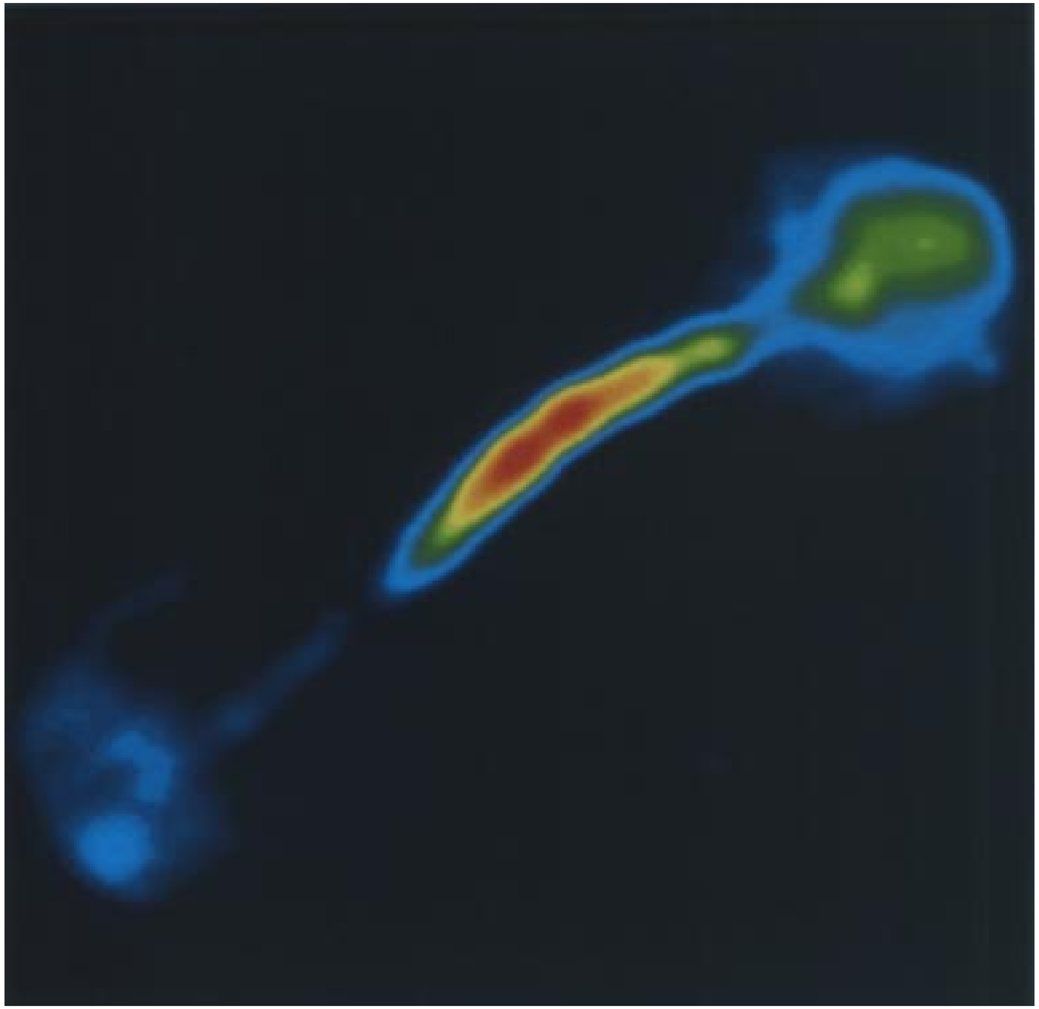
Synchrotron Radiation I

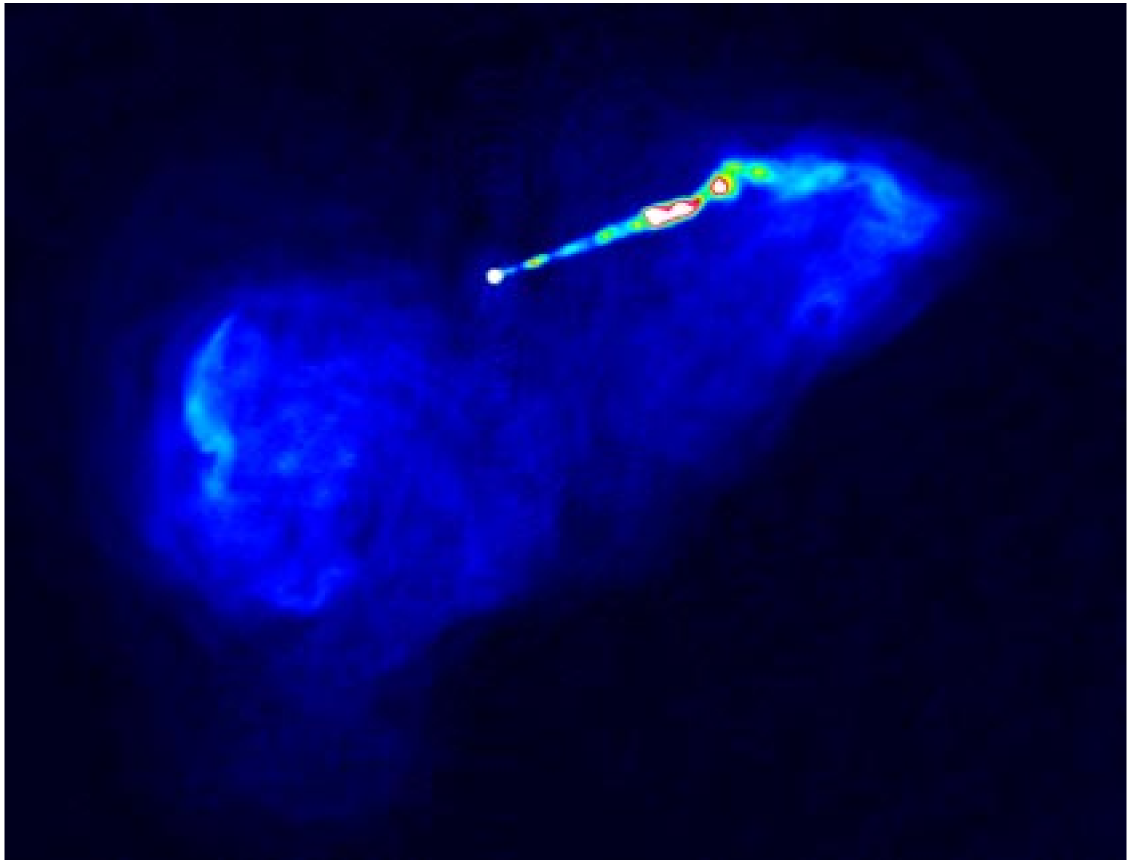
1 Examples of synchrotron emitting plasma

Following are some examples of astrophysical objects that are emitting synchrotron radiation. These include radio galaxies, quasars and supernova remnants.

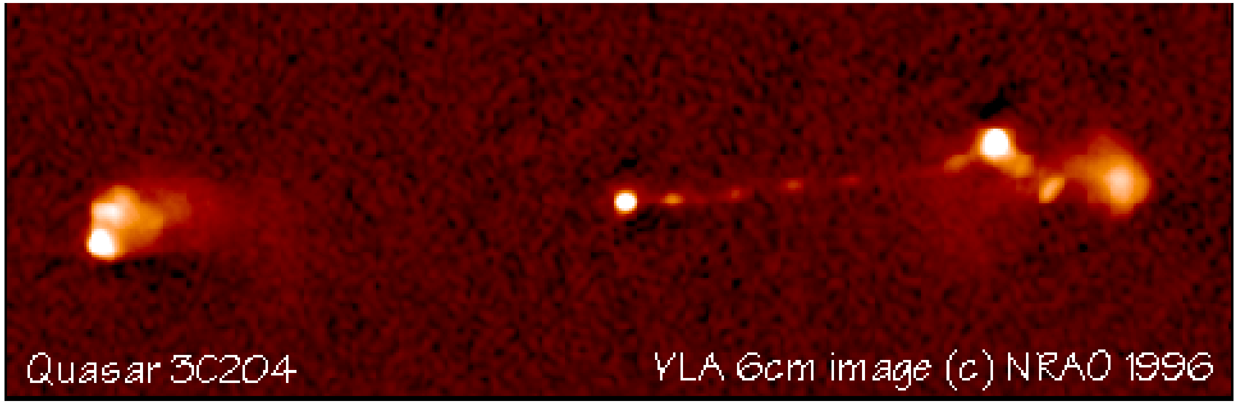


A 20 cm radio image from the VLA of the radio galaxy IC 4296. See Killeen, Bicknell & Ekers, ApJ **325**, 180.

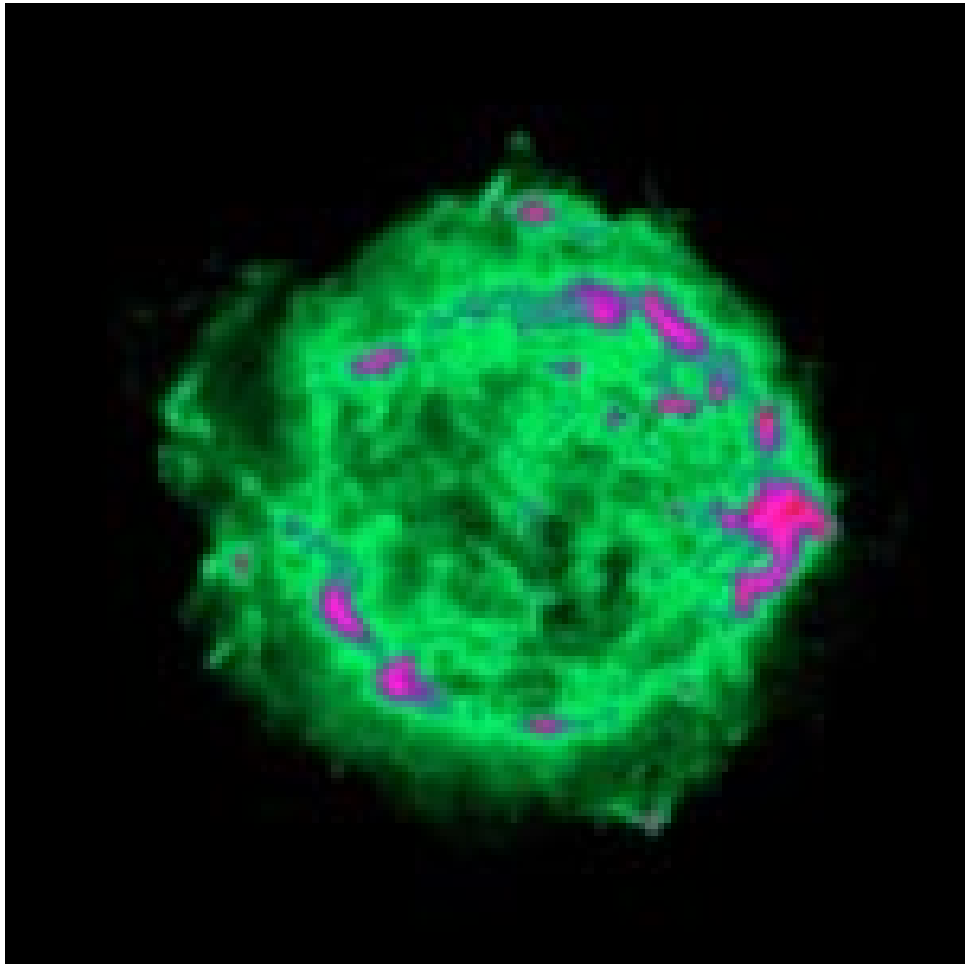
This images shows the jets and lobes of the radio emission corresponding to a relatively nearby giant elliptical galaxy.



The inner part of the radio galaxy M87 from a 2cm VLA image
by Biretta, Zhou & Owen.



VLA 4.9 GHz image of the $z = 1.112$ quasar 3C204. The linear size is $159/h$ kpc Note the periodic knotty structure in jets that may be the result of internal shocks and the very bright core. See http://www.cv.nrao.edu/~abridle/images.htm



The Supernova Remnant Cassiopeia A

Cassiopeia A gets its name from radio astronomers, who “rediscovered” it in 1948 as the strongest radio source in the constellation of Cassiopeia. About 5 years later optical astronomers found the faint wisps, and it was determined that Cas A is the remnant of an explosion that occurred about 300 years ago. The radio emission is synchrotron emission.

The X-ray image of the Cassiopeia A supernova remnant on the left was obtained by the Chandra X-ray Observatory using the Advanced CCD Imaging Spectrometer (ACIS). Two shock waves are visible: a fast outer shock and a slower inner shock. The inner wave is believed to be due to the collision of the ejecta from the supernova explosion with a shell of material, heating it to a temperature of ten million degrees. The outer shock wave is analogous to a sonic boom resulting from the explosion.

