0 c}\right)K(\Omega_0 \sin\vartheta)\left(\frac{2}{3}\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{-\frac{(a-1)}{2}}$$

$$\times \int_{x_2}^{x_1} x^{\frac{a-3}{2}}[F(x) \pm G(x)]dx$$

$$= \frac{3^{\frac{a}{2}} 2^{-\frac{(a-1)}{2}}}{128\pi^3}\left(\frac{e^2}{\varepsilon_0 c}\right)K(\Omega_0 \sin\vartheta)^{\frac{a+1}{2}}\omega^{-\frac{(a-1)}{2}}$$

$$\times \int_{x_2}^{x_1} x^{\frac{a-3}{2}}[F(x) \pm G(x)]dx$$

The integrand involves a power multiplied by the functions $F(x)$ and $G(x)$.

To begin with, we assume that the frequency of interest is such that the values of x_1 and x_2 are well outside the region $x \sim 1$ which dominates the integral. Consider

$$x_1 = \frac{\omega}{\omega_c(\gamma_1)} \qquad x_2 = \frac{\omega}{\omega_c(\gamma_2)}$$

Hence, if $\omega \gg \omega_c(\gamma_1)$ then $x_1 \to \infty$ and if $\omega \ll \omega_c(\gamma_2)$, then $x_2 \to 0$. That is, if the frequency of interest is well inside the the range of critical frequencies defined by γ_1 and γ_2 then we may take $x_2 = 0$ and $x_1 = \infty$.

We then use the following results (derived from the integral properties of Bessel functions):

$$\int_0^{\infty} x^{\mu} F(x) dx = \frac{2^{\mu+1}}{\mu+2} \Gamma\left(\frac{\mu}{2} + \frac{7}{3}\right) \Gamma\left(\frac{\mu}{2} + \frac{2}{3}\right)$$

$$\int_0^{\infty} x^{\mu} G(x) dx = 2^{\mu} \Gamma\left(\frac{\mu}{2} + \frac{4}{3}\right) \Gamma\left(\frac{\mu}{2} + \frac{2}{3}\right)$$

In the integrals we have encountered above,

$$\mu = \frac{a-3}{2} \qquad \mu + 1 = \frac{a-1}{2} \qquad \mu + 2 = \frac{a+1}{2}$$

$$\frac{\mu}{2} + \frac{7}{3} = \frac{a}{4} + \frac{19}{12} \qquad \frac{\mu}{2} + \frac{2}{3} = \frac{a}{4} - \frac{1}{12} \qquad \frac{\mu}{2} + \frac{4}{3} = \frac{a}{4} + \frac{7}{12}$$

4.3 Total emissivity

The total emissivity is:

$$
\begin{aligned}
j(\omega) &= j_{\perp}(\omega) + j_{\|}(\omega) \\
&= \frac{3^{\frac{a}{2}} 2^{-\frac{(a-1)}{2}}}{64\pi^3} \left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{\frac{a+1}{2}} \omega^{-\frac{(a-1)}{2}} \\
&\quad \times \int_0^{\infty} x^{\frac{a-3}{2}} F(x) dx
\end{aligned}
$$

Using the above integral relations and a little bit of algebra:

$$j(\omega) = \frac{3^{a/2}}{32\pi^3} \frac{\Gamma\left(\frac{a}{4} + \frac{19}{12}\right)\Gamma\left(\frac{a}{4} - \frac{1}{12}\right)}{a+1} \left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{\frac{(a+1)}{2}}$$

$$\times \omega^{-\frac{(a-1)}{2}}$$

4.4 Spectral index

One immediately obvious feature of the above expression is the spectral index of the emission, given by the exponent of ω, i.e.

$$\alpha = \frac{a-1}{2} \Rightarrow a = 2\alpha + 1$$

Hence a value of $a = 2$ corresponds to a spectral index of 0.5. Typically spectral indices in synchrotron sources are about $0.6 - 0.7$ corresponding to values of $a \approx 2.2 - 2.4$. In these cases, we do not need to worry about the divergence of expressions for the energy density if we take $\gamma_2 = \infty$.

