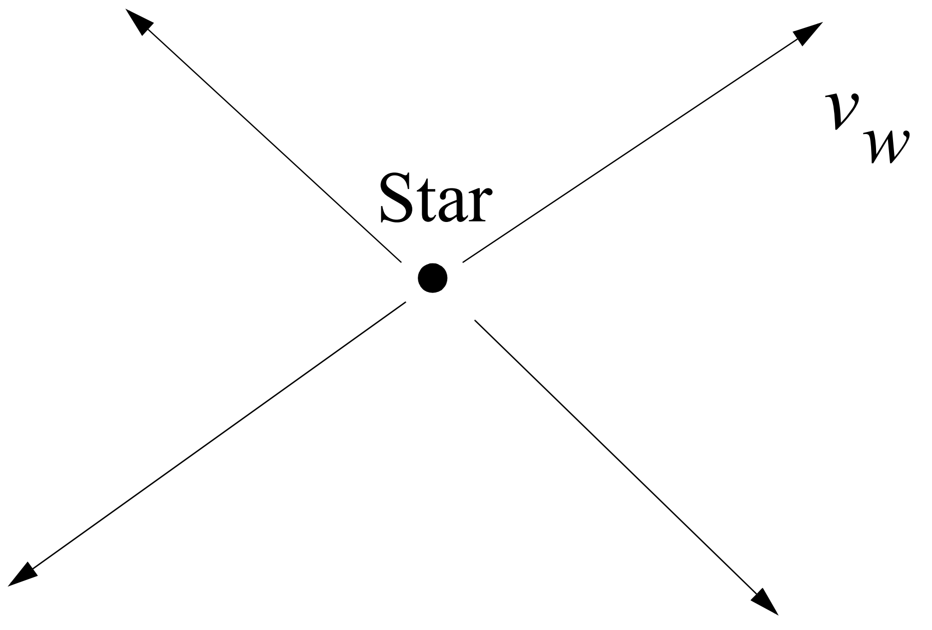
Stellar Winds



1 Characteristics of stellar winds

Solar wind

Velocity at earth's orbit:

$$v \approx 400 \text{ km/s} \tag{1}$$

Density:

$$n \approx 10^7 \text{ m}^{-3} \tag{2}$$

Temperature:

$$T \approx 10^5 \text{ K} \tag{3}$$

Speed of sound:

$$c_s = 50 \text{ km/s} \tag{4}$$

Mass flux (spherically symmetric wind):

$$\dot{M} = 4\pi\mu n m v_r r^2 = 3\times10^{-14}\,\text{solar masses /yr} \qquad (5)$$

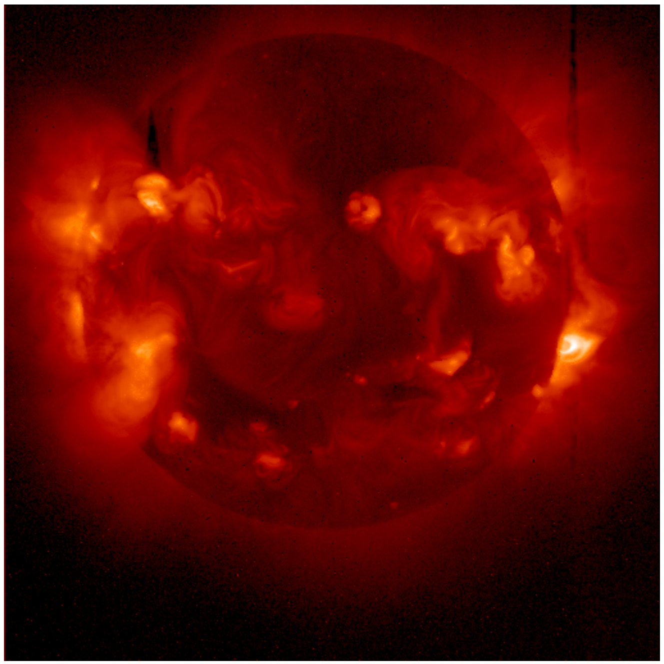
Other stars

Red giants: $\dot{M} \approx 10^{-11} M_{\text{sun}}/\text{yr}$

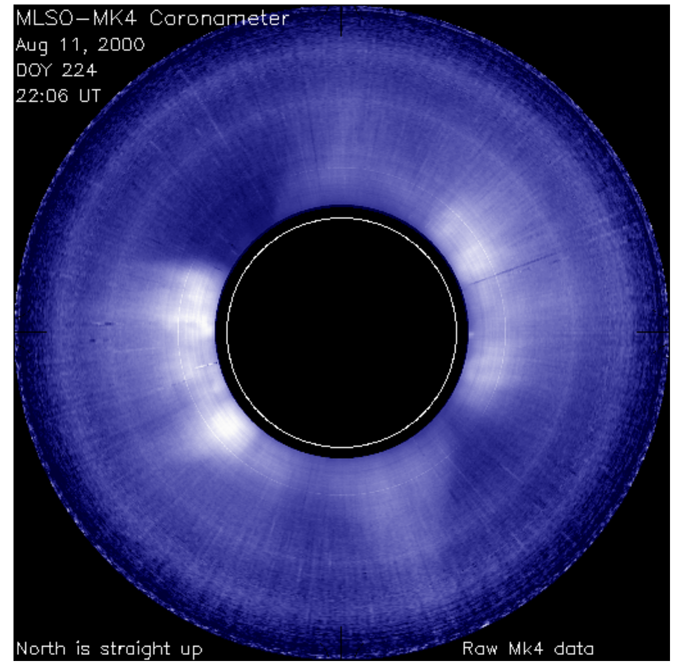
O&B type stars: $\dot{M} \approx (10^{-7} - 10^{-6}) M_{\text{sun}}/\text{yr}$