2 What can we learn from synchrotron emission from an astrophysical object?

- Estimates of total radiative luminosity
- Total minimum energy (particles plus field) in the radio emitting particles
- Direction of the magnetic field
- Constraints on energy density of emitting particles and estimates of magnetic field
- Total amount of matter converted into energy (relevant for radio galaxies and quasars)
- Accretion rate onto central black hole (radio galaxies and quasars)
- Constraints on the mass of the black hole (radio galaxies and quasars)
- Information relating to source dynamics, e.g. strengths of shock

waves; velocities of jets

3 Background theory

3.1 Electric field

We have the expression for the Fourier component of electric field of the pulse from a moving electron derived from the Lienard-Weichert potentials:

$$r\boldsymbol{E}(\omega) = \frac{-i\omega q}{4\pi c\varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta}') \times \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right)\right] dt \qquad (1)$$

The Fourier components of a transverse radiation field can be expressed in terms of unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ perpendicular to the wave direction such that

$$\boldsymbol{E}(\omega) = E_1(\omega)\boldsymbol{e}_1 + E_2(\omega)\boldsymbol{e}_2 \qquad (2)$$

In any particular application we choose $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ so as to simplify the expression for the resultant electric field and Stokes parameters, defined by:

$$
\begin{aligned}
I_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) + E_2(\omega)E_2^*(\omega)] \\
Q_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1(\omega)E_1^*(\omega) - \langle E_2(\omega)E_2^*(\omega)\rangle] \\
U_\omega &= \frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) + E_1(\omega)E_2^*(\omega)] \\
V_\omega &= \frac{1}{i}\frac{c\varepsilon_0}{\pi T}[E_1^*(\omega)E_2(\omega) - E_1(\omega)E_2^*(\omega)]
\end{aligned}
\qquad (3)
$$

Interpretation of T

Another way to interpret the parameter T which is appropriate when considering radiation from a distribution of particles is:

$$T = \text{time between pulses} \tag{4}$$

or

$$T^{-1} = \text{No of pulses per unit time} \tag{5}$$

In this case each pulse originates from a different electron in the distribution.

Nomenclature

We refer to the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ components of the electric field as the two ***modes*** of polarisation.

3.2 Emissivities

The basic relation is:

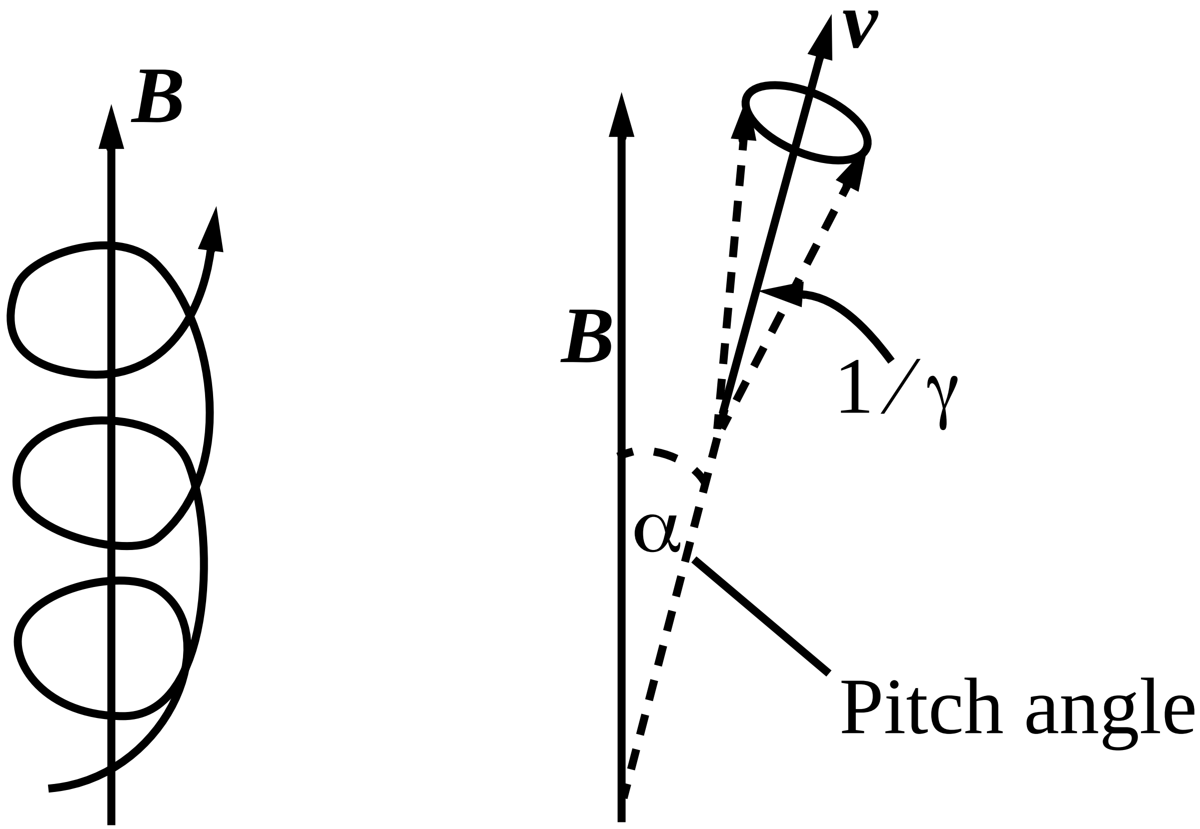
$$\frac{dW_{\alpha\beta}}{d\Omega d\omega dt} = \frac{c\varepsilon_0}{\pi T} r^2 E_\alpha(\omega) E^*_\beta(\omega) \qquad (6)$$

In terms of I, Q, U, V

$$
\begin{aligned}
j_I(\omega) &= \frac{dW_{11}}{d\Omega d\omega dt} + \frac{dW_{22}}{d\Omega d\omega dt} \\
j_Q(\omega) &= \frac{dW_{11}}{d\Omega d\omega dt} - \frac{dW_{22}}{d\Omega d\omega dt} \\
j_U(\omega) &= \frac{dW_{12}}{d\Omega d\omega dt} + \frac{dW^*_{12}}{d\Omega d\omega dt} \\
j_V(\omega) &= i\frac{dW_{12}}{d\Omega d\omega dt} - \frac{dW^*_{12}}{d\Omega d\omega dt}
\end{aligned}
\qquad (7)
$$

3.3 Helical motion of a relativistic particle

For synchrotron emission, we evaluate the above integrals using the expression for helical motion:



$$\boldsymbol{v} = c\beta \begin{bmatrix} \sin\alpha\cos(\Omega_B t' + \phi_0) \\ -\eta\sin\alpha\sin(\Omega_B t' + \phi_0) \\ \cos\alpha \end{bmatrix} \tag{8}$$

$$\begin{aligned} \Omega_B &= \text{Gyrofrequency} = \frac{|q|B}{\gamma m} = \gamma^{-1}\frac{|q|B}{m} = \gamma^{-1}\Omega_0 \\ \Omega_0 &= \text{Non-relativistic gyrofrequency} \\ \eta &= \frac{q}{|q|} = \text{sign of charge} \\ \alpha &= \text{pitch angle} \end{aligned} \tag{9}$$

The derivation of the equations for the helical trajectory of a charged particle in a magnetic field are given in the background notes.

Usually we are concerned with electrons ($\eta = -1$) so that:

$$\boldsymbol{v} = c\beta \begin{bmatrix} \sin\alpha\cos(\Omega_B t' + \phi_0) \\ \sin\alpha\sin(\Omega_B t' + \phi_0) \\ \cos\alpha \end{bmatrix} \qquad (10)$$

Note that the motion of the velocity vector is anti-clockwise for an electron.

Coordinates of particle

Integrating the velocity gives for the position vector:

$$\boldsymbol{x} = c\beta \begin{bmatrix} \dfrac{\sin\alpha}{\Omega_B}\sin(\Omega_B t + \phi_0) \\ \dfrac{\eta\cos\alpha}{\Omega_B}\cos(\Omega_B t + \phi_0) \\ t\cos\alpha \end{bmatrix} + \boldsymbol{x}_0 \qquad (11)$$

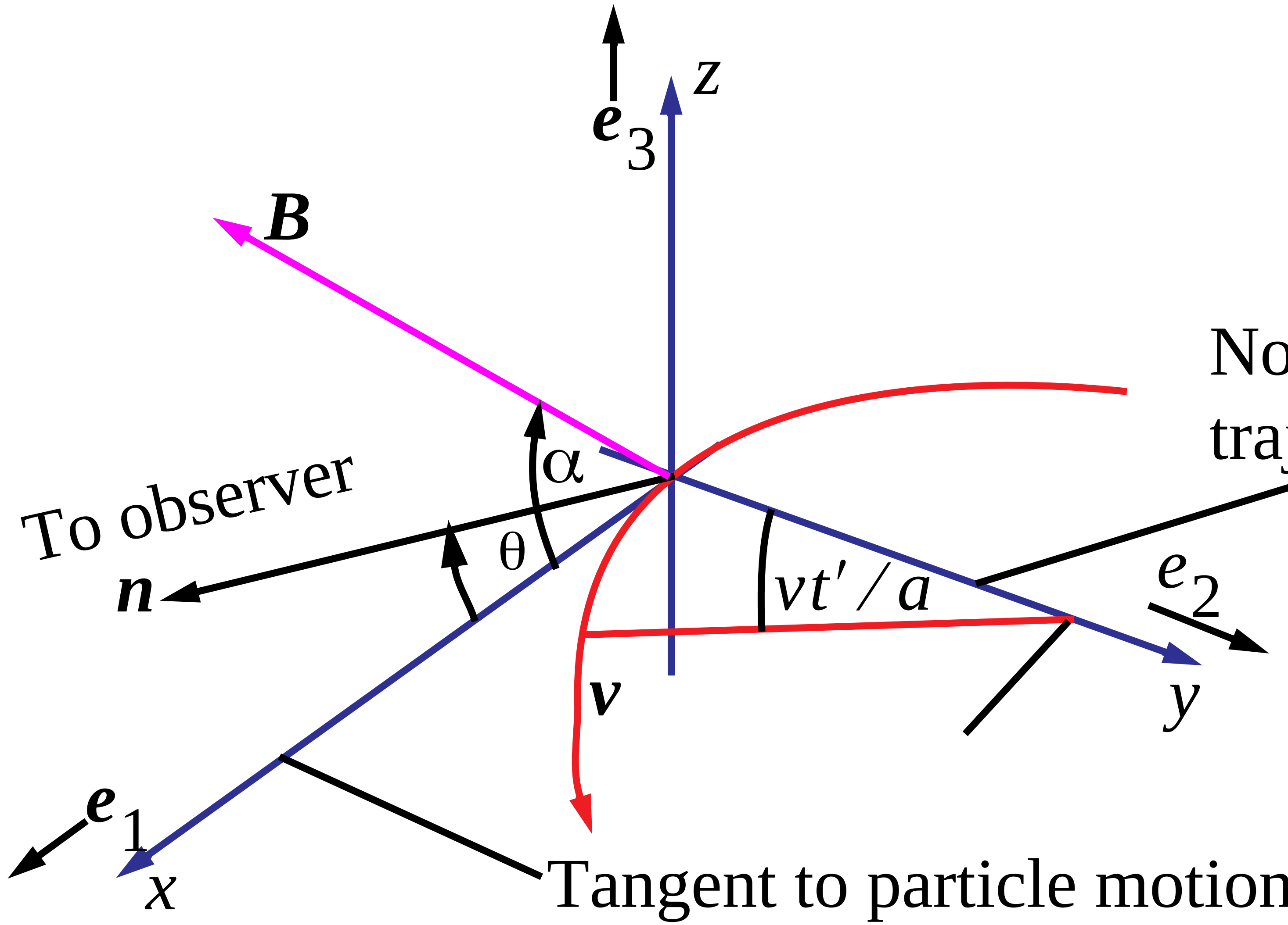
This consists of a linear motion in the z-direction superimposed upon a circular motion in the $x-y$ plane. The radius of the circular motion is the gyroradius

$$r_G = \frac{c\beta\sin\alpha}{\Omega_B} = \frac{\gamma mc\beta\sin\alpha}{\Omega_B} \tag{12}$$

3.4 Approximations for ultrarelativistic particles

- $\gamma \gg 1 \Rightarrow \beta \approx 1 - \dfrac{1}{2\gamma^2}$. Sometimes $|\beta| \approx 1$ is appropriate.
- Most of the radiation beamed into a cone of half-angle $1/\gamma$. This means that we can expand functions in terms of small powers of the angle between the particle velocity and the direction of emission.

We set up the calculation in the following way (following Rybicki & Lightman):



Tangent to particle motion at $t' = 0$.

• The origin of time $t' = 0$ is defined as the instant where the particle’s velocity comes closest to the direction of the observer.

- The x–axis represents the instantaneous direction of motion at $t' = 0$.
- The y–axis represents the direction of the centre of curvature.
- The z–axis completes the system of coordinates (i.e. $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$).
- $\boldsymbol{n}$ is the direction of the observer and makes an angle θ to the direction of the electron at $t' = 0$. We take θ small.
- Since $t' = 0$ represents the instant at which the angle between the velocity and the observer’s direction is a minimum, the component of $\boldsymbol{n}$ in the direction of the normal, $\boldsymbol{m}$ is zero.
- The parameter a is the instantaneous radius of curvature of the particle

- Formally, the pulse of radiation reaching the observer originates from the entire trajectory of the particle. However, most of this radiation originates from a very small region of the particle's orbit near the origin of the above coordinate system.