4.5 Polarisation

The emissivity in Stokes Q is given by

$$
\begin{aligned}
j_Q(\omega) &= j_\perp(\omega) - j_{||}(\omega) \\
&= \frac{3^{\frac{a}{2}} 2^{-\frac{(a-1)}{2}}}{64\pi^3} \left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{\frac{(a+1)}{2}} \omega^{-\frac{(a-1)}{2}} \\
&\quad \times \int_{-\infty}^{\infty} x^{\frac{a-3}{2}} G(x) dx
\end{aligned}
\qquad 88
$$

The only difference between this expression and that for the total emissivity is the integral; all of the leading terms are the same. Hence, the fractional polarisation is:

$$q(\omega) = \frac{j_Q}{j_I} = \frac{\int_{-\infty}^{\infty} x^{\frac{a-3}{2}} G(x)dx}{\int_{-\infty}^{\infty} x^{\frac{a-3}{2}} F(x)dx}$$

Using the above expressions for the integrals:

$$q = \frac{\mu+2}{2}\frac{\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)}{\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{2}{3}\right)}$$

The Γ function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

and satisfies the recurrence relation

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

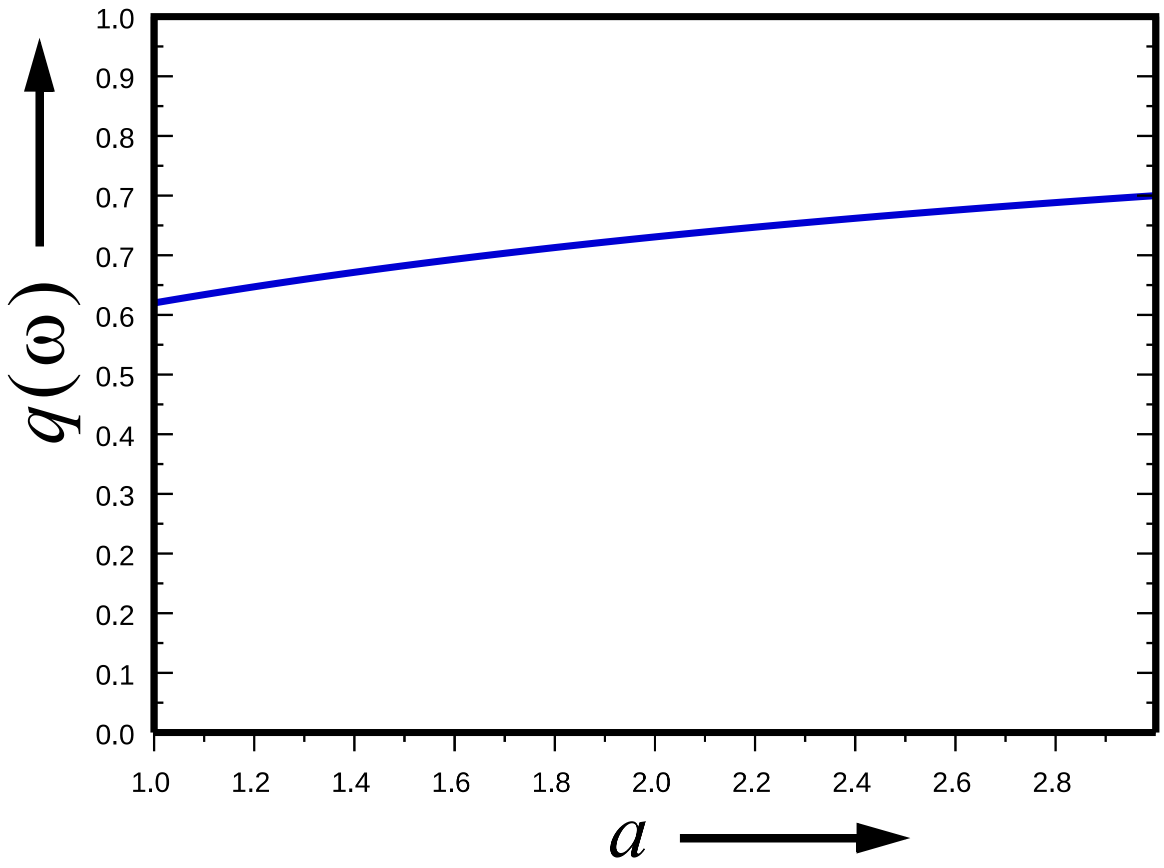
Hence,

$$\Gamma\left(\frac{\mu}{2}+\frac{7}{3}\right) = \left(\frac{\mu}{2}+\frac{4}{3}\right)\Gamma\left(\frac{\mu}{2}+\frac{4}{3}\right)$$

and

$$q(\omega) = \frac{\mu+2}{\mu+\frac{2}{3}} = \frac{a+1}{a+7/3}$$

Three features of synchrotron polarisation:



1. The polarisation is frequency independent.

2. The major axis of the polarisation ellipse is perpendicular to the projection of the magnetic field on the sky.

3. The polarisation is high – about 69% for $a = 2$. This plot represents polarisation as a function of a.

5 Simplified expression for j_ν

5.1 Standard expression

We have the expression for the total emissivity $j(\omega)$:

$$j(\omega) = \frac{3^{a/2}}{32\pi^3} \frac{\Gamma\left(\frac{a}{4}+\frac{19}{12}\right)\Gamma\left(\frac{a}{4}-\frac{1}{12}\right)}{a+1}$$

$$\times \left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)\left(\frac{\omega}{\Omega_0 \sin\vartheta}\right)^{-\frac{(a-1)}{2}}$$

which we write in the following way in order to separate the physical parameters from simple numerical ones:

$$j(\omega) = \frac{3^{a/2}}{32\pi^3} \frac{\Gamma\left(\frac{a}{4}+\frac{19}{12}\right)\Gamma\left(\frac{a}{4}-\frac{1}{12}\right)}{a+1}$$

$$\times \left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{\frac{a+1}{2}} \omega^{-\frac{(a-1)}{2}}$$

Notes

1. The magnetic field dependence enters through $\Omega_0 = \frac{eB}{m}$

2. The particle density through K ($N(\gamma) = K\gamma^{-a}$).

Emissivity in terms of linear frequency

For relation to observations we require $j_\nu = 2\pi j(\omega)$ and we also put $\omega = 2\pi\nu$ in the above expression. This gives:

$$j_\nu = C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}}$$

$$= C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K(\Omega_0 \sin\vartheta)^{1+\alpha} \nu^{-\alpha}$$

where:

$$C_1(a) = 3^{a/2} 2^{-\frac{(a+7)}{2}} \pi^{-\frac{(a+3)}{2}} \frac{\Gamma\left(\frac{a}{4}+\frac{19}{12}\right)\Gamma\left(\frac{a}{4}-\frac{1}{12}\right)}{a+1}$$

5.2 Randomly oriented magnetic field

An approximation which is often used for synchrotron sources is that the magnetic field is randomly oriented (or "tangled") along the line of sight through the source. This is a justifiable approximation if the mean direction of the magnetic field varies significantly along the line of sight or if the plasma is highly turbulent so that the magnetic field direction changes direction significantly from point to point.

For a magnetic field which varies in direction we compute an ensemble-averaged value of the emissivity as follows. The magnetic field direction enters through the factor $\sin^{(a+1)/2}\vartheta$ in the expression for the emissivity. We take ϑ as a random variable and compute the mean over solid angle:

$$\langle \sin^{(a+1)/2}\vartheta \rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \sin^{\frac{a+1}{2}}\vartheta \sin\vartheta \, d\vartheta \, d\phi$$