Protostars: $\dot{M} \approx 10^{-4} M_{\text{sun}}/\text{yr}$

2 Why winds?

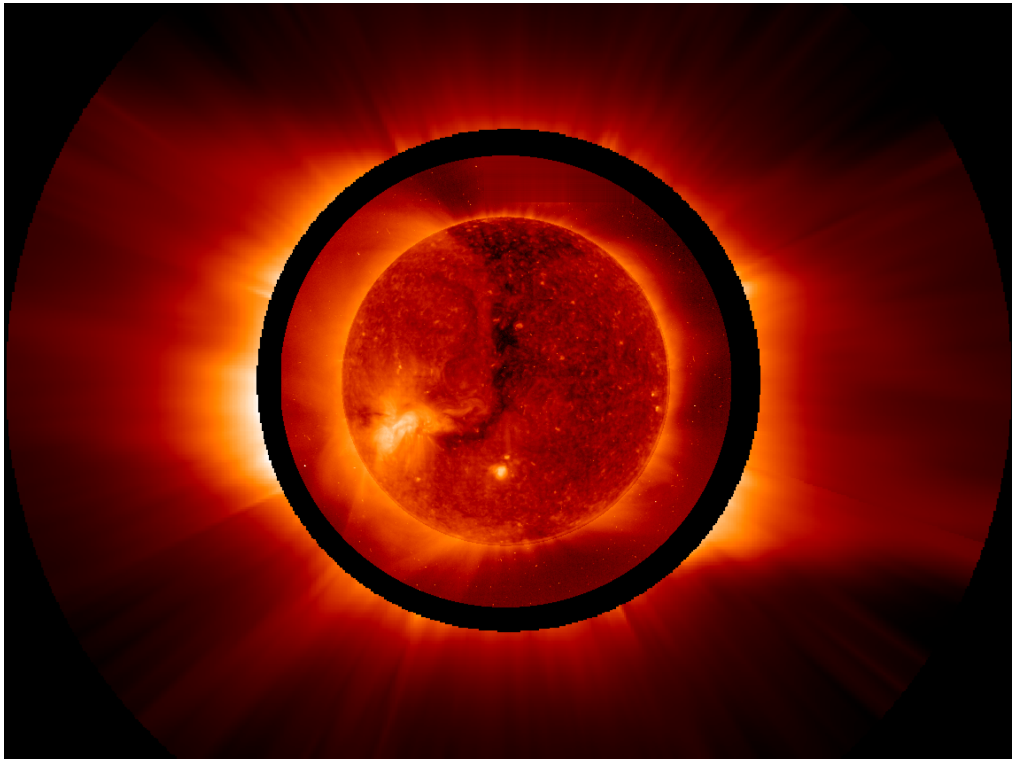


Yohkoh Soft X-ray Telescope (SXT) full-field images from the Hiraiso Solar Terrestrial Research Center / CRL



White-light Mk. 4 coronameter images

http://umbra.nascom.nasa.gov/images/



The Sun's outer atmosphere as it appears in ultraviolet light emitted by ionized oxygen flowing away from the Sun to form the solar wind (region outside black circle), and the disk of the Sun in light emitted by ionized iron at temperatures near two million degrees Celsius (region inside circle). This composite image taken by two instruments (UVCS, outer region and EIT, inner region) aboard the SOHO spacecraft shows dark areas called coronal holes at the poles and across the disk of the Sun where the highest speed solar wind originates. UVCS has discovered that the oxygen atoms flowing out of these regions have extremely high energies corresponding to temperatures of over 200 million degrees Celsius and accelerate to supersonic outflow velocities within 1.5 solar radii of the solar surface. The structure of the corona is controlled by the Sun's magnetic field which forms the bright active regions and the ray-like structures originating in the coronal holes. The composite image allows one to trace these structures from the base of the corona to millions of kilometers above the solar surface. (http://sohowww.nascom.nasa.gov/gallery/UVCS/)

2.1 Hydrostatic atmospheres

Analysis of the X-ray and radio emission from the solar corona indicates a temperature $\sim 10^6\,^\circ$K. Gas at this temperature cannot be held in by the gravitational field of the Sun.To show this, we first consider *hydrostatic atmospheres* and show that an hydrostatic atmosphere is not feasible.

The momentum equations are:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \qquad (6)$$

$$\phi = \text{Gravitational potential} \qquad (7)$$

Hydrostatic solutions

$$v_i = 0$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial p}{\partial x_i} = -\frac{\partial \phi}{\partial x_i} \tag{8}$$

Spherical symmetry

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{d\phi}{dr} \tag{9}$$

Suppose we consider isothermal solutions with the pressure defined by the following equation:

Boltzmanns constant $\approx 1.38\times10^{-23}\,\mathrm{J\,K^{-1}}$

Temperature $\sim 10^{6\circ}\mathrm{K}$

$$p = \frac{\rho k T}{\mu m} \tag{10}$$

Atomic mass unit $\approx 1.66\times10^{-27}$ Kg

Molecular weight ≈ 0.62

The potential exterior to a spherical star is

Newtons gravitational constant

$$= 6.67\times10^{-11}\ \text{SI units}$$

$$\phi = -\frac{GM}{r}$$

Mass of sun

$$= 2.0\times10^{30}\ \text{kg} \qquad (11)$$

For an isothermal atmosphere, the hydrostatic equation becomes:

$$\frac{kT}{\mu m}\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{d\phi}{dr}$$

$$\text{Integrating} \Rightarrow \ln\rho = -\frac{\mu m}{kT}\phi + \text{Constant} \tag{12}$$

Suppose the density at a point r_0 within the corona is ρ_0. Then the constant of integration is

$$\text{Constant} = \ln\rho_0 + \frac{\mu m}{kT}\phi_0$$

$$\Rightarrow \ln\frac{\rho}{\rho_0} = -\frac{\mu m}{kT}(\phi - \phi_0) = \frac{\mu GMm}{kT}\left(\frac{1}{r} - \frac{1}{r_0}\right) \tag{13}$$

$$\Rightarrow \frac{\rho}{\rho_0} = \exp\left[\frac{\mu GMm}{kT}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]$$

Since the pressure is proportional to the density, the pressure in this atmosphere is given by:

$$\frac{p}{p_0} = \exp\left[\frac{\mu GMm}{kT}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right] = \exp\left[-\frac{\mu GMm}{kTr_0}\left(1 - \frac{r_0}{r}\right)\right] \quad (14)$$

The pressure at $r = \infty$ – the pressure cooker model

The pressure described by the above equation decreases as $r \to \infty$. ***However, it does not decrease to zero.*** The asymptotic value is

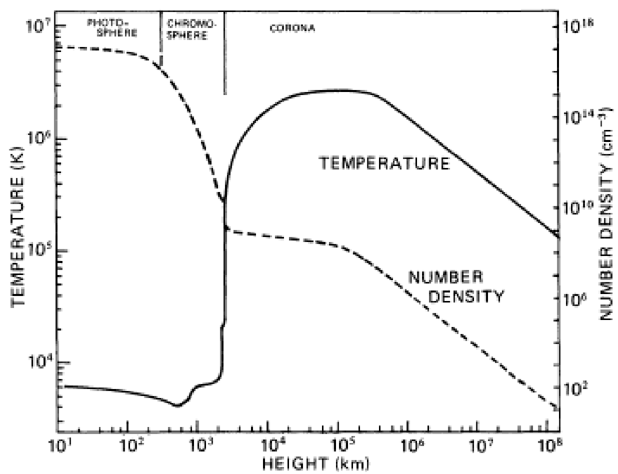
$$\frac{P_\infty}{P_0} = \exp\left[-\frac{\mu GMm}{kTr_0}\right] \quad (15)$$

The interpretation of this pressure is that this is what is required in addition to the gravitational field to confine the atmosphere. In the case of the sun, this confining pressure has to be provided by the interstellar medium.