3.5 Aside – Radius of curvature

Summary of the theory of 3 dimensional curves:

$$\boldsymbol{x} = \boldsymbol{x}(t) \qquad \text{Velocity vector} = \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$$

$$\text{Unit tangent vector} = \boldsymbol{t} = \frac{\boldsymbol{v}}{v} \tag{13}$$

$$\text{Arc length:} \qquad \frac{ds}{dt} = v$$

The curvature (κ), radius of curvature (a) and curvature normal ($\boldsymbol{m}$) are given by:

$$\frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{m} = \frac{1}{a}\boldsymbol{m} \qquad \frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}}{dt}\frac{dt}{ds} = \frac{1}{v}\frac{d\boldsymbol{t}}{dt} \tag{14}$$

Using the previous expression for velocity:

$$\begin{aligned} \boldsymbol{v} &= c\beta\left[\sin\alpha\cos(\Omega_B t) \;\; \sin\alpha\sin(\Omega_B t) \;\; \cos\alpha\right] \\ \boldsymbol{t} &= \frac{\boldsymbol{v}}{v} = \left[\sin\alpha\cos(\Omega_B t) \;\; \sin\alpha\sin(\Omega_B t) \;\; \cos\alpha\right] \end{aligned} \tag{15}$$

Since the particle passes through the origin at $t = 0$ the position vector is:

$$\boldsymbol{x} = \left[\frac{c\beta\sin\alpha}{\Omega_B}\sin(\Omega_B t') \;\; \frac{c\beta\sin\alpha}{\Omega_B}(1-\cos(\Omega_B t')) \;\; c\beta t'\cos\alpha\right] \tag{16}$$

$$= \left[r_G\sin(\Omega_B t') \;\; r_G(1-\cos(\Omega_B t')) \;\; c\beta t'\cos\alpha\right]$$

Remember that the gyroradius

$$r_G = \frac{c\beta\sin\alpha}{\Omega_B} \tag{17}$$

The curvature is determined from:

$$\frac{1}{v}\frac{d\boldsymbol{t}}{dt} = \kappa\boldsymbol{m} = \frac{\Omega_B \sin\alpha}{c\beta}\left[-\sin(\Omega_B t) \;\; \cos(\Omega_B t) \;\; 0\right] \qquad (18)$$

This implies

$$\boldsymbol{m} = \left[-\sin(\Omega_B t) \;\; \cos(\Omega_B t) \;\; 0\right]$$

$$\kappa = \frac{\Omega_B \sin\alpha}{c\beta} \qquad a = \frac{c\beta}{\Omega_B \sin\alpha} = \frac{r_G}{\sin^2\alpha} \qquad (19)$$

NB. The radius of curvature is not the same as the radius of the projection of the orbit onto the plane perpendicular to $\boldsymbol{B}$. As one expects, $a = r_G$ when $\alpha = \pi/2$.

3.6 The magnetic field in the particle-based system of coordinates

- Since the normal vector $\boldsymbol{m} = \begin{bmatrix} -\sin(\Omega_B t) & \cos(\Omega_B t) & 0 \end{bmatrix}$ is perpendicular to the magnetic field, then the magnetic field lies in the plane of $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. i.e. in the same plane as $\boldsymbol{n}$.
- Also since the velocity makes an angle of α with the magnetic field, the direction of $\boldsymbol{B}$ is as shown in the diagram.

4 Semi-quantitative treatment of synchrotron emission

4.1 Small angle approximation

Recall the expression for the electric field of a radiating charge:

$$\begin{aligned} \boldsymbol{E}_{\mathrm{rad}} \cdot \boldsymbol{n}' &= \frac{q}{4\pi c \varepsilon_0 r'} \frac{[(1-\dot{\boldsymbol{\beta}}' \cdot \boldsymbol{n}')(\boldsymbol{n}' \cdot \dot{\boldsymbol{\beta}}') - \boldsymbol{n}' \cdot \dot{\boldsymbol{\beta}}(1-\dot{\boldsymbol{\beta}}' \cdot \boldsymbol{n}')]}{(1-\boldsymbol{\beta}' \cdot \boldsymbol{n}')^3} \\ &= 0 \end{aligned} \qquad (20)$$

For a relativistically moving particle, the electric field is highly peaked in the direction of motion, i.e. when

$(1 - \beta \cdot \boldsymbol{n}')$ is close to zero. Since θ is the angle between β and n

$$(1 - \boldsymbol{\beta}' \cdot \boldsymbol{n}') = (1 - \beta \cos\theta) \approx 1 - \left(1 - \frac{1}{2\gamma^2}\left(1 - \frac{1}{2}\theta^2\right)\right)$$
$$= \frac{1}{2\gamma^2} + \frac{1}{2}\theta^2 = \frac{1}{2\gamma^2}(1 + \gamma^2\theta^2) \tag{21}$$

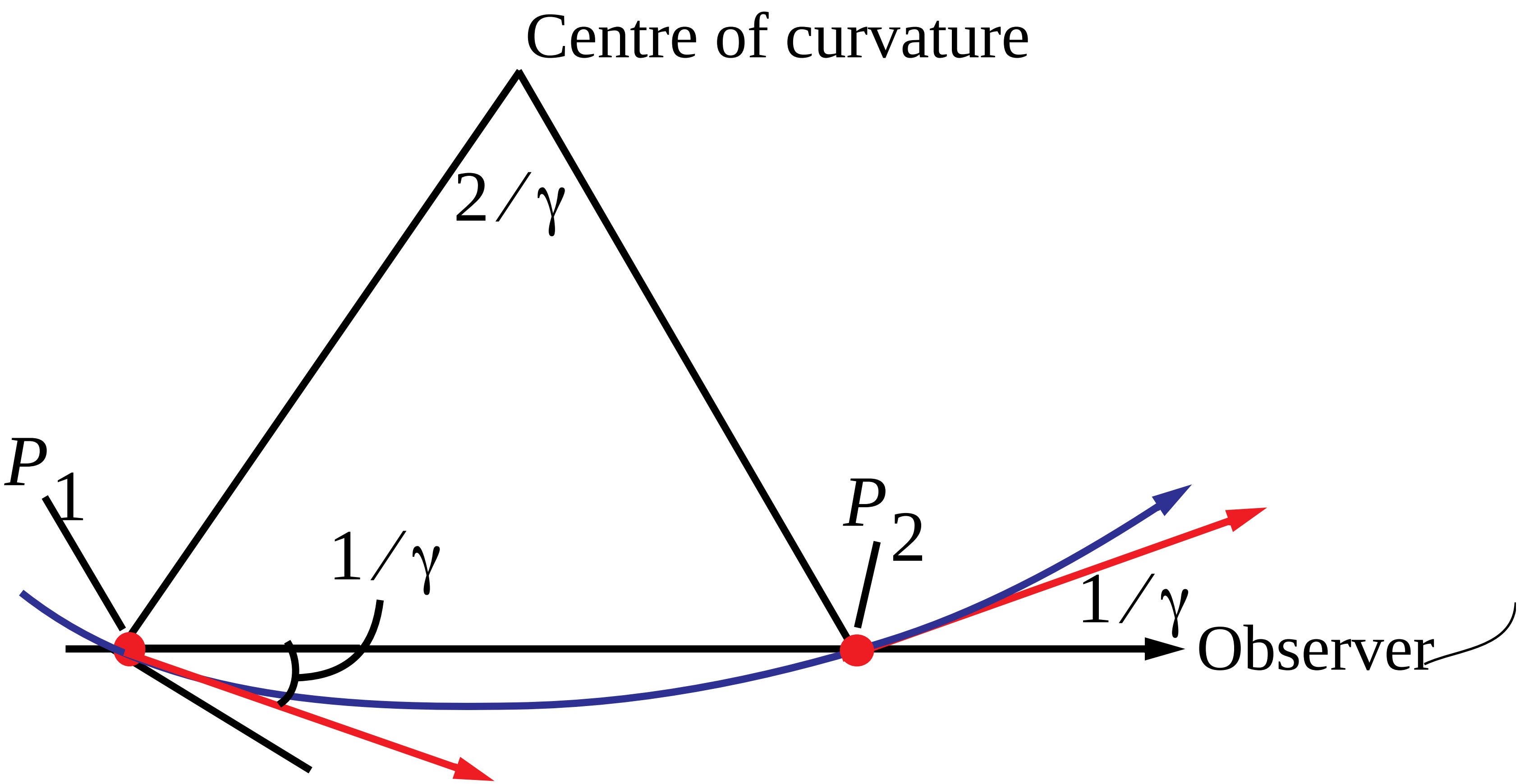
So $1 - \boldsymbol{\beta}' \cdot \boldsymbol{n}'$ is a minimum at $\theta = 0$ and increases rapidly for $\gamma\theta > 1$, i.e. for $\theta > 1/\gamma$. Since $\gamma \gg 1$, then the angles we are dealing with are very small.

The expression for $1 - \boldsymbol{\beta}' \cdot \boldsymbol{n}'$ introduces the variable

$$\theta_\gamma = (1 + \gamma^2\theta^2)^{1/2} \tag{22}$$

This variable occurs frequently in the following theory.

4.2 Estimate of critical frequency



The angle turned through and the arc length of the particle orbit are related by:

$$\frac{d\psi}{ds} = \frac{1}{a} \tag{23}$$

where a is the previously determined radius of curvature.

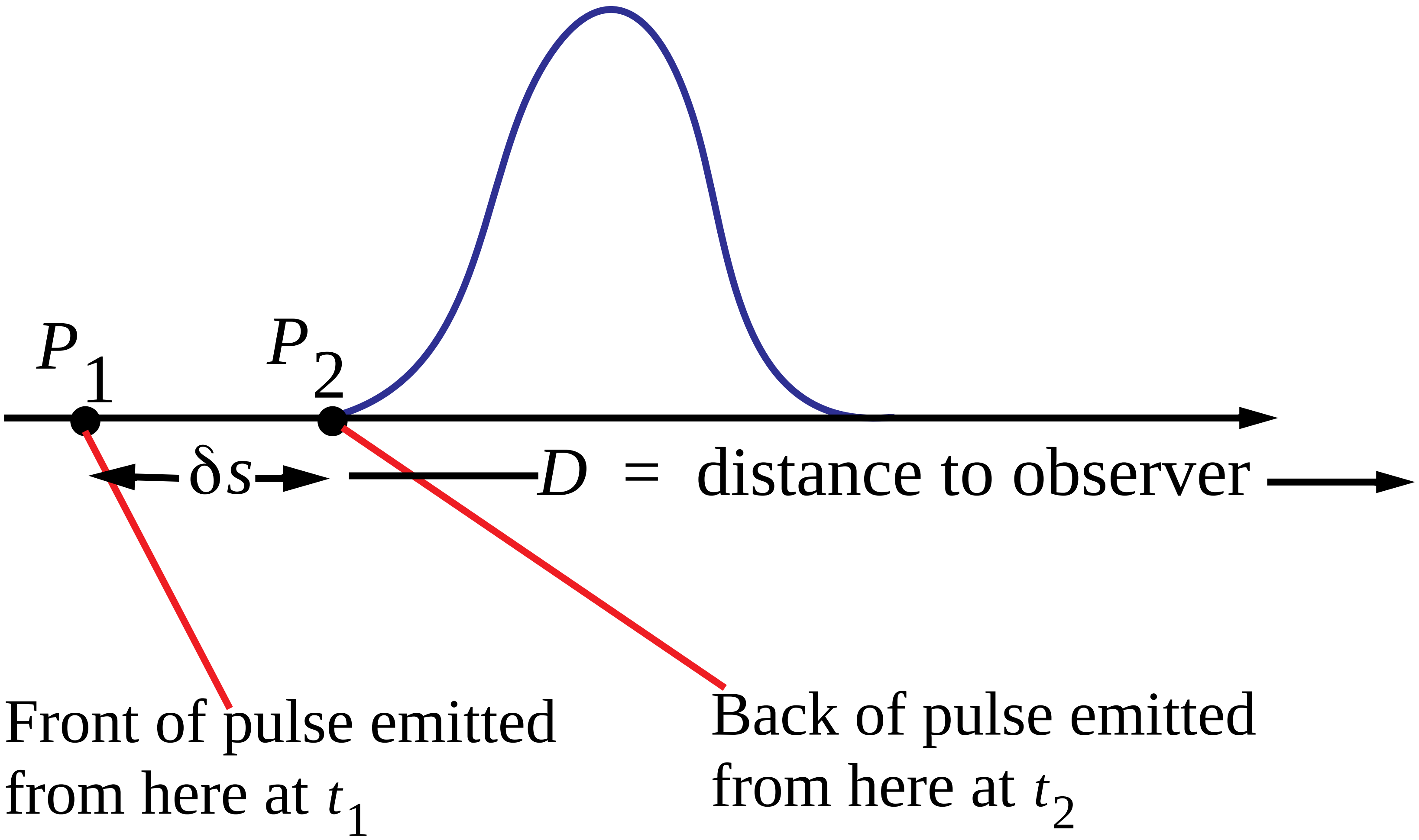
From the above geometry

$$\delta\psi = \frac{2}{\gamma} \Rightarrow \delta s = a\delta\psi = 2\frac{a}{\gamma} \tag{24}$$

Time taken for trajectory to sweep through relevant angle

$$t_2 - t_1 = \frac{\delta s}{c} = \frac{2a}{c\gamma} \tag{25}$$

This is not the period for which the pulse is observed



Times of arrival of pulse:

$$t_1' = t_1 + \frac{D + \delta s}{c} \qquad t_2' = t_2 + \frac{D}{c}$$

$$\Rightarrow t_2' - t_1' = t_2 - t_1 - \frac{\delta s}{c} = (t_2 - t_1) - \frac{v}{c}(t_2 - t_1) \qquad (26)$$

$$= \left(1 - \frac{v}{c}\right)(t_2 - t_1)$$

i.e. the time between pulses is reduced because the trailing part of the pulse has less distance to travel. This is one example of the effect of time retardation.