$$= \frac{1}{2}\int_0^{\pi} \sin^{\frac{a+3}{2}}\vartheta \, d\vartheta$$

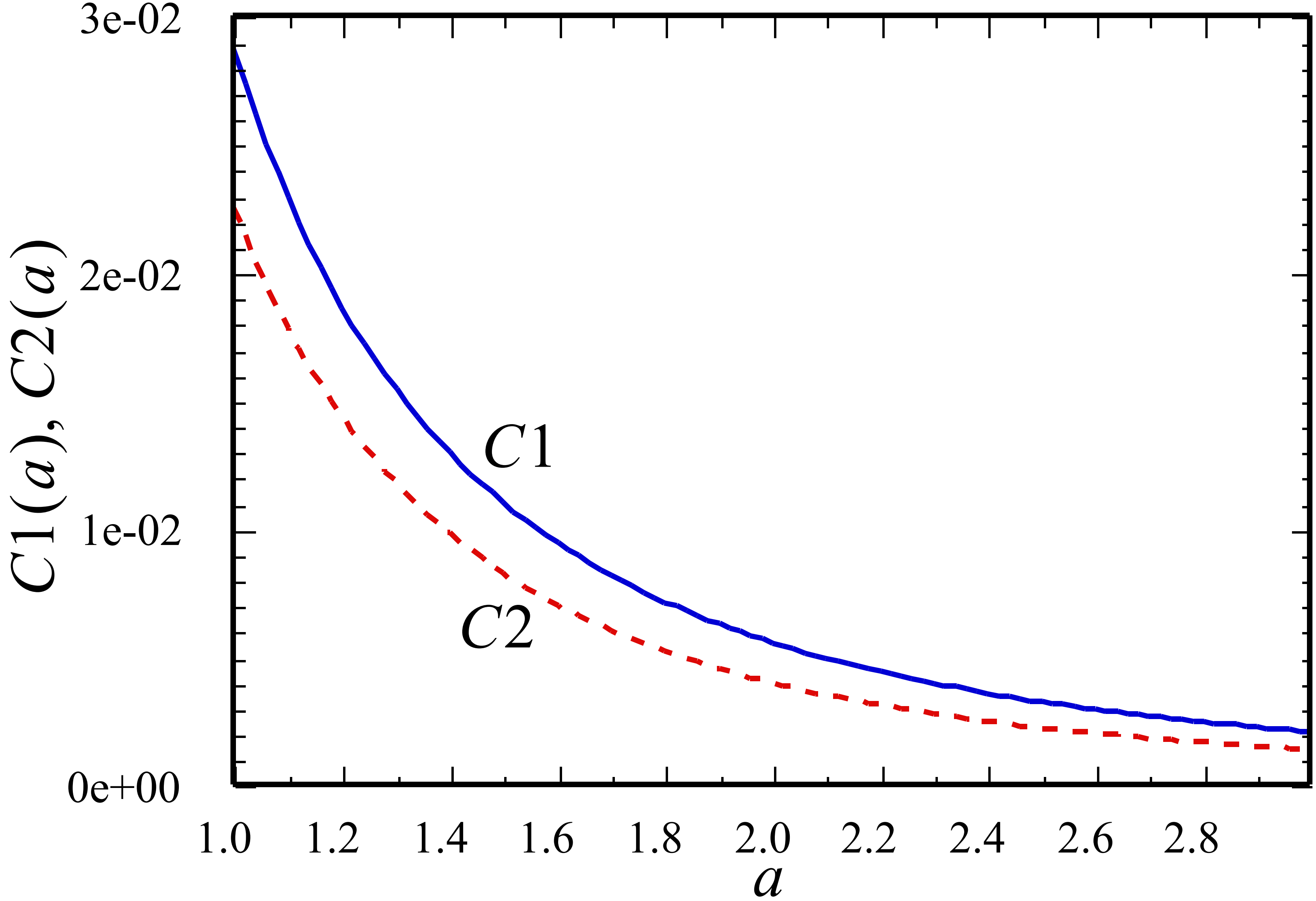
$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{5+a}{4}\right)}{\Gamma\left(\frac{7+a}{4}\right)}$$

The angle-averaged emissivity for a randomly inclined magnetic field is given by:

$$\langle j_\nu \rangle = C_2(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K \Omega_0^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}}$$