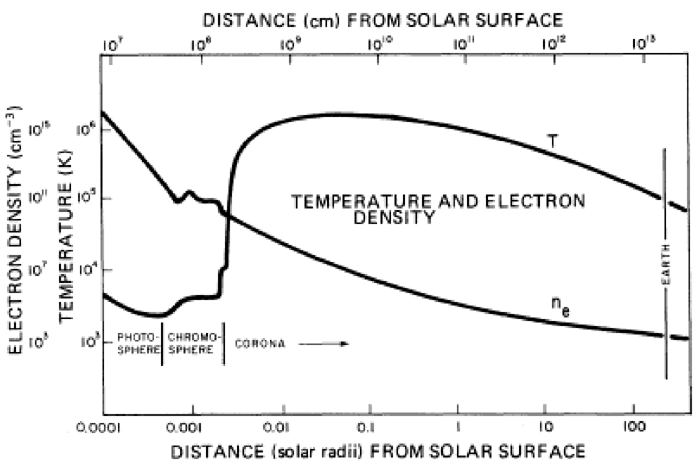
2.2 Some numbers

The Sun (*cont.*)

Temperature and density as a function of distance from the solar surface. (Courtesy of G. Withbroe, Harvard/Smithsonian Center for Astrophysics.)



Solar temperature and electron density. (Adapted from Carrigan, A. L. & Skrivanek, *Aerospace Environment*, Air Force Cambridge Research Laboratories, Massachusetts, 1974.)



Excerpt from the Handbook of Astronomy and Astrophysics, available from:

http://adsabs.harvard.edu/books/hsaa/idx.html

For indicative calculations let us insert some numbers into the above equations, with the view in mind of determining what pressure we would need to hold the sun's corona in. The following parameters relate to the corona at a solar radius.

$$
\begin{aligned}
\mu &= 0.62 \qquad G = 6.67\times10^{-11}\,\mathrm{SI} \\
m &= 1.66\times10^{-27}\,\mathrm{kg} \\
T &= 1.5\times10^{6}\,{}^{\circ}\mathrm{K} \qquad \text{Solar radius} = 6.96\times10^{8}\ \mathrm{m} \\
\text{Electron density} &= 1.55\times10^{8}\,\mathrm{cm}^{-3} = 1.55\times10^{14}\ \mathrm{m}^{-3}
\end{aligned}
\tag{16}
$$

For order of magnitude purposes, it is not necessary to distinguish between electron density and total number density. However, it is useful to go though the exercise, linking electron density to total number density and mass density.

Number density, electron density and mass density in a fully ionised plasma

Suppose that the plasma consists of ionised Hydrogen, and fully ionised Helium and the associated electrons. Then,

$$\begin{aligned} \text{Electron density} = n_e &= n_H + 2n_{He} \\ &= n_H\left(1 + 2\frac{n_{He}}{n_H}\right) \end{aligned} \tag{17}$$

The total number density is:

$$
\begin{aligned}
n &= n_H + n_{He} + n_e \\
&= (2n_H + 3n_{He}) \\
&= n_H\left(2 + 3\frac{n_{He}}{n_H}\right)
\end{aligned}
\tag{18}
$$

The mass density is given by:

$$
\begin{aligned}
\rho &= n_H m + 4n_{He} m \\
&= n_H m\left(1 + 4\frac{n_{He}}{n_H}\right)
\end{aligned}
\tag{19}
$$

If we adopt a solar abundance of Helium, then

$$\frac{n_{He}}{n_H} \approx 0.085 \Rightarrow n_e \approx 1.17 n_H \qquad n \approx 2.26 n_H \tag{20}$$

$$\rho \approx 1.34 n_H m \approx 0.59 m$$

Usually, based on more accurate composition and allowing for metals, i.e. elements heavier than Helium, we adopt $\mu = 0.62$ and

$$\rho = 0.62 nm \tag{21}$$

Also from the above numbers:

$$n \approx 1.9 n_e \tag{22}$$

Required pressure for the pressure-cooker model

$$\frac{p_\infty}{p_0} = \exp\left[-\frac{\mu G M m}{k T r_0}\right] \qquad (23)$$

At a solar radius $r_0 = 6.96 \times 10^8$ m, we have

$$\frac{\mu G M m}{k T r_0} = 7.2 \Rightarrow \frac{p_\infty}{p_0} \approx 7.5 \times 10^{-4} \qquad (24)$$

The pressure at the base of the corona, is

$$p_0 = nkT = 1.9 n_e kT = 8.1 \times 10^{-3}\ \mathrm{N\ m^{-2}} \qquad (25)$$

Therefore, the ISM pressure required by the pressure cooker model is:

$$p_\infty = 7.5\times10^{-4} \times 8.1\times10^{-3}\ \mathrm{N\ m^{-2}} = 6.1\times10^{-6}\ \mathrm{N\ m^{-2}} \qquad (26)$$

The ISM is a multi-phase medium, with all phases in pressure equilibrium. Take the warm phase, for example:

$$\begin{gathered} n \approx 1\ \mathrm{cm^{-3}} \approx 10^6\ \mathrm{m^{-3}} \qquad\qquad T \approx 10^4\,{}^\circ\mathrm{K} \\ p_{\mathrm{ISM}} \approx 1.4\times10^{-13}\ \mathrm{N\ m^{-2}} \end{gathered} \qquad (27)$$

The ISM pressure fails by a factor of about 4×10^7 ***to contain the solar atmosphere!*** Hence, the solar atmosphere flows out in a wind.

2.3 Preliminary estimate of mass flux

In order to gain some idea of the mass flux we might expect, let us suppose that the solar wind flows out spherically from a solar radius at the escape velocity from the Sun.

$$\text{Escape velocity} = \left(\frac{2GM}{r_0}\right)^{1/2} = 6.2\times10^5 \text{ m s}^{-1} \qquad (28)$$

$$= 620 \text{ km s}^{-1}$$

$$
\begin{aligned}
\text{Mass flux} &= 4\pi \times r_0^2 \times \rho_0 \times V_{\text{esc}} \\
&= 4\pi \times (6.96\times 10^8)^2 \times 1.2 n_e m \times 6.2\times 10^5 \\
&= 1.2\times 10^{12}\ \text{kg s}^{-1} \\
&= 2.0\times 10^{-11}\ \text{solar masses per year}
\end{aligned}
\tag{29}
$$

This estimate is out by about three orders of magnitude mainly because our estimate of the density of the outflowing wind is in error. It is interesting to see how the correct theory makes allowance for this.

3 Analysis of spherically symmetric winds

3.1 Fundamental equations

Euler equations

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \qquad (30)$$

$$\text{Spherical symmetry} \Rightarrow v_r \frac{\partial v_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\partial \phi}{\partial r}$$

See Landau & Lifshitz *Fluid Mechanics* for expressions for the gas dynamics equations in cylindrical and spehrical coordinates.

For a spherical star of mass M

$$\phi = -\frac{GM}{r}$$
$$-\frac{\partial\phi}{\partial r} = -\frac{GM}{r^2} \tag{31}$$

Equation of state

$$\text{Adiabatic:} \quad p = K(s)\rho^\gamma$$
$$\text{Polytropic:} \quad p = C\rho^\gamma \tag{32}$$

In the latter, γ is not necessarily c_p/c_v. $\gamma < 5/3 \Rightarrow$ Heating (in an expanding flow).

Isothermal equation of state

$$p = \frac{kT}{\mu m}\rho \qquad T = \text{constant} \tag{33}$$

Source of heat for the solar wind: Dissipation by waves in solar wind generated in photosphere/chromosphere.