The factor $1 - \frac{v}{c} = 1 - \beta$ can be expanded in powers of γ as follows:

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \beta^2 = 1 - \frac{1}{\gamma^2} \Rightarrow \beta \approx 1 - \frac{1}{2\gamma^2}$$
$$\Rightarrow 1 - \beta \approx \frac{1}{2\gamma^2} \qquad (27)$$

The ***observed*** duration of the pulse is therefore

$$\left(1 - \frac{v}{c}\right)(t_2 - t_1) \approx \frac{1}{2\gamma^2}(t_2 - t_1) \approx \frac{a}{c\gamma^3} = \frac{c}{\Omega_B \sin\alpha} \times \frac{1}{c\gamma^3}$$
$$= \frac{1}{\Omega_B \gamma^3 \sin\alpha} \qquad (28)$$

Fourier theory tells us that if the length of a pulse is Δt then the typical frequencies present in such a pulse are $\omega \sim (\Delta t)^{-1}$. Hence the typical frequencies expected in a synchrotron pulse are given by:

$$\omega \sim \Omega_B \gamma^3 \sin\alpha \tag{29}$$

The ***critical frequency*** of the exact synchrotron theory (to follow) is close to this estimate:

$$\omega_c = \frac{3}{2}\Omega_B \sin\alpha\gamma^3 = \frac{3}{2}\frac{|q|B}{m\gamma}\gamma^3 \sin\alpha = \frac{3}{2}\frac{|q|B}{m}\gamma^2 \sin\alpha \tag{30}$$

The critical frequency which characterises synchrotron emission is a factor of γ^2 higher than the cyclotron frequency $\frac{|q|B}{m}$ and a factor of γ^3 higher than the gyrofrequency at which particles gyrate around the magnetic field lines.

4.3 Typical example

$$B = 10^{-5}\,\mathrm{G} = 1\,\mathrm{nT} \qquad \gamma = 10^4$$

$$\Omega_B = \frac{eB}{\gamma m_e} = 1.8\times 10^{-2}\,\mathrm{Hz}$$

$$\omega_c = \frac{3}{2}\frac{eB}{m}\gamma^2 \sin\alpha = 1.3\times 10^{10} \sin\alpha\,\mathrm{Hz} \qquad (31)$$

$$r_G = \frac{c\beta\sin\alpha}{\Omega_B} = 1.7\times 10^{10} \sin\alpha \;\; \mathrm{m}$$

Note that the critical frequency in this example is about 10 GHz – a microwave radio frequency.

4.4 Relation between frequency and Lorentz factor

$$\begin{aligned}
\omega &= 2\pi\nu = \frac{3}{2}\frac{eB}{m}\gamma^2\sin\alpha \\
\Rightarrow \gamma &= \left(\frac{4\pi}{3}\frac{m}{e}\right)^{1/2}(B\sin\alpha)^{-1/2}\nu^{1/2} \\
&= 4.9\times10^3\left(\frac{B\sin\alpha}{\mathrm{nT}}\right)^{-1/2}\left(\frac{\nu}{\mathrm{GHz}}\right)^{1/2} \\
&= 4.9\times10^3\left(\frac{B\sin\alpha}{10\,\mu\mathrm{G}}\right)^{-1/2}\left(\frac{\nu}{\mathrm{GHz}}\right)^{1/2}
\end{aligned} \tag{32}$$

5 Detailed development of synchrotron emission

Angle turned through in time t':

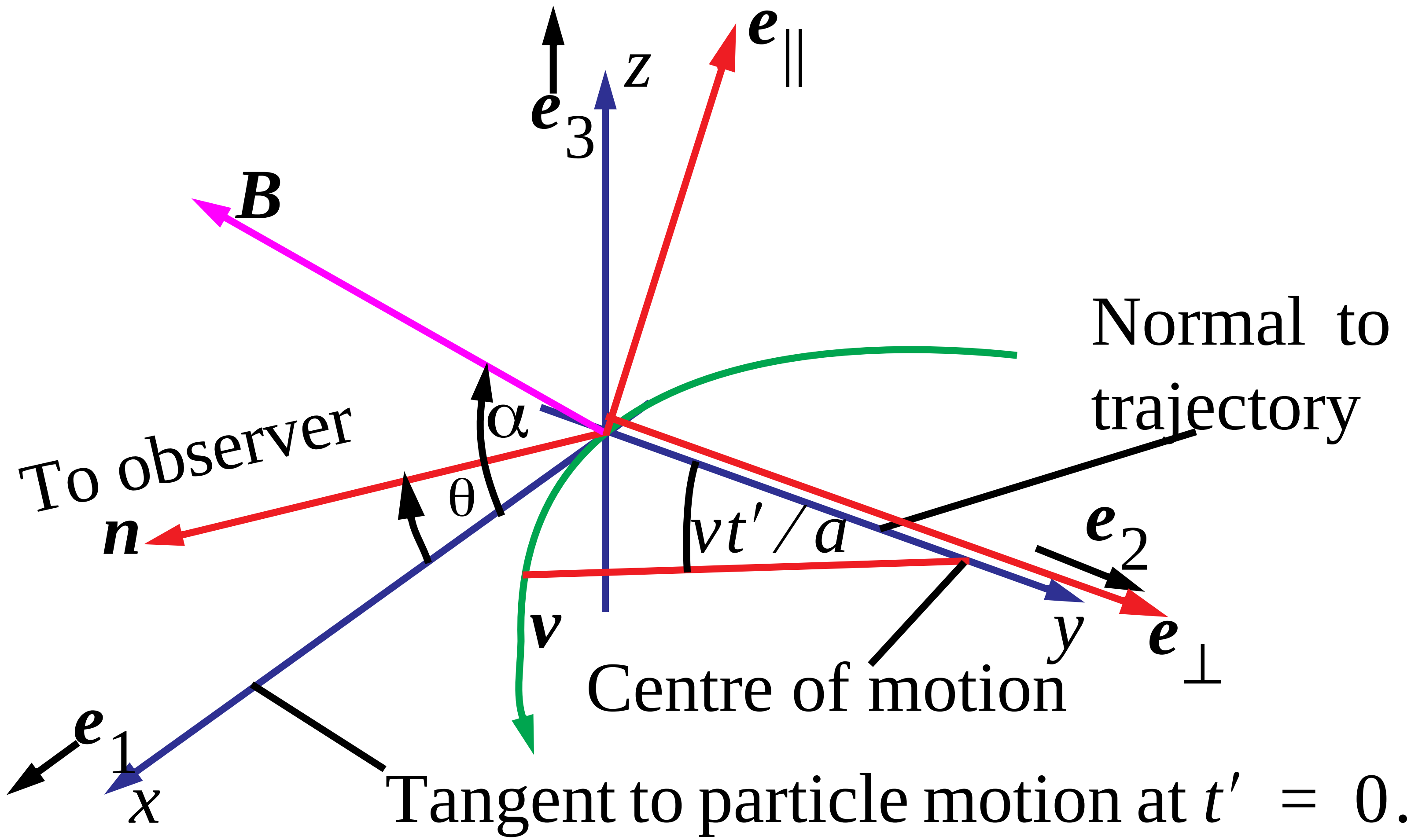
$$\frac{d\psi}{ds} = \frac{1}{a}$$

$$\frac{1}{v}\frac{d\psi}{dt} = \frac{1}{a} \Rightarrow \psi = \frac{vt'}{a} \tag{33}$$

In the new set of axes, the motion is circular with no component in the z–direction. The velocity in these axes is therefore:

$$\begin{aligned} \boldsymbol{v} &= c\beta\left[\cos\frac{vt'}{a} \;\; \sin\frac{vt'}{a} \;\; 0\right] \\ \text{i.e.} \qquad \beta &= \beta\left[\cos\frac{vt'}{a} \;\; \sin\frac{vt'}{a} \;\; 0\right] \end{aligned} \tag{34}$$

5.1 Axes based upon instantaneous orbital plane



5.2 Significance of $\boldsymbol{e}_{\|}$ and $\boldsymbol{e}_{\perp}$

The axes $\boldsymbol{e}_{\|}$ and $\boldsymbol{e}_{\perp}$ are respectively parallel and perpendicular to the projection of the magnetic field on the plane perpendicular to the line of sight given by the vector $\boldsymbol{n}$. This is the plane of propagation of the radiation.

5.3 Integral to evaluate

We have our expression for the Fourier component of the electric field of a pulse:

$$
r\boldsymbol{E}(\omega) = \frac{-i\omega q}{4\pi c\varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta}) \times \exp\left[i\omega\left(t' - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c} \right) \right] dt' \tag{35}
$$

5.4 Determination of $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta})$

We first calculate a simple expression for the quantity $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta})$.

Use:

$$\begin{aligned} \boldsymbol{n} &= \cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_3 \\ \beta &= \beta\cos\left(\frac{vt'}{a}\right)\boldsymbol{e}_1 + \beta\sin\left(\frac{vt'}{a}\right)\boldsymbol{e}_2 \end{aligned} \tag{36}$$

This gives:

$$\begin{aligned} \boldsymbol{n}\times\beta &= \beta\cos\left(\frac{vt'}{a}\right)\boldsymbol{n}\times\boldsymbol{e}_1 + \beta\sin\left(\frac{vt'}{a}\right)\boldsymbol{n}\times\boldsymbol{e}_2 \\ &= \beta\cos\left(\frac{vt'}{a}\right)\sin\theta\,\boldsymbol{e}_\perp + \beta\sin\left(\frac{vt'}{a}\right)\boldsymbol{e}_\parallel \end{aligned} \tag{37}$$

$$\boldsymbol{n}\times(\boldsymbol{n}\times\beta) = \beta\sin\theta\cos\left(\frac{vt'}{a}\right)\boldsymbol{e}_\parallel - \beta\sin\left(\frac{vt'}{a}\right)\boldsymbol{e}_\perp$$

We make the following approximations:

$$|\beta| \approx 1 \qquad \cos\left(\frac{vt'}{a}\right) \approx 1 \qquad \sin\left(\frac{vt'}{a}\right) \approx \frac{vt'}{a} \tag{38}$$

so that

$$\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta}) \approx \theta \boldsymbol{e}_{\parallel} - \left(\frac{vt'}{a}\right)\boldsymbol{e}_{\perp} = \theta \boldsymbol{e}_{\parallel} - \frac{ct'}{a}\boldsymbol{e}_{\perp} \tag{39}$$

Note:

In order to determine terms that are relevant to the small amount of circular polarisation that emerges from a synchrotron source, we would have to expand to the next order in $1/\gamma$.

The term $i\omega\left[t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right]$

This term in the argument of the exponential involves $\boldsymbol{X}(t')$. Since

$$\frac{d\boldsymbol{X}}{dt'} = \boldsymbol{v} = c\beta\left[\cos\frac{vt'}{a}\;\;\sin\frac{vt'}{a}\;\;0\right] \tag{40}$$

and $\boldsymbol{X}(0) = \boldsymbol{0}$

then

$$\boldsymbol{X}(t') = \frac{c\beta a}{v}\left[\sin\frac{vt'}{a} \;\left(1-\cos\frac{vt'}{a}\right)\; 0\right]$$

$$= a\left[\sin\frac{vt'}{a} \;\left(1-\cos\frac{vt'}{a}\right)\; 0\right] \qquad (41)$$

giving

$$t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c} = t' - \left[\cos\theta \;\; 0 \;\; \sin\theta\right]\cdot\frac{a}{c}\left[\sin\frac{vt'}{a} \;\; 1-\cos\frac{vt'}{a} \;\; 0\right] \qquad (42)$$

$$= t' - \frac{a}{c}\cos\theta\sin\frac{vt'}{a}$$

5.5 Expansion of $t' - \frac{\mathbf{n} \cdot \mathbf{X}(t')}{c}$ in small powers of θ and t':

$$\cos\theta \approx \left(1 - \frac{1}{2}\theta^2\right) \qquad \sin\left(\frac{vt'}{a}\right) \approx \frac{vt'}{a} - \frac{1}{6}\frac{v^3 t'^3}{a^3} \tag{43}$$

then

$$t' - \frac{a}{c}\cos\theta\sin\left(\frac{vt'}{a}\right) = t'\left(1 - \frac{v}{c}\right) + \frac{1}{2}\frac{v}{c}t'\theta^2 + \frac{1}{6}\frac{v^3 t'^3}{a^2 c} \tag{44}$$

In the first term, we use $1-\frac{v}{c} = \frac{1}{2\gamma^2}$ and put $v = c$ in the remainder, giving:

$$t' - \frac{a}{c}\cos\theta\sin\left(\frac{vt'}{a}\right) \approx \frac{1}{2\gamma^2}t' + \frac{1}{2}\theta^2 t' + \frac{1}{6}\frac{c^2 t'^3}{a^2}$$

$$t' - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c} = \frac{1}{2\gamma^2}\left[t'(1+\gamma^2\theta^2) + \frac{1}{3}\frac{c^2\gamma^2 t'^3}{a^2}\right] \qquad (45)$$

The characteristic angle $\theta \sim 1/\gamma$ enters into the expression.