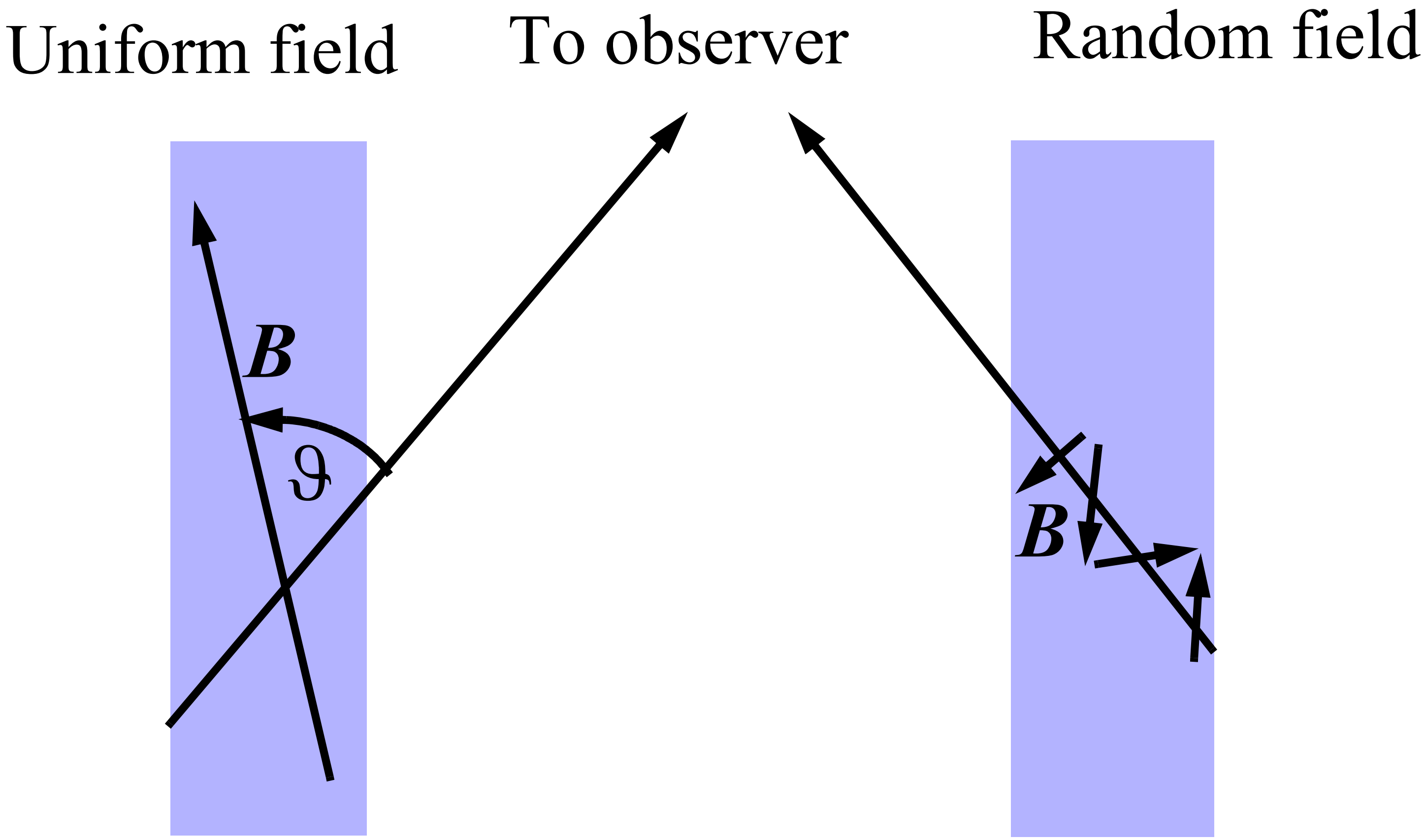
$$C_2(a) = C_1(a) \times \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{5+a}{4}\right)}{\Gamma\left(\frac{7+a}{4}\right)}$$

$$= \frac{3^{a/2}}{2^{\left(\frac{a+13}{2}\right)} \pi^{\left(\frac{a+2}{2}\right)}} \frac{\Gamma\left(\frac{a}{4}+\frac{1}{4}\right)\Gamma\left(\frac{a}{4}+\frac{19}{12}\right)\Gamma\left(\frac{a}{4}-\frac{1}{12}\right)}{\Gamma\left(\frac{a}{4}+\frac{7}{4}\right)}$$


3e-02
2e-02
1e-02
0e+00
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
a
C1(a), C2(a)
C1
C2

6 Models for a synchrotron-emitting plasma

6.1 Slab model



The diagram illustrates the different approximations we use for radiative transfer in a synchrotron-emitting region. When the emission is optically thin:

$$I_\nu = \int_0^L j_\nu ds \approx j_\nu L$$

the latter approximation being valid for a region with approximately constant properties.