Speed of sound

$$a_s^2 = \frac{dp}{d\rho} = \gamma C \rho^{\gamma - 1} \tag{34}$$

where the derivative is no longer at constant entropy. We also define the

$$\text{Isothermal sound speed} = \sqrt{\frac{kT}{\mu m}} \tag{35}$$

Radial momentum equation

Pressure gradient:

$$\frac{dp}{dr} = \frac{dp}{d\rho}\frac{d\rho}{dr} = a_s^2\frac{d\rho}{dr} \tag{36}$$

so that

$$v_r\frac{dv_r}{dr} = -\frac{a_s^2}{\rho}\frac{d\rho}{dr} - \frac{GM}{r^2} \tag{37}$$

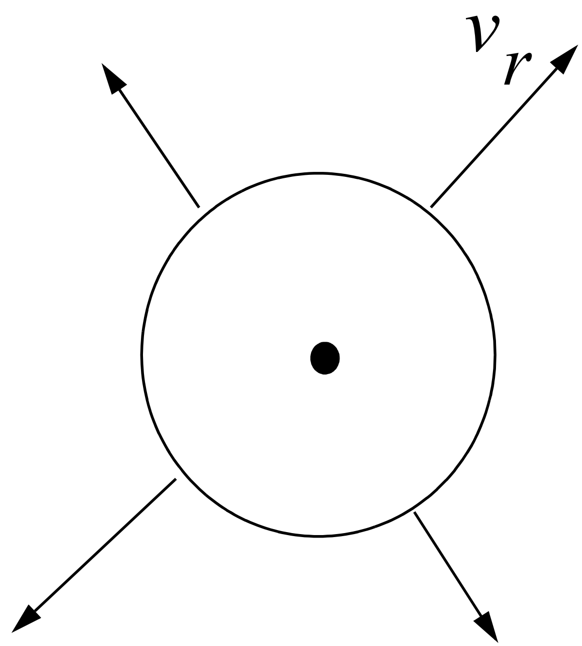
Mass flux

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = \frac{1}{r^2}\frac{d}{dr}(r^2\rho v_r) = 0$$

$$\Rightarrow \rho r^2 v_r = \text{Constant} = \frac{\dot{M}}{4\pi} \tag{38}$$

$$\dot{M} = \int_{\text{Sphere}} \rho v_i n_i dS = 4\pi r^2 \rho v_r \tag{39}$$

From now on $v_r = v$.



Also note that this equation gives us a way of estimating the density given the velocity. Solving for ρ:

$$\rho = \frac{\dot{M}}{4\pi v r^2} \tag{40}$$

3.2 Sonic point

An important feature of all winds is the existence of a sonic point where the flow makes a transition from subsonic to supersonic flow. To show the existence of this we consider both the mass flux equation and the momentum equation.

Mass flux

$$4\pi\rho v R^2 = \dot{M} \tag{41}$$

Taking logs and differentiating:

$$\ln 4\pi + \ln\rho + \ln v + 2\ln R = \ln\dot{M}$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} + \frac{2}{r} = 0 \tag{42}$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} = -\left(\frac{1}{v}\frac{dv}{dr} + \frac{2}{r}\right)$$

Substitute $\rho^{-1}\frac{d\rho}{dr}$ into momentum equation:

$$v\frac{dv}{dr} = \frac{a_s^2}{v}\frac{dv}{dr} + 2\frac{a_s^2}{r} - \frac{GM}{r^2}$$

$$(v^2 - a_s^2)\frac{dv}{dr} = v\left(2\frac{a_s^2}{r} - \frac{GM}{r^2}\right) \qquad (43)$$

$$\frac{dv}{dr} = \frac{v\left(2\frac{a_s^2}{r} - \frac{GM}{r^2}\right)}{(v^2 - a_s^2)}$$

The equation for v has a ***critical point*** where the numerator and denominator of the right hand side of this equation are both zero. That is, where:

$$
\begin{aligned}
v &= \pm a_s = \sqrt{\frac{\gamma k T_c}{\mu m}} \\
2\frac{a_s^2}{r} &= \frac{GM}{r^2} \Rightarrow r_c = \frac{GM}{2a_s^2} = \frac{\mu GMm}{2\gamma k T_c}
\end{aligned}
\tag{44}
$$

e.g. the Sun ($\gamma \approx 1$):

$$v_c \approx 170 \text{ km/s}\left(\frac{T}{10^6}\right)^{1/2}$$

$$r_c \approx 4.8\times10^9 \left(\frac{T}{10^6}\right)^{-1} \text{metres} = 6.9 R_{\text{sun}}\left(\frac{T}{10^6}\right)^{-1} \tag{45}$$

Solar radius: $R_{\text{sun}} = 6.96\times10^8$ m .

The actual measured coronal temperature is approximately $2\times10^{6}\,°\mathrm{K}$. This implies that:

$$
\begin{aligned}
r_c &\approx 2.0\times10^{9}\ \mathrm{m} \;=\; 3.5\ \text{solar radii} \\
v_c &\approx 340\ \mathrm{km\ s^{-1}}
\end{aligned}
\tag{46}
$$

3.3 A better estimate of mass flux

Let us suppose that the corona of the Sun is approximately determined by the hydrostatic solution out to the sonic point. Then

$$
\frac{\rho}{\rho_0} = \exp\left[-\frac{\mu G M m}{k T r_0}\left(1-\frac{r_0}{r}\right)\right] \tag{47}
$$

We have

$$\frac{\mu GMm}{kTr_0} \approx 7.2 \qquad \frac{r}{r_0} \approx 3.5 \Rightarrow \frac{\rho}{\rho_0} \approx 2.1\times10^{-3} \tag{48}$$

Hence our mass flux estimate becomes:

$$\begin{aligned} \dot{M} &\approx 4\pi \times (3.5 \times 6.96\times10^{8})^{2} \times 2.1\times10^{-3} \\ &\quad \times (1.2 \times 1.55\times10^{14} m)3.4\times10^{5} \\ &= 2.6\times10^{-13} \text{ solar masses per year} \end{aligned} \tag{49}$$

We are now only about an order of magnitude out!

4 Critical point analysis

4.1 Splitting equations

If we want a wind solution which accelerates to supersonic then we need to negotiate the critical point:

$$\frac{dv}{dr} = \frac{v\left(\frac{2a_s^2}{r} - \frac{GM}{r^2}\right)}{(v^2 - a_s^2)} \tag{50}$$

Split this equation into 2 by introducing a parameter u and write the equations as:

$$\frac{dr}{du} = r^2(v^2 - a_s^2) = f_1(r, v, \rho)$$
$$\frac{dv}{du} = v(2a_s^2 r - GM) = f_2(r, v, \rho) \qquad (51)$$

These equations have no potential infinities anywhere and are much better behaved numerically and easier to analyse mathematically.