Summary so far

$$\boldsymbol{n} \times (\boldsymbol{n} \times \beta) \approx \theta \boldsymbol{e}_{\parallel} - \left(\frac{vt'}{a}\right)\boldsymbol{e}_{\perp} = \theta \boldsymbol{e}_{\parallel} - \frac{ct'}{a}\boldsymbol{e}_{\perp}$$

$$i\omega\left[t - \frac{\boldsymbol{n} \cdot \boldsymbol{X}(t')}{c}\right] = \frac{i\omega}{2\gamma^2}\left[(1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2}{3a^2}t'^3\right] \qquad (46)$$

New variables

We put these equations into a dimensionless form by defining new variables

$$\theta_\gamma^2 = 1 + \gamma^2\theta^2 \qquad y = \frac{c\gamma}{a\theta_\gamma} t' \qquad \eta = \frac{\omega a}{3\gamma^3 c}\theta_\gamma^3 \qquad (47)$$

- The purpose of introducing these variables is to make quantities in the integrand of the expression for the Fourier transform of order unity.
- In so doing it helps to represent the new dimensionless variables in a way that makes their physical significance apparent.

Significance of y

The quantity $a/(c\gamma)$ is the time taken for the particle's velocity to swing through an angle $1/\gamma$. Hence

$$y = \frac{t'}{\text{Characteristic time} \times \theta_\gamma} \qquad (48)$$

Note:

- y is a dimensionless variable
- One expects the major contribution to come from the region of the integrand wherein $y \sim 1$.
- Integration over t' will be replaced by integration over y.

Significance of η

$$\eta = \frac{\omega a}{3\gamma^3 c}\theta_\gamma^3 = \frac{1}{3}\frac{\omega}{[\gamma^3 c/a]}\theta_\gamma^3$$

$$\frac{a}{c} = \frac{1}{c}\frac{c\beta}{\Omega_B \sin\alpha} \approx \frac{1}{\Omega_B \sin\alpha} \qquad (49)$$

$$\Rightarrow \eta = \frac{1}{3}\frac{\omega}{\gamma^3 \Omega_B \sin\alpha}\theta_\gamma^3 = \frac{1}{2}\left(\frac{\omega}{\omega_c}\right)\theta_\gamma^3$$

Since the radiation is concentrated at the critical frequency, the dominant contribution to the result will be around $\eta \sim 1$.

Rearrangement of integrand

For $\boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{\beta})$:

$$\boldsymbol{n} \times (\boldsymbol{n} \times \beta) \approx \theta \boldsymbol{e}_{\|} - \frac{ct'}{a}\boldsymbol{e}_{\perp} = \theta \boldsymbol{e}_{\|} - \frac{c}{a}\frac{a\theta_{\gamma}}{c\gamma} y \boldsymbol{e}_{\perp}$$

$$= \theta \boldsymbol{e}_{\|} - \frac{\theta_{\gamma}}{\gamma} y \boldsymbol{e}_{\perp} \tag{50}$$

For $i\omega\left[t - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right]$

$$i\omega\left[t - \frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right] = \frac{i\omega}{2\gamma^2}\left[(1+\gamma^2\theta^2)t' + \frac{c^2\gamma^2}{3a^2}t'^3\right]$$

$$= \frac{i\omega}{2\gamma^2}\left[\theta_\gamma^2 \times \frac{a\theta_\gamma}{c\gamma}y + \frac{c^2\gamma^2}{3a^2} \times \frac{a^3\theta_\gamma^3}{c^3\gamma^3}y^3\right] \qquad (51)$$

$$= \frac{i\omega}{2\gamma^2}\left[\frac{a\theta_\gamma^3}{c\gamma}y + \frac{1}{3}\frac{a\theta_\gamma^3}{c\gamma}y^3\right]$$

$$i\omega\left[t-\frac{\boldsymbol{n}\cdot\boldsymbol{X}(t')}{c}\right]=\frac{i\omega a\theta_\gamma^3}{2c\gamma^3}\left[y+\frac{1}{3}y^2\right]$$

$$= i\frac{3}{2}\left(\frac{\omega a\theta_\gamma^3}{3c\gamma^3}\right)\left[y+\frac{1}{3}y^2\right] \qquad (52)$$

$$= \frac{3}{2}i\eta\left[y+\frac{1}{3}y^3\right]$$

For t' – variable of integration:

$$t' = \frac{a\theta_\gamma}{c\gamma}y \Rightarrow dt' = \frac{a\theta_\gamma}{c\gamma}dy \qquad (53)$$

6 Evaluation of Fourier components of electric field

$$rE(\omega) = \frac{-i\omega q}{4\pi c\varepsilon_0} e^{i\omega r/c} \int_{-\infty}^{\infty} n \times (n \times \beta') \times \exp\left[i\omega\left(t' - \frac{n \cdot X(t')}{c}\right)\right] dt'$$

$$n \times (n \times \beta) \approx \theta e_{\|} - \frac{\theta_\gamma}{\gamma} y e_{\perp}$$

$$i\omega\left[t - \frac{n \cdot X(t')}{c}\right] \approx \frac{3}{2} i\eta\left[y + \frac{1}{3}y^3\right] \qquad dt' = \frac{a\theta_\gamma}{c\gamma} dy$$

Therefore we have a parallel component and a perpendicular component of the electric field:

$$rE_{\|}(\omega) = -\frac{i\omega q}{4\pi c\varepsilon_0}\left(\frac{a\theta\theta_\gamma}{c\gamma}\right)e^{i\omega r/c}\int_{-\infty}^{\infty}\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy$$

$$rE_{\perp}(\omega) = \frac{i\omega q}{4\pi c\varepsilon_0}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)e^{i\omega r/c}\int_{-\infty}^{\infty}y\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right]dy \tag{54}$$

The integrals over y have real and imaginary parts:

$$\exp\left[\frac{3}{2}i\eta\left(y+\frac{1}{3}y^3\right)\right] = \cos\left[\frac{3}{2}\eta\left(y+\frac{1}{3}y^3\right)\right] + i\sin\left[\frac{3}{2}\eta\left(y+\frac{1}{3}y^3\right)\right] \tag{55}$$

Parallel component

In the integral for $rE_{||}(\omega)$ the real part only contributes since the imaginary part is odd and the different contributions over $[-\infty, 0]$ and $[0, \infty]$ cancel.

Therefore, we evaluate:

$$\int_{-\infty}^{\infty} \exp\left[\frac{3}{2}i\eta\left(y + \frac{1}{3}y^3\right)\right]dy = \int_{-\infty}^{\infty} \cos\left[\frac{3}{2}\eta\left(y + \frac{1}{3}y^3\right)\right]dy$$

Perpendicular component

In the integral for $rE_{\perp}(\omega)$ the imaginary part only contributes since the exponential term is multiplied by y. We therefore evaluate:

$$\int_{-\infty}^{\infty} y \exp\left[\frac{3}{2} i\eta\left(y + \frac{1}{3}y^3\right)\right] dy = i\int_{-\infty}^{\infty} y \sin\left[\frac{3}{2}\eta\left(y + \frac{1}{3}y^3\right)\right] dy \quad (56)$$

These integrals are expressed in terms of Bessel functions:

$$\int_{-\infty}^{\infty} \cos\left[\frac{3}{2}\eta\left(y + \frac{1}{3}y^3\right)\right] dy = \frac{2}{\sqrt{3}} K_{1/3}(\eta)$$
$$\int_{-\infty}^{\infty} y \sin\left[\frac{3}{2}\eta\left(y + \frac{1}{3}y^3\right)\right] = -\frac{2}{\sqrt{3}} K_{2/3}(\eta) \quad (57)$$

where $K_{1/3}(\eta)$ and $K_{2/3}(\eta)$ are modified Bessel functions of order $1/3$ and $2/3$ respectively. These integrals are evaluated in the Appendix.

Final expressions

$$rE_{\|}(\omega) = -\frac{i\omega q}{2\sqrt{3}\pi c\varepsilon_0}\left(\frac{a\theta\theta_\gamma}{c\gamma}\right)e^{\frac{i\omega r}{c}}K_{1/3}(\eta)$$

$$rE_{\perp}(\omega) = \frac{\omega q}{2\sqrt{3}\pi c\varepsilon_0}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)e^{\frac{i\omega r}{c}}K_{2/3}(\eta) \qquad (58)$$

7 Ellipticity of single electron emission

The waves corresponding to the above Fourier components are, for the parallel component:

$$
\begin{aligned}
E_{\|}(r,t) &= \frac{1}{2\pi}\left(E_{\|}(\omega)e^{-i\omega t} + E_{\|}(-\omega)e^{i\omega t}\right) \\
&= \frac{1}{2\pi}\left(E_{\|}(\omega)e^{-i\omega t} + E_{\|}^{*}(\omega)e^{i\omega t}\right)
\end{aligned}
\tag{59}
$$

(using the reality condition of the Fourier transform).

and for the perpendicular component

$$\begin{aligned} E_{\perp}(r,t) &= \frac{1}{2\pi}(E_{\perp}(\omega)e^{-i\omega t} + E_{\perp}(-\omega)e^{i\omega t}) \\ &= \frac{1}{2\pi}(E_{\perp}(\omega)e^{-i\omega t} + E_{\perp}^{*}(\omega)e^{i\omega t}) \end{aligned} \tag{60}$$

Using the expressions for the Fourier components:

$$E_{\|}(r,t) = \frac{\omega q}{4\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta\theta_\gamma}{c\gamma}\right)K_{1/3}(\eta) \times \left[-ie^{i\omega\left(\frac{r}{c}-t\right)} + ie^{-i\omega\left(\frac{r}{c}-t\right)}\right] \qquad (61)$$

$$= -\frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta\theta_\gamma}{c\gamma}\right)K_{1/3}(\eta)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

$$E_{\perp}(r,t) = \frac{\omega q}{4\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)K_{2/3}(\eta)$$

$$\times\left[e^{i\omega\left(\frac{r}{c}-t\right)} + e^{-i\omega\left(\frac{r}{c}-t\right)}\right] \tag{62}$$

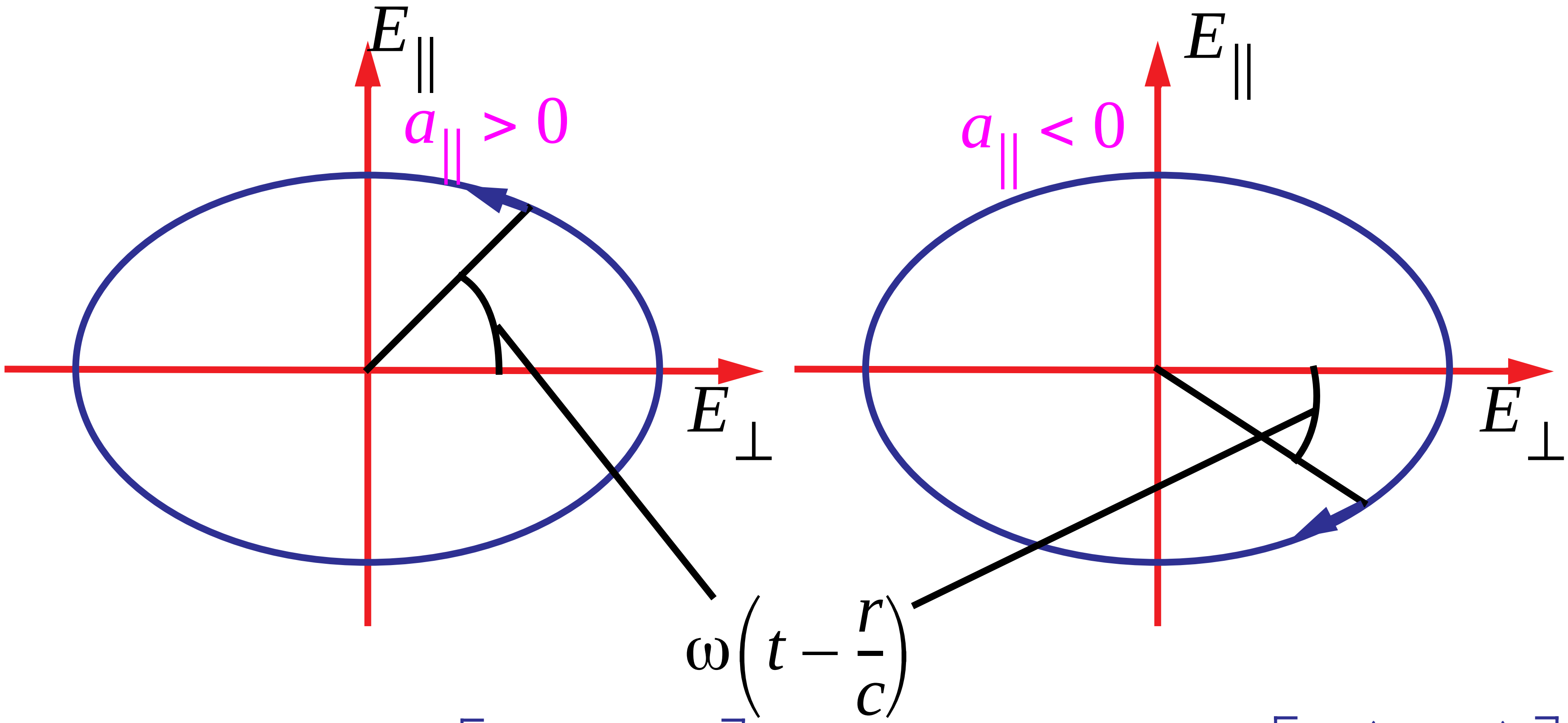
$$= \frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)K_{2/3}(\eta)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]$$

Now consider the path traced out with time by the $\boldsymbol{E}$–vector at a fixed r. Write

$$E_{\|}(r,t) = a_{\|}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]$$

$$E_{\perp}(r,t) = a_{\perp}\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \tag{63}$$