For a constant magnetic field direction:

$$I_\nu \approx C_1(a)\left(\frac{e^2}{\varepsilon_0 c}\right) KL(\Omega_0 \sin\vartheta)^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}}$$

For a random magnetic field:

$$I_\nu \approx C_2(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K L \Omega_0^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}}$$

7 Estimation of magnetic fields and number densities

7.1 Minimum energy density and minimum energy – the last resort of rogues and scoundrels

The surface brightness or total power alone does not give us enough information to determine the parameters of a source – specifically the number density of emitting particles and the magnetic field. However, one can minimise the total energy density subject to the constraints provided by the synchrotron emission to get an estimate of the minimum energy in particles and field. The corresponding values of energy density in particles and the magnetic field have often been used in research as estimates for the source parameters, albeit with little detailed justification. One needs another independent diagnostic to reliably determine plasma parameters. This is provided by the observation of inverse Compton emission – to be considered later.

7.2 Minimum energy density from surface brightness observations

Take the expression for the surface brightness of a plasma with an embedded random magnetic field:

$$I_\nu \approx C_2(a)\left(\frac{e^2}{\varepsilon_0 c}\right) K L \Omega_0^{\frac{a+1}{2}} \nu^{-\frac{(a-1)}{2}}$$

The parameter K can be related to the relativistic energy density by:

$$\varepsilon_e = \frac{K m_e c^2}{a-2} \gamma_1^{-(a-2)} \left[1 - \left(\frac{\gamma_2}{\gamma_1} \right)^{-(a-2)} \right]$$

$$\rightarrow \frac{K m_e c^2}{a-2} \gamma_1^{-(a-2)} \quad \text{for} \quad \gamma_2 \gg \gamma_1$$

$$\Rightarrow \left(\frac{\varepsilon}{m_e c^2} \right) = K f(a, \gamma_1, \gamma_2)$$

$$f(a, \gamma_1, \gamma_2) = (a-2)^{-1} \gamma_1^{-(a-2)} \left[1 - \left(\frac{\gamma_2}{\gamma_1} \right)^{-(a-2)} \right]$$

Hence K is expressed in terms of the electron density by:

$$K = \frac{\varepsilon_e}{m_e c^2} f^{-1}(a, \gamma_1, \gamma_2)$$

We allow for the possible presence of other particles by taking the total particle density

$$\varepsilon_p = (1 + c_E)\varepsilon_e$$

Notes

1. For an electron/positron plasma $c_E = 0$, since the surface brightness and linear polarisation are the same for positrons as for electrons.

2. If there are relativistic protons in the plasma, c_E could be of order 100.

3. Thermal plasma can also contribute to the particle energy density. This can have other observable consequences through internal Faraday depolarisation.

4. In extragalactic radio sources we frequently take $c_E \approx 0, 1$.

5. Supernova remnants are frequently regarded as the source of cosmic rays in the galaxy. The percentage of relativistic electrons in cosmic rays is typically 1%. Therefore, for supernova remnants, $c_E \sim 100$ may be appropriate.

Back to calculation:

The total energy density is:

$$\varepsilon_{\mathrm{tot}} = \varepsilon_p + \frac{B^2}{2\mu_0} = (1 + c_E)\varepsilon_e + \frac{B^2}{2\mu_0}$$

For a given surface brightness, the electron energy density is a function of the magnetic field. Differentiating the total energy density with respect to the magnetic field to determine the parameters of the minimum calculation:

$$\frac{\partial \varepsilon_{\mathrm{tot}}}{\partial B} = \frac{\partial \varepsilon_p}{\partial B} + \frac{B}{\mu_0} = 0$$

$$\Rightarrow \frac{\partial \varepsilon_p}{\partial B} = -\frac{B}{\mu_0}$$

Now consider the equation for the surface brightness in the form:

$$I_\nu = C_2\left(\frac{e^2}{\varepsilon_0 c}\right)\left(\frac{\varepsilon_e}{m_e c^2}\right) f^{-1}(a, \gamma_1, \gamma_2)\Omega_0^{\frac{a+1}{2}} L \nu^{-\frac{(a-1)}{2}}$$

Solving for the electron energy density times a power of Ω_0:

$$\left(\frac{\varepsilon_e}{mc^2}\right)\Omega_0^{\frac{a+1}{2}} = C_2^{-1}(a)\left(\frac{e^2}{\varepsilon_0 c}\right)^{-1}\left(\frac{I_\nu \nu^\alpha}{L}\right) f(a, \gamma_1, \gamma_2)$$