4.2 Critical point

This is now defined by:

$$\frac{dr}{du} = \frac{dv}{du} = 0 \tag{52}$$

Expansion in neighbourhood of critical point:

$$r = r_c + r' \qquad v = v_c + v'$$
$$\frac{dr}{du} = \frac{d}{du}r' \qquad \frac{dv}{du} = \frac{d}{du}v' \tag{53}$$

$$\frac{d}{du}\begin{bmatrix} r' \\ v' \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial v} \end{bmatrix} \begin{bmatrix} r' \\ v' \end{bmatrix} \qquad (54)$$

where

$$f_1 = r^2(v^2 - a_s^2) \qquad f_2 = v(2a_s^2 r - GM) \qquad (55)$$

The functions f_1 and f_2 involve the function a_s^2 and in order to evaluate the partial derivatives of these functions, we need to evaluate the partial derivatives of a_s^2. This involves evaluating Bernoulli's equation.

4.3 Bernoulli's equation:

Since

$$p = C\rho^{\gamma} \tag{56}$$

then

$$\frac{1}{\rho}\frac{dp}{dr} = \gamma C\rho^{\gamma-2}\frac{d\rho}{dr} \tag{57}$$

and

$$a_s^2 = \frac{dp}{d\rho} = \gamma C \rho^{\gamma - 1} \tag{58}$$

Hence

$$v\frac{dv}{dr} = -\gamma C\rho^{\gamma-2}\frac{d\rho}{dr} - \frac{GM}{r^2}$$

$$\Rightarrow \frac{d}{dr}\left(\frac{1}{2}v^2\right) = -\gamma C\rho^{\gamma-2}\frac{d\rho}{dr} - \frac{GM}{r^2}$$

$$\frac{1}{2}v^2 = -\frac{\gamma C}{\gamma-1}\rho^{\gamma-1} + \frac{GM}{r} + \text{constant} \quad (59)$$

$$\frac{1}{2}v^2 + \frac{C\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = \text{constant}$$

$$a_s^2 = (\gamma-1)\left(\text{constant} + \frac{GM}{r} - \frac{1}{2}v^2\right)$$

The last equation implies:

$$\frac{\partial}{\partial r}a_s^2 = -(\gamma - 1)\frac{GM}{r^2} \qquad \frac{\partial}{\partial v}a_s^2 = -(\gamma - 1)v \qquad (60)$$

Since

$$f_1 = r^2(v^2 - a_s^2) \qquad f_2 = v(2a_s^2 r - GM) \qquad (61)$$

then, at the critical point:

$$\frac{\partial f_1}{\partial r} = 2r(v^2 - a_s^2) - r^2\frac{\partial}{\partial r}a_s^2 = (\gamma - 1)GM \qquad (62)$$

$$\frac{\partial f_1}{\partial v} = r^2\left(2v - \frac{\partial}{\partial v}a_s^2\right) = r^2(\gamma + 1)v \qquad (63)$$

$$\frac{\partial f_2}{\partial r} = v\left(2a_s^2 + 2r\frac{\partial}{\partial r}a_s^2\right) = v\left(\frac{GM}{r} - 2(\gamma - 1)\frac{GM}{r}\right)$$

$$\frac{\partial f_2}{\partial r} = (3 - 2\gamma)v\frac{GM}{r} \qquad (64)$$

$$\frac{\partial f_2}{\partial v} = (2a_s^2 r - GM) + 2vr\frac{\partial}{\partial v}a_s^2 = -2(\gamma - 1)a_s^2 r$$

$$\frac{\partial f_2}{\partial v} = -(\gamma - 1)GM \qquad (65)$$

$$
\begin{bmatrix} \dfrac{\partial f_1}{\partial r} & \dfrac{\partial f_1}{\partial v} \\[2ex] \dfrac{\partial f_2}{\partial r} & \dfrac{\partial f_2}{\partial v} \end{bmatrix} = \begin{bmatrix} (\gamma - 1)GM & (\gamma + 1)v_c r_c^2 \\[1ex] \dfrac{GM}{r_c} v_c (3 - 2\gamma) & -(\gamma - 1)GM \end{bmatrix} \qquad (66)
$$

and

$$
\frac{d}{du}\begin{bmatrix} r' \\ v' \end{bmatrix} = \begin{bmatrix} (\gamma - 1)GM & (\gamma + 1)v_c r^2 \\[1ex] \dfrac{GM}{r_c} v_c (3 - 2\gamma) & -(\gamma - 1)GM \end{bmatrix} \begin{bmatrix} r' \\ v' \end{bmatrix} \qquad (67)
$$

The solution of these equations requires the eigenvalues and eigenvectors of the matrix. Denoting the eigenvalues by λ:

$$\begin{vmatrix} \lambda-(\gamma-1)GM & -(\gamma+1)v_c r_c^2 \\ -\dfrac{GM}{r_c}v_c(3-2\gamma) & \lambda+(\gamma-1)GM \end{vmatrix} = 0 \qquad (68)$$

This simplifies to:

$$\begin{aligned} \lambda^2 &= (GM)^2\frac{(5-3\gamma)}{2} \\ \Rightarrow \lambda &= \pm GM\left(\frac{5-3\gamma}{2}\right)^{1/2} \end{aligned} \qquad (69)$$

Eigenvectors:

$$\begin{bmatrix} \lambda - (\gamma - 1)GM & -(\gamma + 1)vr_c^2 \\ X & Y \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (70)$$

$$\Rightarrow [\lambda - (\gamma - 1)GM]u_1 - (\gamma + 1)v_c r_c^2 u_2 = 0$$

Take

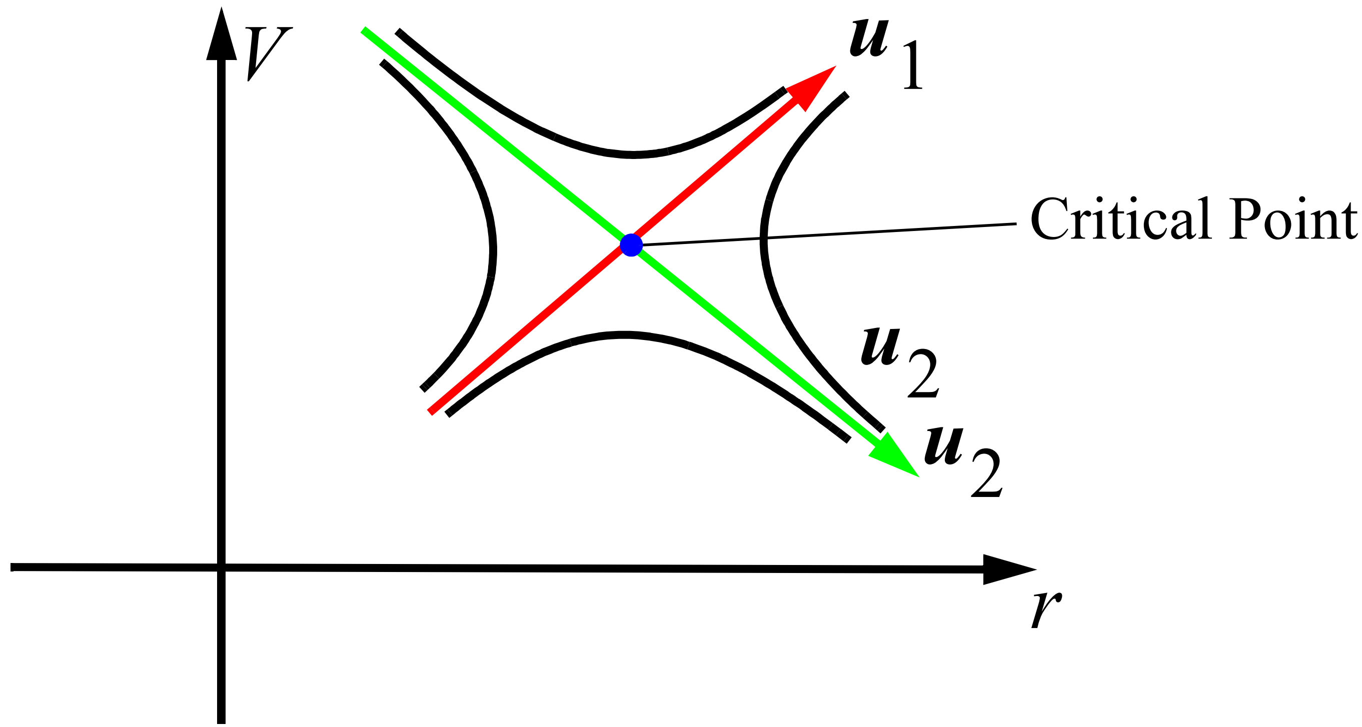
$$u_2 = 1 \Rightarrow u_1 = \frac{(\gamma + 1)\dfrac{v_c r_c^2}{GM}}{\pm\left(\dfrac{5 - 3\gamma}{2}\right)^{1/2} - (\gamma - 1)} \qquad (71)$$