This describes an ellipse in the $(\boldsymbol{e}_{\perp}, \boldsymbol{e}_{\|})$ coordinate system. We shall see that $a_{\perp} > 0$ but that $a_{\|}$ can be positive or negative.



$$E_{\|}(r,t) = a_{\|}\sin\left[\omega\left(t - \frac{r}{c}\right)\right] \quad E_{\perp}(r,t) = a_{\perp}\cos\left[\omega\left(t - \frac{r}{c}\right)\right]$$

$$a_{\parallel} = -\frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta\theta_\gamma}{c\gamma}\right)K_{1/3}(\eta)$$

$$= \frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a}{c\gamma^2}\right)(-\gamma\theta\theta_\gamma)K_{1/3}(\eta)$$

$$a_{\perp} = \frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a\theta_\gamma^2}{c\gamma^2}\right)K_{2/3}(\eta)$$

$$= \frac{\omega q}{2\sqrt{3}\pi^2 c\varepsilon_0 r}\left(\frac{a}{c\gamma^2}\right)(\theta_\gamma^2)K_{2/3}(\eta) \tag{64}$$

- The semi-major and semi-minor axes of the electric vector depend upon θ (through the variable $\gamma\theta$) and we have to bear in mind the fact that

$$\eta = \frac{1}{2}\frac{\omega}{\omega_c}(1+\gamma^2\theta^2)^{3/2} \tag{65}$$

• The rotation of the electric vector and the shape of the polarisation ellipse depends upon θ

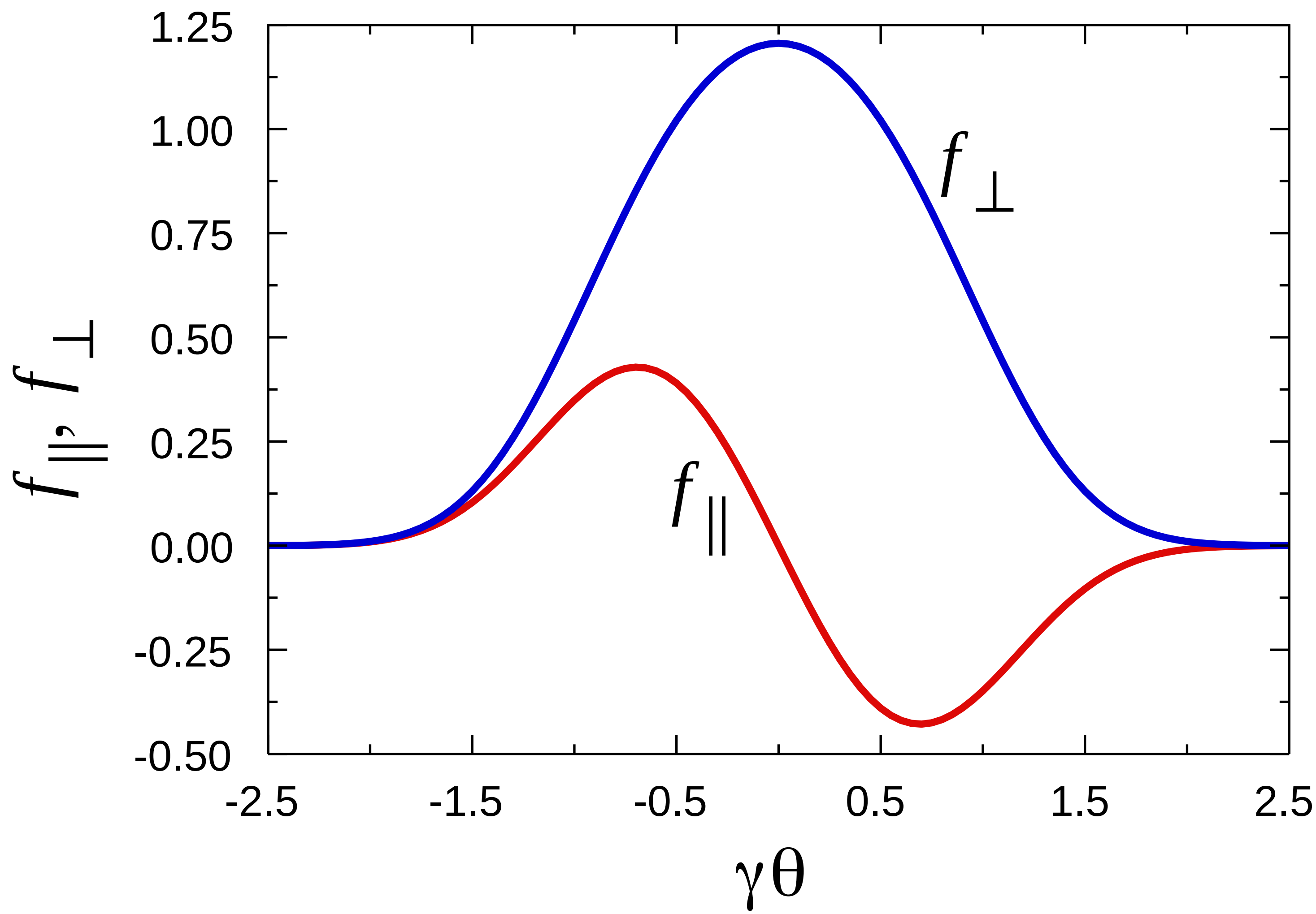
• The angular dependence of the radiation field can be understood from the plots of the functions

$$\begin{aligned} f_{\|}(x, \gamma\theta) &= (-\gamma\theta\theta_\gamma)K_{1/3}(\eta) \\ f_{\perp}(x, \gamma\theta) &= \theta_\gamma^2 K_{2/3}(\eta) \end{aligned} \tag{66}$$

where

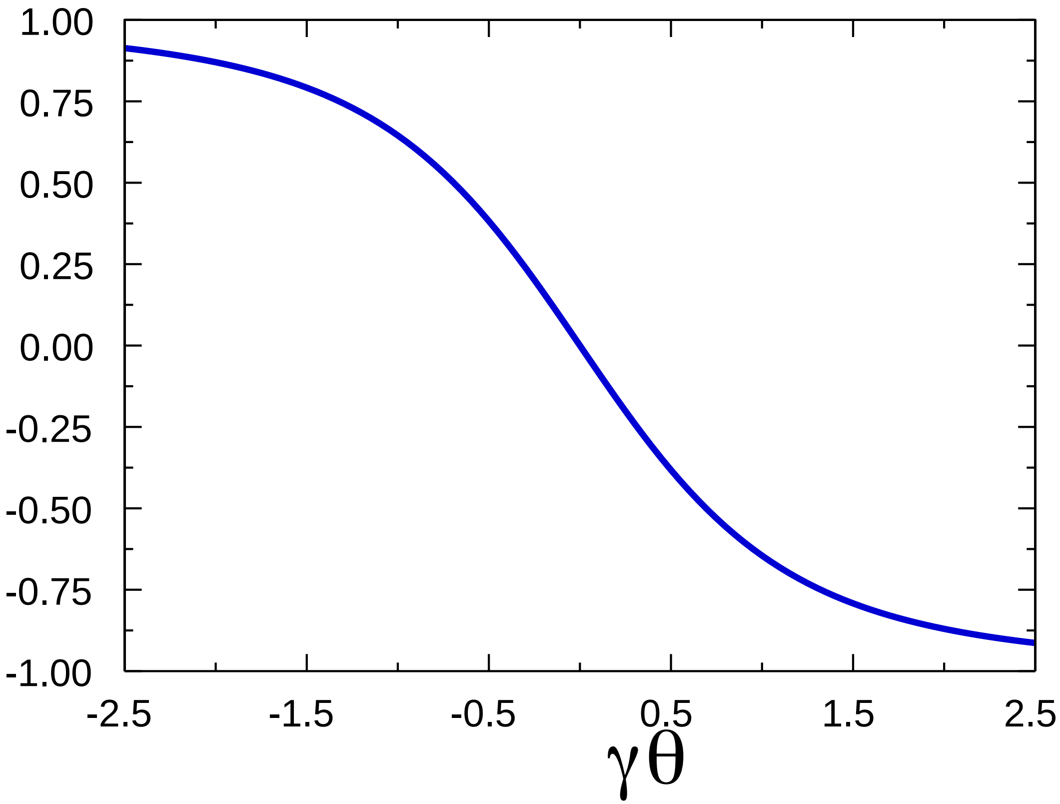
$$x = \frac{\omega}{\omega_c} \tag{67}$$

These are plotted below for $x = 1$. The variation is similar for all x.


1.25
1.00
0.75
0.50
0.25
0.00
-0.25
-0.50
-2.5
-1.5
-0.5
0.5
1.5
2.5
γθ
f||, f⊥
f⊥
f||

As can be seen both the parallel and perpendicular components of the electric field vanish rapidly for $\gamma\theta > 1$.

Axes of ellipse



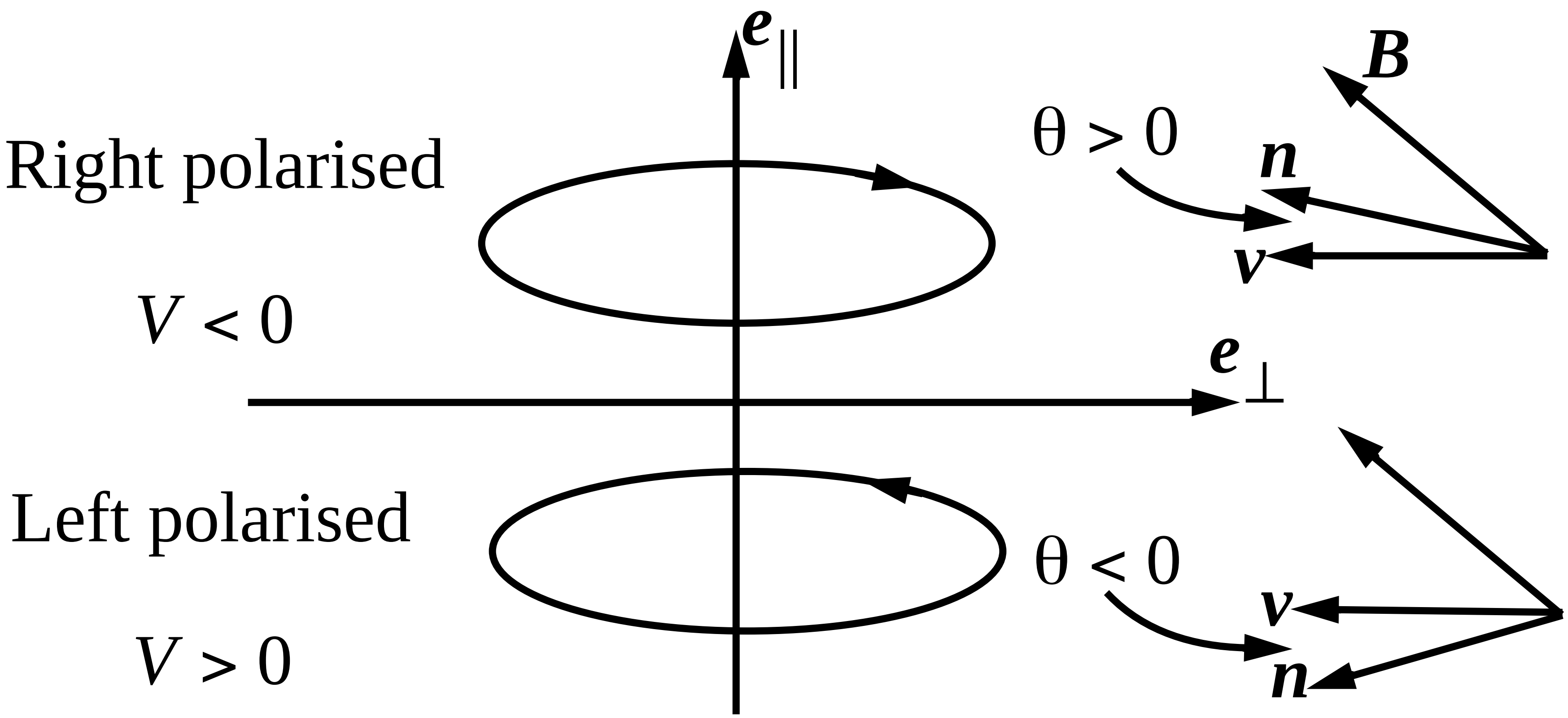
This plot shows the variation of the ratio of $a_{\|}/a_{\perp} = f_{\|}/f_{\perp}$ for $x = 1$. Since this ratio is always less than 1 for all $\gamma\theta$ the parallel axis is always the minor axis and the perpendicular axis is always the major axis.

Summary of polarisation properties of single electron emission

From the above:

- $a_{||}$ is positive for $\gamma\theta < 0$ and negative for $\gamma\theta > 0$
- Therefore the radiation is always left polarised for $\theta < 0$ and right polarised for $\theta > 0$
- The ratio $a_{||}/a_{\perp} < 1$ so that the major axis is always $\boldsymbol{e}_{\perp}$.
- That is the major axis of the polarisation ellipse is perpendicular to the projection of the magnetic field on the plane of propagation.

The following figure summarises the above. Note that $\theta = 0$ corresponds to looking exactly along the direction of the electron velocity.