The particle energy density is just $(1+c_E)$ times ε_e so that:

$$\left(\frac{\varepsilon_p}{mc^2}\right)\Omega_0^{\frac{a+1}{2}} = (1+c_E)C_2^{-1}(a)\left(\frac{e^2}{\varepsilon_0 c}\right)^{-1}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a,\gamma_1,\gamma_2)$$

The right hand side of the last equation is independent of B. Taking logs and using $\Omega_0 = eB/m_e$, gives:

$$\ln\varepsilon_p + \frac{a+1}{2}\ln B = \ln(\text{Parameters independent of } B)$$

$$\Rightarrow \frac{1}{\varepsilon_p}\frac{\partial\varepsilon_p}{\partial B} + \frac{a+1}{2}\times\frac{1}{B} = 0$$

Since,

$$\frac{\partial \varepsilon_p}{\partial B} = -\frac{B}{\mu_0}$$

then

$$-\frac{1}{\varepsilon_p}\frac{B}{\mu_0} + \frac{a+1}{2} \times \frac{1}{B} = 0$$

$$\Rightarrow \varepsilon_p = \left(\frac{4}{a+1}\right)\frac{B^2}{2\mu_0}$$

For the values of a which are normally relevant, the particle energy density is comparable to the magnetic energy density, i.e. the particles and field are close to equipartition. Exact equipartition is represented by

$$\varepsilon_p = \frac{B^2}{2\mu_0}$$

Minimum energy parameters

Since,

$$\left(\frac{\varepsilon_p}{m_e c^2}\right)\Omega_0^{\frac{a+1}{2}} = (1+c_E)C_2^{-1}(a)\left(\frac{e^2}{\varepsilon_0 c}\right)^{-1}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a, \gamma_1, \gamma_2)$$

and

$$\varepsilon_p = \frac{4}{a+1}\left(\frac{B^2}{2\mu_0}\right) = \frac{2}{a+1}\frac{m_e^2}{\mu_0 e^2}\Omega_0^2$$

then

$$\frac{2}{a+1}\frac{m_e^2}{\mu_0 e^2}\frac{1}{m_e c^2}\Omega_0^{\frac{a+5}{2}} = (1+c_E)C_2^{-1}(a)\left(\frac{e^2}{\varepsilon_0 c}\right)^{-1}$$

$$\times\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a, \gamma_1, \gamma_2)$$

Hence,

$$\Omega_0^{\frac{a+5}{2}} = \frac{a+1}{2}(1+c_E)C_2^{-1}(a)\frac{\mu_0\varepsilon_0 c^3}{m_e}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a, \gamma_1, \gamma_2)$$

Using $\mu_0\varepsilon_0 = c^{-2}$ and $B = \frac{m_e}{e}\Omega_0$ gives:

$$B_{\min} = \frac{m_e}{e}\left[\frac{a+1}{2}(1+c_E)C_2^{-1}(a)\frac{c}{m_e}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a, \gamma_1, \gamma_2)\right]^{\frac{2}{a+5}}$$

The corresponding particle density

$$\varepsilon_{p,\,\min} = \frac{4}{a+1}\left(\frac{B^2}{2\mu_0}\right) = \frac{2}{a+1}\frac{B^2}{\mu_0}$$

$$= \frac{2}{a+1}\frac{m_e^2}{\mu_0 e^2} \times$$

$$\times \left[\frac{a+1}{2}(1+c_E)C_2^{-1}(a)\frac{c}{m_e}\left(\frac{I_\nu \nu^\alpha}{L}\right)f(a,\gamma_1,\gamma_2)\right]^{\frac{4}{a+5}}$$

Notes

1. For $a \sim 2$, the values of B_{min} and $\varepsilon_{p,\,\text{min}}$ only depend weakly upon the input parameters.

2. Since $f \propto \gamma_1^{-(a-2)}$, the dependence upon γ_1 is very weak.

3. *As estimates of particle energy density and magnetic field these estimates have to be treated with caution.* The only *physical* argument that these parameters are close to reality is that if a plasma is given enough time, the particles and field will come into equipartition as a result of the interchange of energy between the particles and field.