The general solution in the neighbourhood of the critical point is:

$$\begin{bmatrix} r' \\ v' \end{bmatrix} = A_1 \boldsymbol{u}_1 \exp \lambda_1 u + A_2 \boldsymbol{u}_2 \exp \lambda_2 u \qquad (72)$$

where $\boldsymbol{u}_{\mathbf{1}}$ and $\boldsymbol{u}_{\mathbf{2}}$ are the two independent eigenvectors.

Slopes of lines through critical point and topology of the critical point:



$$\frac{dv}{dr} = \frac{u_2}{u_1} = \frac{2v_c}{r_c}\left[\frac{\pm\left(\frac{5-3\gamma}{2}\right)^{1/2} - (\gamma - 1)}{\gamma + 1}\right] \tag{73}$$

5 Scaling of wind equations for numerical solutions

When we have analytical solutons of differential (or other) equations, the dependence on physical parameters is evident. However, when we construct numerical solutions, it is best to scale the equations in order to derive solutions depending upon the smallest number of physical parameters. Numerical solutions of the wind equations were given in section 5. The physical equations should be scaled before numerical solutions can be determined. One reason for this scaling is that the numerical solutions can then be presented using the smallest number of parameters. Various parameters (specifically the velocity, radius and density at the critical point) then enter as scaling parameters.

The wind equations in physical units are:

$$\begin{aligned}\frac{dr}{du} &= r^2(v^2 - a_s^2) \\ \frac{dv}{du} &= v(2a_s^2 r - GM)\end{aligned} \tag{74}$$

For numerical purposes we scale these by the values of velocity and radius at the critical point.

Hence

$$r = r_c r' \qquad v = v_c v' \qquad a_s = v_c a_s' \qquad \rho = \rho_c \rho' \tag{75}$$

NB The primed variables here are not the same as the perturbations that were used to study solutions in the neighbourhood of the critical point.

5.1 Scaling of the sound speed

The solutions that were shown in this lecture are all constrained to have the same mass flux. Hence,

$$\frac{\dot{M}}{4\pi} = \rho v r^2 = \rho_c v_c r_c^2 \Rightarrow \frac{\rho}{\rho_c} = \left(\frac{v}{v_c}\right)^{-1}\left(\frac{r}{r_c}\right)^{-2} \tag{76}$$

i.e.

$$\rho' = v'^{-1} r'^{-2} \tag{77}$$

Also,

$$a_s^2 = \gamma C \rho^{\gamma - 1} \qquad a_c^2 = \gamma C \rho_c^{\gamma - 1} \Rightarrow \left(\frac{a_s}{a_c}\right)^2 = \left(\frac{\rho}{\rho_c}\right)^{\gamma - 1} \tag{78}$$

i.e.

$$a_s'^2 = \rho'^{(\gamma - 1)} \qquad (79)$$

5.2 Normalisation of the differential equations

$$
\begin{aligned}
r_c \frac{dr}{du} &= (r_c^2 v_c) r'^2 (v'^2 - a_s'^2) \\
\Rightarrow \frac{1}{r_c v_c^2} \frac{dr'}{du} &= r'^2 (v'^2 - a_s'^2) \\
v_c \frac{dv'}{du} &= r_c v_c^3 v \left(2 a_s'^2 r' - \frac{GM}{r_c v_c^2} \right) \qquad (80) \\
&= 2 r_c v_c^3 v (a_s'^2 r' - 1) \\
\Rightarrow \frac{1}{r_c v_c^2} \frac{dv'}{du} &= 2 (a_s'^2 r' - 1)
\end{aligned}
$$

The last equation follows from the condition at the critical point

$$\frac{GM}{r_c v_c^2} = 2 \tag{81}$$

We now simply make the transformation of the parameter u:

$$du' = r_c v_c^2 du \tag{82}$$

and the normalised equations become:

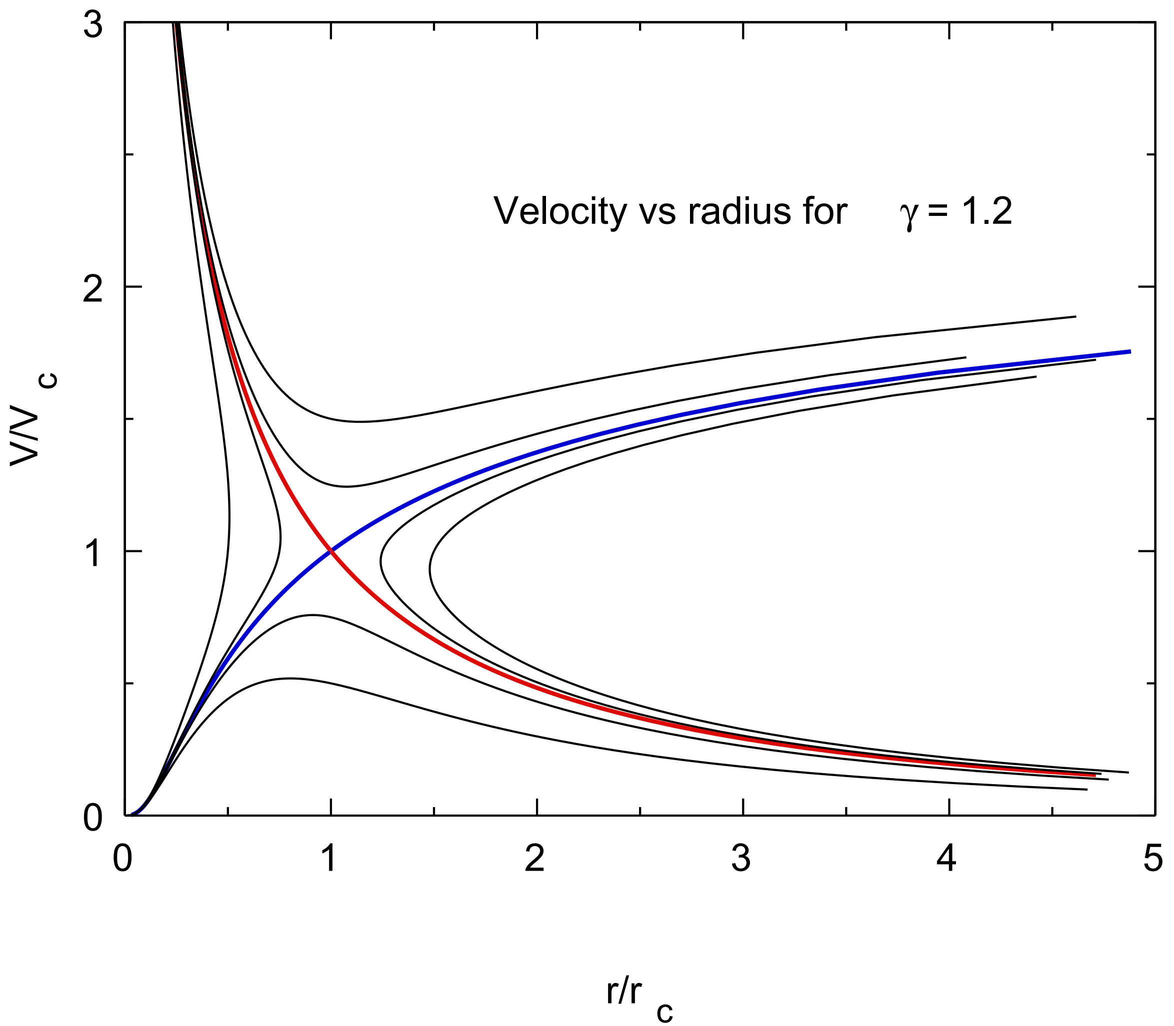
$$\begin{aligned} \frac{dr'}{du'} &= r'^2(v'^2 - a_s'^2) \\ \frac{dv'}{du'} &= 2(a_s'^2 r' - 1) \end{aligned} \tag{83}$$

with

$$\rho' = v'^{-1} r'^{-2} \qquad a_s'^2 = \rho'^{\gamma - 1} \tag{84}$$

These can be easily solved with a numerical differential equation solver (I simply use the NAG library). To start solutions near the critical point, the perturbation equations, developed in this lecture, need to be used.

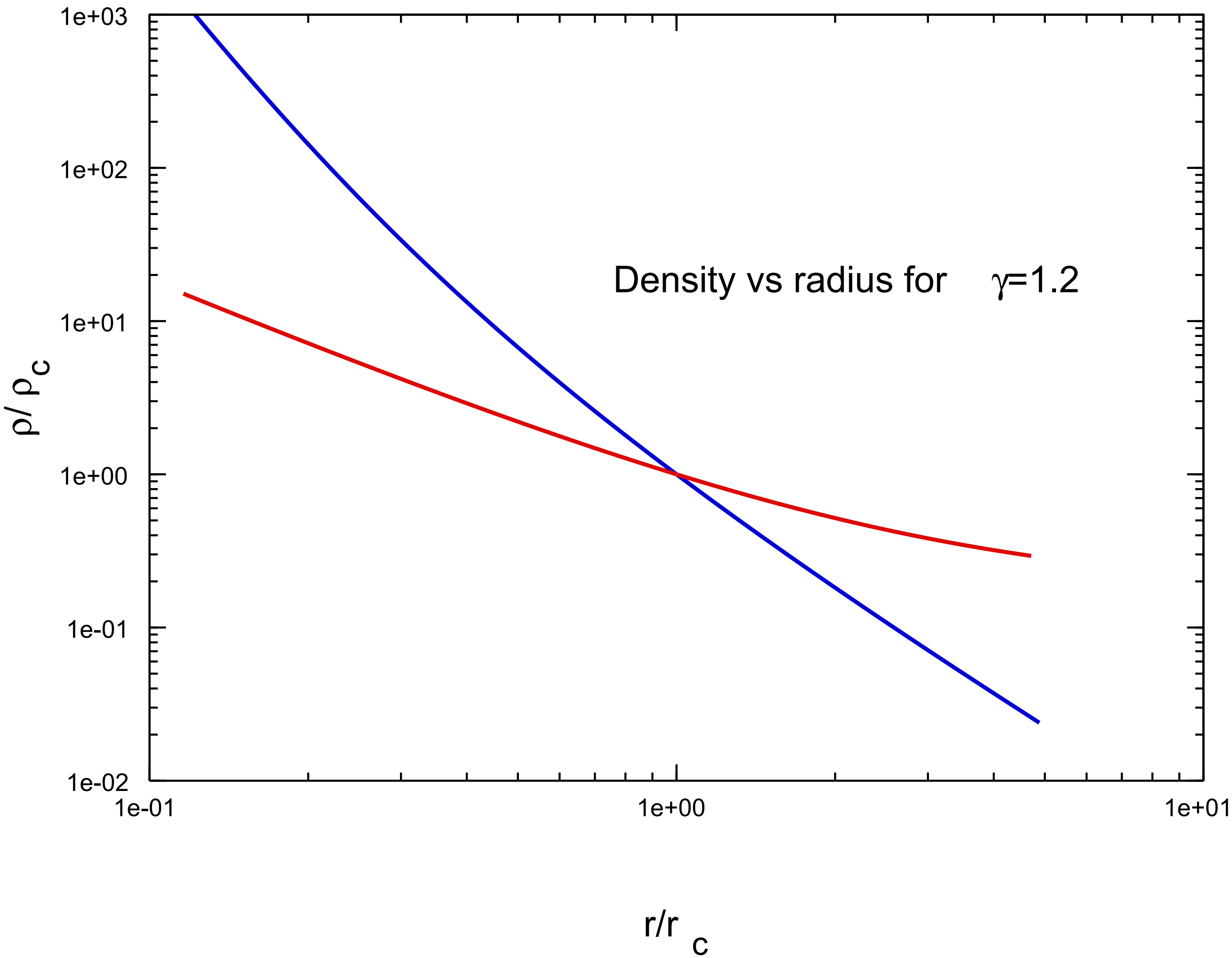
6 Numerical solution of equations



This plot shows the numerical integration of the two curves through the critical point and the numerical solution of other curves not passing through the critical point.

Blue curve: wind

Red curve: accretion



Density for the two critical curves

Blue: wind

Red: accretion

7 Mass flux

We can estimate the mass flux from the sun with this simple model.

Estimate parameters at the critical point assuming a temperature of $T_c = 10^6$K there.