8 Integrated single electron emissivity

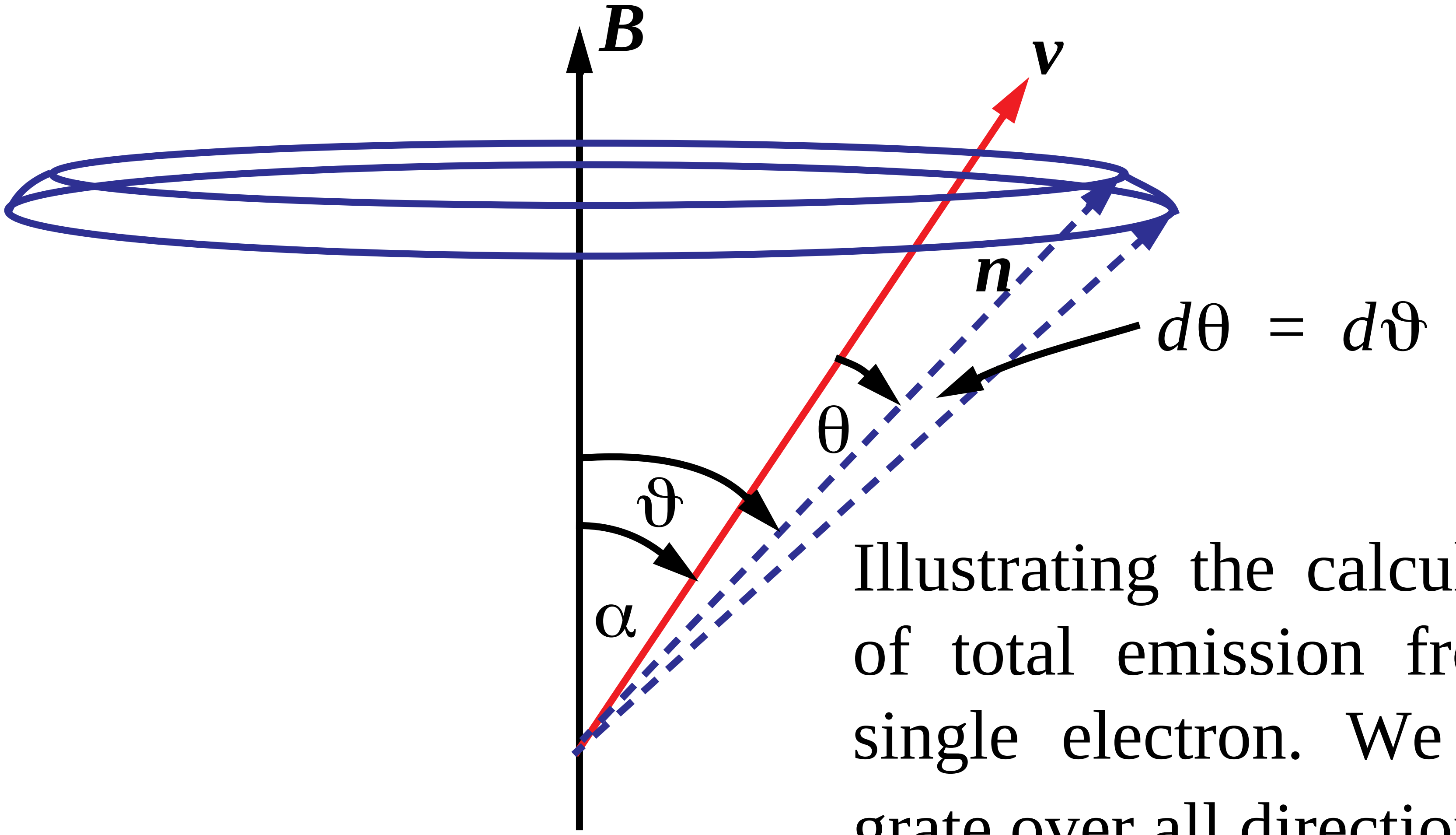
- This is a good example of how the integrated emission from a source can exhibit polarisation properties that are different from the individual components.
- A qualitative feature that one can note from the above is that the

integrated emission from a single electron will have a low circular polarisation because of the positive and negative contributions to V from negative and positive θ

- Note that $Q \approx 0$ because the axis of the ellipse is in the $\boldsymbol{e}_{\perp}$ direction.

Recall our expression for the emissivity corresponding to the various components of the polarisation tensor:

$$\frac{dW_{\alpha\beta}}{d\Omega d\omega dt} = \frac{c\varepsilon_0}{\pi T} r^2 E_{\alpha}(\omega) E_{\beta}^{*}(\omega) \tag{68}$$



Illustrating the calculation of total emission from a single electron. We integrate over all directions ***n***.

The spectral power of the pulse, per unit solid angle is:

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{dW_{11}}{d\Omega dt} + \frac{dW_{22}}{d\Omega dt} \\ &= \frac{dW_{\parallel}}{d\Omega dt} + \frac{dW_{\perp}}{d\Omega dt}\end{aligned} \tag{69}$$

The two terms correspond to emission into the two modes. We can now use this expression to determine the total power emitted by the electron.

We first consider the total energy per unit solid angle per unit circular frequency emitted over one entire gyration of the particle. Begin with the energy radiated per unit solid angle, at a particular point on the orbit, that we have just calculated:

$$
\begin{aligned}
\frac{dW_{\|}}{d\Omega d\omega} &= \frac{c\varepsilon_0}{\pi} r^2 E_{\|}(\omega) E_{\|}^{*}(\omega) \\
&= \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2 (\theta_\gamma \gamma\theta)^2 K_{1/3}^2(\eta) \\
\frac{dW_{\perp}}{d\Omega d\omega} &= \frac{c\varepsilon_0}{\pi} r^2 E_{\perp}(\omega) E_{\perp}^{*}(\omega) \\
&= \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2 \theta_\gamma^4 K_{2/3}^2(\eta)
\end{aligned}
\tag{70}
$$

These expressions are integrated over all the directions $\boldsymbol{n}$. Let ϑ, ϕ be the polar angles for the direction $\boldsymbol{n}$. The solid angle for emission around this direction is:

$$d\Omega = \sin\vartheta d\vartheta d\phi \tag{71}$$

We can use the solid angle corresponding to a complete orbit since the spectrum of a pulse is independent of ϕ.

For a complete orbit:

$$d\Omega = 2\pi \sin\vartheta d\vartheta \tag{72}$$

We have

$$\vartheta = \alpha + \theta \tag{73}$$

where θ is the angle we have used in the previous calculations. Since the contribution to the integral over solid angle mainly comes from $\theta \sim 1/\gamma$, then we can put

$$\begin{aligned} \vartheta &\approx \alpha \\ d\vartheta &= d\theta \\ d\Omega &\approx 2\pi \sin\alpha \, d\theta \end{aligned} \tag{74}$$

Hence, the total energy emitted over one gyrational period into the parallel and perpendicular modes is:

$$\frac{dW_{\|,\perp}}{d\omega} \approx 2\pi \sin\alpha \int_0^{\pi} \frac{dW_{\|,\perp}}{d\Omega\, d\omega} d\vartheta$$
$$\approx 2\pi \sin\alpha \int_{-\infty}^{\infty} \frac{dW_{\|,\perp}}{d\Omega\, d\omega} d\theta \quad (75)$$

The integral over $\vartheta = [0, 2\pi]$ is replaced by an integral over $\theta = (-\infty, \infty)$ because of the concentration of $dW_{\|,\perp}/d\Omega d\omega$ around $\theta = 0$ and the rapid vanishing of the Bessel functions for $|\gamma\theta| \gg 1$.

These expressions give the amount of energy radiated per orbit. The energy radiated per unit time is given by dividing by the period of the orbit

$$T = \frac{2\pi}{\Omega_B} = \frac{2\pi\gamma m}{|q|B} \Rightarrow \frac{1}{T} = \frac{|q|B}{2\pi\gamma m} \tag{76}$$

Hence, the power emitted in the parallel mode is

$$P_{\parallel}(\omega) = \frac{\omega^2 q^2}{12\pi^3 c\varepsilon_0}\left(\frac{a}{c\gamma^2}\right)^2 \times 2\pi\sin\alpha \times \frac{|q|B}{2\pi\gamma m} \times \int_{-\infty}^{\infty}(\gamma\theta)^2\theta_\gamma^2 K_{1/3}^2(\eta)\,d\theta \tag{77}$$

We change the variable of integration to $\gamma\theta$, thereby introducing another factor of γ into the denominator. We also use, for the local radius of curvature,

$$a = \frac{c}{\Omega_B \sin\alpha} \Rightarrow \frac{a^2}{c^2} = \frac{1}{\Omega_B^2 \sin^2\alpha} \qquad (78)$$

Various factors in the numerical factor in from of the expression for $P_{\|}(\omega)$ combine to give ω_c^2 in the denominator, where

$$\omega_c^2 = \frac{9}{4}\Omega_B^2 \gamma^6 \sin^2\alpha \qquad (79)$$

All of this combines to give:

$$P_{\parallel}(\omega) = \frac{3}{16\pi^3 c\varepsilon_0}\frac{q^3 B\sin\alpha}{m}\left(\frac{\omega}{\omega_c}\right)^2 \times\int_{-\infty}^{\infty}(\gamma\theta)^2\theta_\gamma^2 K_{1/3}^2(\eta)\,d(\gamma\theta) \tag{80}$$

The numerical factor in front of the expression for the other power $P_{\perp}(\omega)$ is identical and

$$P_{\perp}(\omega) = \frac{3}{16\pi^3 c\varepsilon_0}\frac{q^3 B\sin\alpha}{m}\left(\frac{\omega}{\omega_c}\right)^2\int_{-\infty}^{\infty}\theta_\gamma^4 K_{2/3}^2(\eta)\,d(\gamma\theta) \tag{81}$$

remembering that

$$\theta_\gamma = (1+\gamma^2\theta^2)^{1/2} \qquad \eta = \frac{1}{2}\frac{\omega}{\omega_c}(1+\gamma^2\theta^2)^{3/2} \tag{82}$$

The two integrals involve integration over the variable $\gamma\theta$ and involve as parameter

$$x = \frac{\omega}{\omega_c} \tag{83}$$

These integrals were evaluated by Westfold (1959) in one of the fundamental papers on synchrotron emission. The results are:

$$\int_{-\infty}^{\infty} (\gamma\theta)^2 \theta_\gamma^2 K_{1/3}^2(\eta)\, d(\gamma\theta) = \frac{\pi}{\sqrt{3}} x^{-2} [F(x) - G(x)]$$

$$\int_{-\infty}^{\infty} \theta_\gamma^4 K_{2/3}^2(\eta)\, d(\gamma\theta) = \frac{\pi}{\sqrt{3}} x^{-2} [F(x) + G(x)] \qquad (84)$$

where the functions are defined by

$$F(x) = x \int_x^{\infty} K_{5/3}(z)\, dz \qquad G(x) = x K_{2/3}(x) \qquad (85)$$

Hence the components of the single electron emissivities per unit frequency are:

$$
\begin{aligned}
P_{\|}(\omega) &= \frac{\sqrt{3}}{16\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}[F(x) - G(x)] \\
P_{\perp}(\omega) &= \frac{\sqrt{3}}{16\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}[F(x) + G(x)]
\end{aligned}
\tag{86}
$$

The total synchrotron power per unit circular frequency from a single electron is:

$$P_{\text{tot}}(\omega) = P_{\|}(\omega) + P_{\perp}(\omega) = \frac{\sqrt{3}}{8\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}F(x) \tag{87}$$

$$x = \frac{\omega}{\omega_c}$$

The functions $F(x)$ and $G(x)$ are plotted on linear and logarithmic scales in the following figures. The asymptotic forms are as follows:

Small x

$$F(x) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)2^{1/3}} x^{1/3}$$

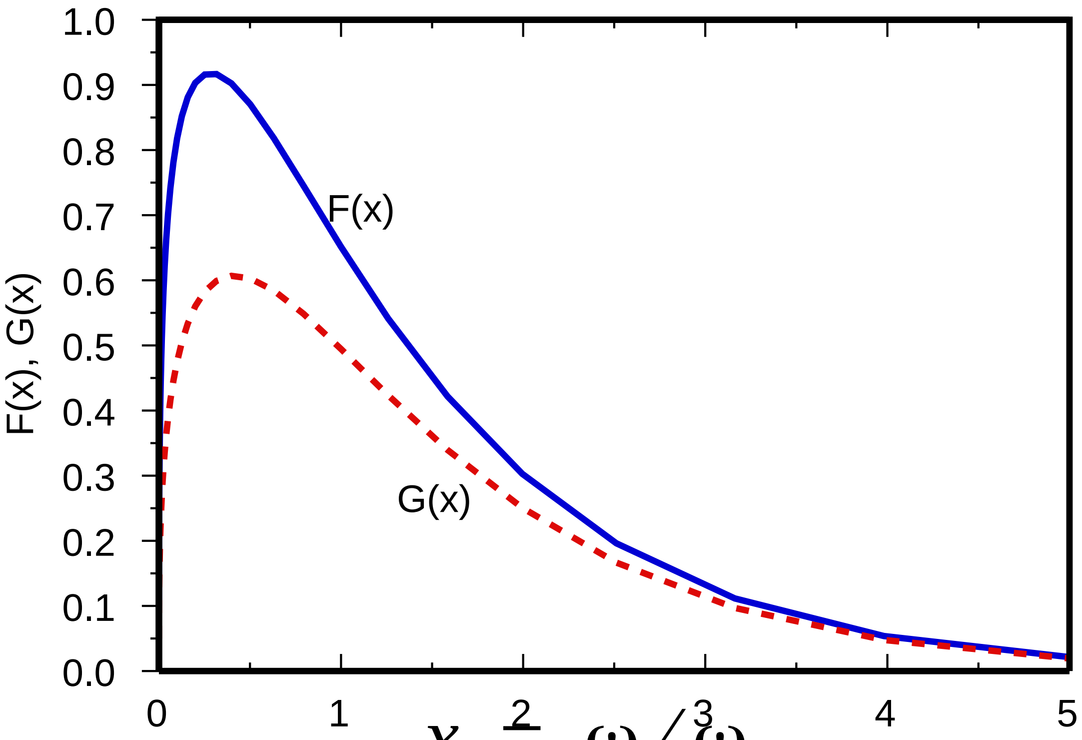
$$G(x) \sim \frac{2\pi}{\sqrt{3}\Gamma(1/3)2^{1/3}} x^{1/3}$$