4. Similar expressions hold for the minimum total ***pressure*** of a synchrotron emitting plasma where the total pressure is defined by

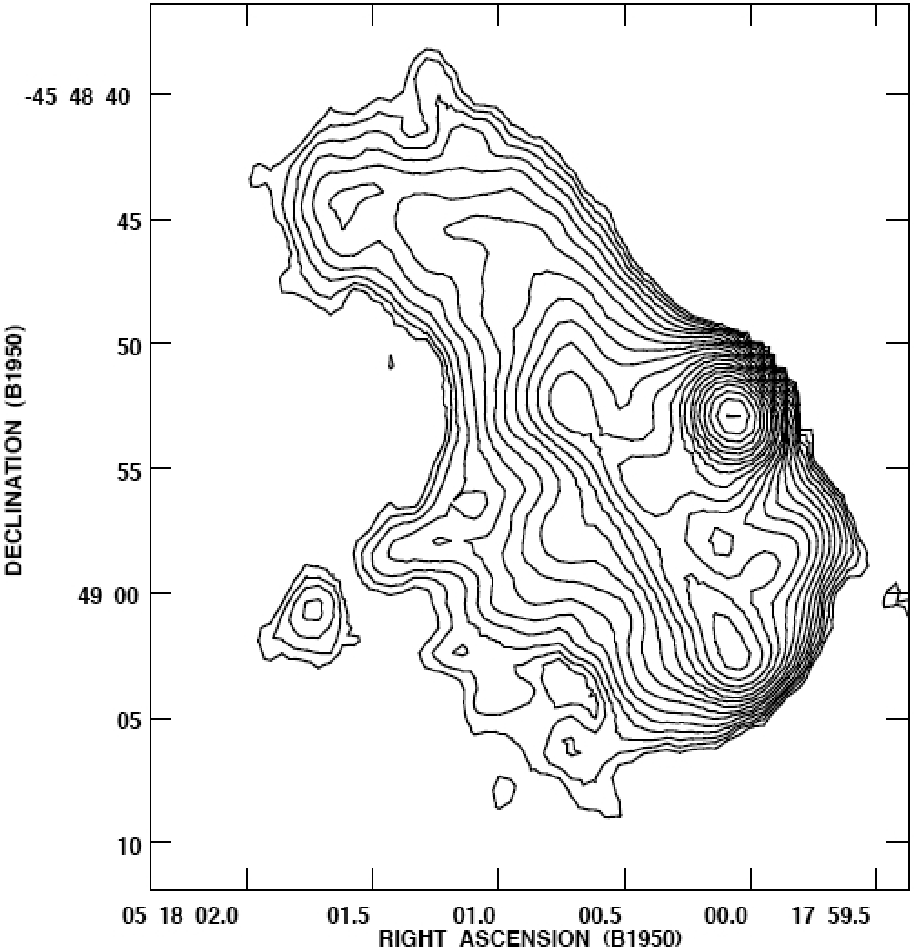
$$p_{\mathrm{tot}} = (1 + c_p) \times \frac{1}{3}\varepsilon_{\mathrm{rel}} + \frac{B^2}{2\mu_0}$$

5. While one has to be careful about using these minimum energy estimates in order to estimate the parameters of radio sources, they certainly provide us with an estimate of

the total minimum energy

$$E_{\min} = \left[\varepsilon_{p,\,\min} + \frac{B_{\min}^2}{2\mu_0} \right] \times \text{Volume}$$

7.3 Example: Minimum energy: Hot-spot of Pictor A



The western hot spot of Pictor A at 3.6 cm wavelength and 1.5″ resolution.

Contours spaced by a factor of $2^{1/2}$ between 0.071 and 70.71 % of the peak intensity of 0.94 Jy/beam.

Figure from Perley et al. 1997, A&A, **328**, 12-32

The calculations for Pictor A are set out in the template spreadsheet Pictor A.xlsx available from the course web page.

Note that for the distance through the hot spot we take the contour corresponding to the FWHM of the peak surface brightness. This is an approximation that could be refined but the actual inferred values of energy density are insensitve to values around this estimate.

The results of this calculation for $c_E = 0$ are:

Minimum energy magnetic field:

$$10.3 \text{ nT} = 1.3\times10^{-4} \text{ G}$$

Minimum energy particle energy density:

$$4.9\times10^{-11}\ \mathrm{J/m^3} = 4.9\times10^{-10}\ \mathrm{ergs\ cm^{-3}}.$$