The mass flux is:

$$\dot{M} = 4\pi\rho_c v_c r_c^2 \tag{85}$$

where the velocity and critical radius are given by

$$v_c = \sqrt{\frac{\gamma k T}{\mu m}} = 1.27 \times 10^2 \text{ km/s} \tag{86}$$

and

$$r_c = \frac{GM}{2a_s^2} = \frac{\mu GMm}{2\gamma kT_c} = 4.14 \times 10^9 \text{ m} = 5.96 \text{ R}_{\text{sun}} \tag{87}$$

Using our numerical solution, we estimate that the density at the solar radius, i.e. at $r = \frac{r_c}{5.96}$ is $230 \times \rho_c$.

Hence, we can write the mass flux as

$$\dot{M} = 4\pi\rho_c v_c r_c^2 = \dot{M} = 4\pi\rho_0\left(\frac{\rho_c}{\rho_0}\right)v_c r_c^2 \tag{88}$$

Earlier we used an electron number density at the base of the corona of $n_e \approx 1.55 \times 10^8 \text{ cm}^{-3}$ so that the density

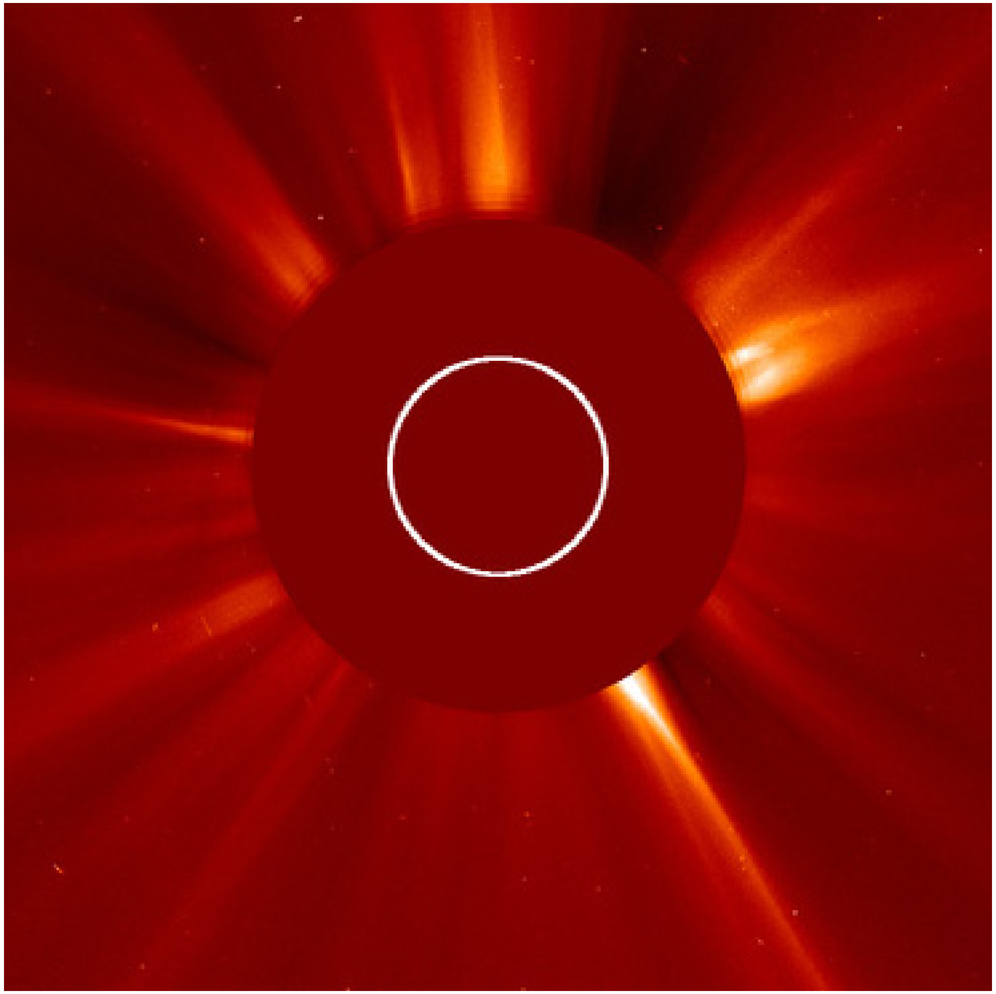
$$\rho_0 = \mu \times 1.9 n_e \times m \approx 3.0 \times 10^{-16} \text{ gm cm}^{-3} \qquad (89)$$

and

$$\dot{M} \approx 4\pi \times \frac{3.0 \times 10^{-16}}{230} \times (1.27 \times 10^7) \times (4.1 \times 10^{11})^2 \qquad (90)$$

$$\dot{M} \approx 3.6 \times 10^{13} \text{ gm s}^{-1} \approx 5.7 \times 10^{-13} \text{ M}_{\text{sun}} \text{ yr}^{-1} \qquad (91)$$

This is about a factor of 20 larger than the observed value. In order to obtain better agreement with observation, theory needs to take into account the effect of magnetic fields and coronal holes.



8 Asymptotic wind solutions

8.1 Deductions from Bernoulli's equation

As $r \to \infty$ the density in the wind goes to zero and therefore since

$$
\begin{aligned}
\frac{1}{2}v^2 + \frac{C\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} &= \text{constant} \\
\frac{1}{2}v^2 + \frac{a_s^2}{\gamma-1} - \frac{GM}{r} &= \text{constant}
\end{aligned}
\tag{92}
$$

We can use Bernoulli's equation to estimate the asymptotic velocity as follows. The constant on the right hand side of the above equation can be estimated from the conditions at the critical point.

$$\text{constant} = \frac{1}{2}v_c^2 + \frac{a_c^2}{\gamma - 1} - \frac{GM}{r_c} = v_c^2\left[\frac{1}{2} + \frac{1}{\gamma - 1} - 2\right] \qquad (93)$$

$$= \frac{5 - 3\gamma}{2(\gamma - 1)}v_c^2$$

As $r \to \infty$, $a_s^2 \to 0$ and $\dfrac{GM}{r} \to 0$ so that

$$v_\infty^2 = \frac{5 - 3\gamma}{(\gamma - 1)}v_c^2 \Rightarrow v_\infty = \sqrt{\frac{5 - 3\gamma}{(\gamma - 1)}}v_c \qquad (94)$$

We have been using $\gamma = 1.2$, in which case

$$v_\infty = 2.65 v_c = 2.65 \times 127 \text{ km s}^{-1} = 320 \text{ km s}^{-1} \qquad (95)$$

Thhis desn't compare too badly with the observed value of 400 km s^{-1} measured at the Earth.

Reasons for the discrepancy between theory and observation

- Magnetic effects are important in the initial phase of the solar wind
- The effective γ outside the solar region is greater than 1.2

8.2 Density in the asymptotic region

Since

$$\dot{M} = 4\pi\rho v r^2$$

then asymptotically

$$\rho = \frac{\dot{M}}{4\pi v_{\infty} r^2} \tag{96}$$

This expression for the density is frequently used for stellar winds well outside the sonic point where the wind can be considered to have achieved its terminal velocity, v_{∞}. As we have sen with the sun, the sonic point is reasonably close to the star.