(88)

Large x

$$F(x) \sim \sqrt{\frac{\pi x}{2}} e^{-x}$$

$$G(x) \sim \sqrt{\frac{\pi x}{2}} e^{-x}$$



$x = \omega/\omega_c$

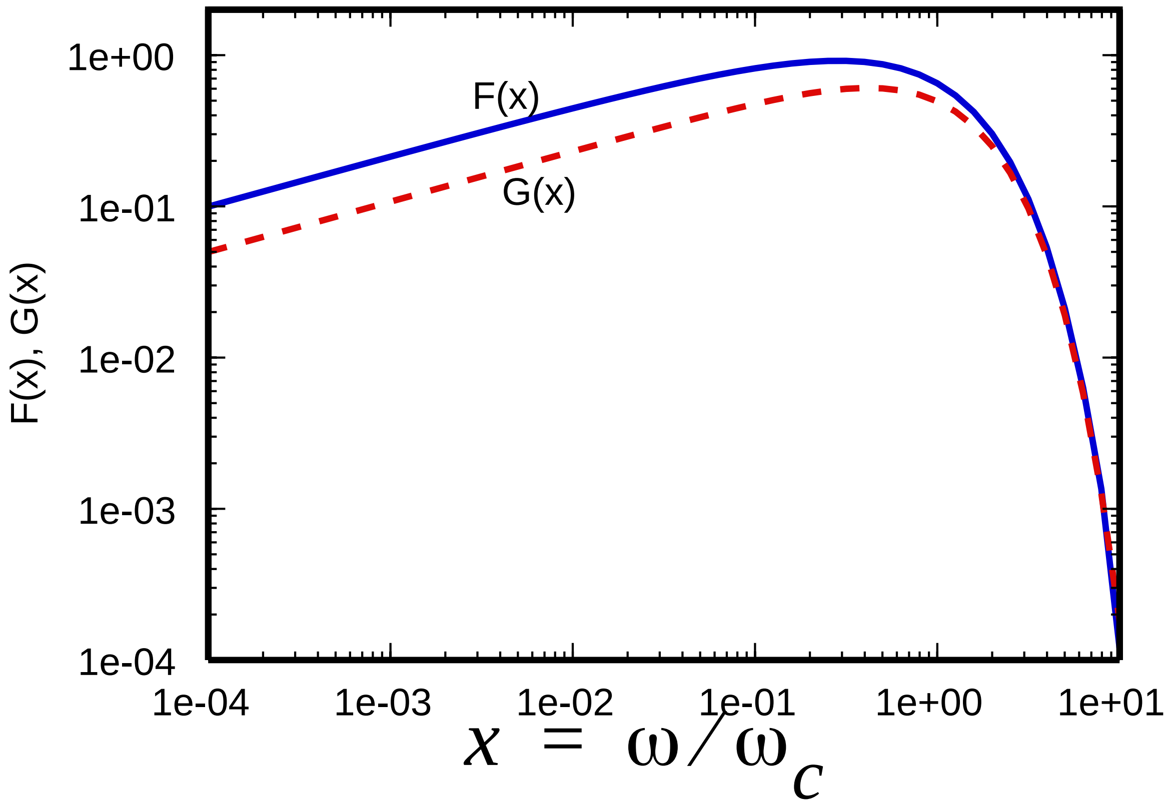
Features

- Peak of $F(x)$ at

$x = \omega/\omega_c \approx 0.29$

- Emission is fairly broadband

with $\dfrac{\Delta\omega}{\omega} \sim 1$



Logarithmic version of the plot of $F(x)$ and $G(x)$, showing the power law dependence of $F(x)$ and $G(x)$ for small x.

9 Synchrotron cooling

9.1 Integration of synchrotron power

A fairly immediate implication of the power radiated by a single electron is the loss of energy by the electron.

Since the power per unit frequency is:

$$P(\omega) = \frac{\sqrt{3}}{8\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}F(x) \tag{89}$$

then the total power radiated over all frequencies is

$$\begin{aligned} P &= \int_0^{\omega} P(\omega)d\omega = \frac{\sqrt{3}}{8\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}\int_0^{\infty} F(x)d\omega \\ &= \frac{\sqrt{3}}{8\pi^2\varepsilon_0 c}\frac{q^3 B\sin\alpha}{m}\omega_c\int_0^{\infty} F(x)dx \end{aligned} \tag{90}$$

The integral

$$\int_0^{\infty} F(x)dx = \int_0^{\infty}\left[x\int_x^{\infty} K_{5/3}(t)dt\right]dx = \frac{8\sqrt{3}\pi}{27} \qquad (91)$$

and

$$\omega_c = \frac{3}{2}\frac{qB}{m}\gamma^2 \sin^2\alpha \qquad (92)$$

Therefore,

$$P = \frac{1}{6\pi\varepsilon_0 c}\frac{q^4 B^2 \sin^2\alpha}{m^2}\gamma^2 \qquad (93)$$

In terms of energy ($E = \gamma mc^2$):

$$P = \frac{1}{6\pi\varepsilon_0}\left(\frac{q^4 B^2 \sin^2\alpha}{m^4 c^5}\right)E^2 \qquad (94)$$

The power of m^4 appearing in this expression is why synchrotron emission from protons is not usually considered.

9.2 Other expressions for total power emitted by electrons

Take $q = e$ so that the total power emitted by an electron is

$$P = \frac{1}{6\pi\varepsilon_0 c}\frac{e^4 B^2 \sin^2\alpha}{m^2}\gamma^2 \qquad (95)$$

This is often expressed in terms of the Thomson cross-section for an electron.

The classical radius of an electron is defined by

$$
\begin{aligned}
\text{Electrostatic potential energy} &= \text{Rest mass energy} \\
\frac{e^2}{4\pi\varepsilon_0 r_0} &= m_e c^2 \\
\Rightarrow r_0 &= \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \\
&= 2.818\times10^{-15}\ \text{m}
\end{aligned}
\qquad (96)
$$

The Thomson cross-section is

$$\sigma_T = \frac{8\pi}{3} r_0^2 = \frac{e^4}{6\pi\varepsilon_0^2 m_e^2 c^4} = 6.65\times 10^{-29}\ \mathrm{m}^2 \qquad (97)$$

This cross-section is important, *inter alia*, when considering the scattering of photons by electrons.

In terms of σ_T the power emitted by an electron is:

$$P = \frac{1}{6\pi\varepsilon_0 c} \frac{e^4 B^2 \sin^2\alpha}{m^2} \gamma^2 = (c\sigma_T)\left(\frac{B^2}{\mu_o}\right) \sin^2\alpha\gamma^2 \qquad (98)$$

9.3 Synchrotron cooling of electrons

By conservation of energy, the rate of energy lost by the electron is

$$\frac{dE}{dt} = -P = -(c\sigma_T)\left(\frac{B^2}{\mu_o}\right)\sin^2\alpha\gamma^2 \qquad (99)$$

Hence, the rate of change of Lorentz factor is:

$$\frac{d\gamma}{dt} = \frac{1}{m_e c^2}\frac{dE}{dt} = -\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_o}\right)\sin^2\alpha\gamma^2 \qquad (100)$$

This equation can be put in the form

$$-\frac{1}{\gamma^2}\frac{d\gamma}{dt} = \left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_o}\right)\sin^2\alpha \qquad (101)$$

Let γ_0 be the value of γ at $t = 0$, then assuming that B and α remain constant during the time t, then

$$\frac{\gamma}{\gamma_0} = \frac{1}{1 + \left[\gamma_0\left(\frac{\sigma_T}{m_e c}\right)\left(\frac{B^2}{\mu_o}\right)\sin^2\alpha\right]t} \qquad (102)$$

This defines a synchrotron cooling time:

$$t_{\mathrm{syn}} = \left(\frac{m_e c}{\sigma_T}\right)\left(\frac{B^2 \sin^2\alpha}{\mu_0}\right)^{-1} \gamma_0^{-1} \qquad (103)$$

Features

- t_{syn} decreases with increasing γ – the higher energy electrons cool the fastest
- t_{syn} decreases with increasing magnetic field.

Example

Take typical values for the lobes of a radio galaxy:

$$B = 10\mu\mathrm{G} = 1\ \mathrm{nT} \qquad \gamma_0 = 10^4 \tag{104}$$

$$t_{\mathrm{syn}} = \frac{9.11\times10^{-31} \times 3\times10^{8}}{6.65\times10^{-29}} \times \left[\frac{(1\times10^{-9})^2}{4\pi \times 10^{-7}}\right]^{-1} \times \gamma_0^{-1}$$
$$= 1.6\times10^{7}\ \mathrm{yrs} \left(\frac{\gamma \sin^2\alpha}{10^4}\right)^{-1} \tag{105}$$

Estimates of the ages of radio galaxies often exceed 10^8 years. Hence there is a need to re-energise particles in the lobes of radio galaxies. This introduces the need for ***particle acceleration***.

10 Appendix: Evaluation of synchrotron integrals

The evaluation of the synchrotron integrals follows from the result (Abramowitz and Stegun, eqn. 10.4.32)

$$I(a, x) = \int_0^\infty \cos[ay^3 + xy]dy = (3a)^{-1/3}\pi Ai((3a)^{-1/3}x) \quad (106)$$

where $Ai(z)$ is the Airy function. In our case,

$$x = \frac{3}{2}\eta \qquad a = \frac{1}{2}\eta \quad (107)$$

Differentiating, $I(a, x)$ with respect to x gives:

$$\begin{aligned}\frac{\partial}{\partial x} I(a, x) &= -\int_0^\infty y \sin(ay^3 + xy) dy \\ &= -(3a)^{-2/3} \pi Ai'[(3a)^{-1/3} x]\end{aligned} \qquad (108)$$

so that

$$\int_0^\infty y \sin(ay^3 + xy) dy = (3a)^{-2/3} \pi Ai'[(3a)^{-1/3} x] \qquad (109)$$

Furthermore, the Airy function and its derivative are related to the modified Bessel functions of order $1/3$ and $2/3$ via:

$$K_{1/3}(\zeta) = \pi\left(\frac{3}{z}\right)^{1/2} Ai(z) \qquad K_{2/3}(\zeta) = -\pi\frac{3^{1/2}}{z} Ai'(z)$$

$$z = \left(\frac{3}{2}\zeta\right)^{2/3} \tag{110}$$

See equations (10.4.26) and (10.4.31) of Abramowitz and Stegun.

For $x = 3\eta/2$ and $a = \eta/2$, the argument of the Airy function and its derivative is

$$z = (3a)^{-1/3}x = \left(\frac{3\eta}{2}\right)^{-1/3}\left(\frac{3\eta}{2}\right) = \left(\frac{3\eta}{2}\right)^{2/3}. \tag{111}$$

Hence, the argument, ζ, of the corresponding Bessel functions is η. Hence,

$$\begin{aligned}
\int_0^\infty \cos[ay^3 + xy]dy &= (3a)^{-1/3}\pi Ai[(3a)^{-1/3}x] \\
&= \left(\frac{3\eta}{2}\right)^{-1/3}\pi Ai\left[\left(\frac{3\eta}{2}\right)^{2/3}\right] \\
&= \left(\frac{3\eta}{2}\right)^{-1/3}\pi \times \\
&\qquad \times \pi^{-1}\left(\frac{3\eta}{2}\right)^{1/3}3^{-1/2}K_{1/3}(\eta) \\
&= \frac{1}{\sqrt{3}}K_{1/3}(\eta)
\end{aligned} \tag{112}$$

and

$$
\begin{aligned}
\int_0^\infty y \sin(ay^3 + xy)\,dy &= (3a)^{-2/3}\pi Ai'[(3a)^{-1/3}x] \\
&= \left(\frac{3\eta}{2}\right)^{-2/3}\pi Ai'\left[\left(\frac{3\eta}{2}\right)^{2/3}\right] \\
&= \left(\frac{3\eta}{2}\right)^{-2/3} \times -3^{-1/2}\left(\frac{3\eta}{2}\right)^{2/3} K_{2/3}(\eta) \\
&= -\frac{1}{\sqrt{3}} K_{2/3}(\eta)
\end{aligned}
\qquad (113)
$$

Finally, the integrals from $-\infty$ to ∞ are twice the integrals from 0 to ∞:

$$\int_{-\infty}^{\infty} \cos\left[\frac{1}{2}y^3 + \frac{3}{2}y\right] dy = \frac{2}{\sqrt{3}} K_{1/3}(\eta)$$

$$\int_{0}^{\infty} y \sin\left(\frac{1}{2}y^3 + \frac{3}{2}y\right) dy = -\frac{1}{\sqrt{3}} K_{2/3}(\eta) \qquad (114)$$