

Sound

References:

L.D. Landau & E.M. Lifshitz: *Fluid Mechanics, Chapter VIII*

F. Shu: *The Physics of Astrophysics, Vol. 2, Gas Dynamics, Chapter 8*

1 Speed of sound

The phenomenon of sound waves is one that can be understood using the fundamental equations of gas dynamics that we have developed so far.

1.1 Perturbation equations

Simplest case: fluid equations without magnetic field and gravity (Euler equations):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}$$

(1)

Consider small perturbations to state $\rho = \rho_0$, and $v_i = v_{0,i}$.
Without loss of generality we may take $v_{0,i} = 0$.

$$\begin{aligned}\rho &= \rho_0 + \delta\rho \\ v_i &= v_{0,i} + \delta v_i\end{aligned}\tag{2}$$

Expand to first order:

$$\begin{aligned}\rho v_i &= (\rho_0 + \delta\rho)\delta v_i = \rho_0\delta v_i + O_2 \\ v_j \frac{\partial v_i}{\partial x_j} &= O_2\end{aligned}\tag{3}$$

The symbol O_2 means second order in the quantities $\delta\rho$, δv_i , i.e. terms such as $(\delta\rho)^2$, $\delta v_i \delta v_j$, $\delta\rho \delta v_i$ etc.

The perturbation to the pressure is determined by the perturbations to the density and entropy:

$$p = p(\rho, s)$$
$$\delta p = \frac{\partial p}{\partial \rho} \delta \rho + \frac{\partial p}{\partial s} \delta s \quad (4)$$

If we take the flow to be adiabatic, then $\delta s = 0$. Define the parameter c_s by:

$$c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_s \Rightarrow \delta p = c_s^2(p_0, \rho_0) \delta \rho \quad (5)$$

The perturbation equations become:

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \frac{\partial}{\partial x_i} \delta v_i = 0$$

$$\rho_0 \frac{\partial}{\partial t} \delta v_i + c_s^2 \frac{\partial}{\partial x_i} \delta \rho = 0$$

(6)

Time derivative of first equation and divergence of second equation:

$$\frac{\partial^2}{\partial t^2} \delta \rho + \rho_0 \frac{\partial^2}{\partial t \partial x_i} \delta v_i = 0$$

$$\rho_0 \frac{\partial^2}{\partial x_i \partial t} \delta v_i + c_s^2 \frac{\partial^2}{\partial x_i \partial x_i} \delta \rho = 0$$

(7)

Subtract \Rightarrow

$$\frac{\partial^2 \delta \rho}{\partial t^2} - c_s^2 \frac{\partial^2 \delta \rho}{\partial x_i \partial x_i} = 0 \quad (8)$$

This is the wave equation for a disturbance moving at velocity c_s .

$$\text{Speed of sound} = c_s \quad (9)$$

1.2 Plane-wave solutions

Take

$$\delta\rho = A \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t]$$

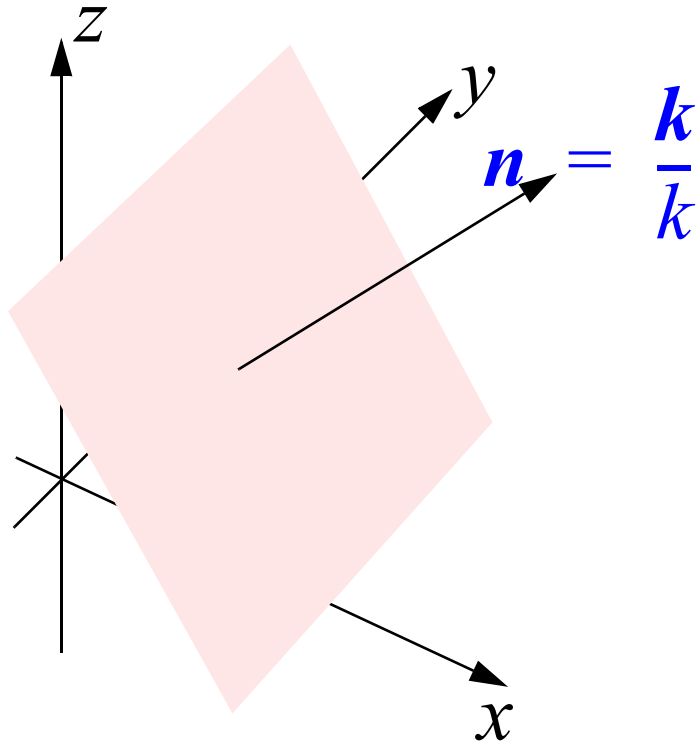
$$\Rightarrow \frac{\partial^2 \delta\rho}{\partial t^2} = -\omega^2 \delta\rho$$

$$\frac{\partial^2 \delta\rho}{\partial x_i \partial x_i} = -k^2 \delta\rho \quad (10)$$

$$\nabla^2 \delta\rho - \frac{1}{c_s^2} \frac{\partial^2 \delta\rho}{\partial t^2} = \left(-k^2 + \frac{\omega^2}{c_s^2} \right) \delta\rho = 0$$

$$\Rightarrow \omega = \pm c_s k$$

What does this plane wave represent?



Consider a surface of constant phase

$$\phi = \mathbf{k} \cdot \mathbf{x} - \omega t = \phi_0 \quad (11)$$

From the expression for the density, this represents a surface of constant density whose location evolves with time. This surface is planar and for our purposes is best represented in the form:

$$\frac{\mathbf{k}}{k} \cdot \mathbf{x} = \frac{\omega}{k} t + \frac{\phi_0}{k} \quad (12)$$

From the above equation, we read off the normal to the plane:

$$\mathbf{n} = \frac{\mathbf{k}}{k} \quad (13)$$

and the perpendicular distance of the plane from the origin is

$$D = \frac{\omega}{k}t + \frac{\phi_0}{k} \quad (14)$$

Therefore, the speed at which the plane is moving away from the origin is:

$$\frac{\omega}{k} = c_s \quad (15)$$

Thus the solution represents a wave moving at a speed c_s in the direction of the wave vector \mathbf{k}

Solution for velocity

The velocity perturbations are derived by considering perturbations of the form:

$$\delta v_i = A_i \exp i[(\mathbf{k} \cdot \mathbf{x}) - \omega t] \quad (16)$$

Note the appearance of a vector A_i for the amplitude since δv_i is a vector.

The time and spatial derivatives of δv_i are:

$$\frac{\partial}{\partial t} \delta v_i = -i\omega A_i \exp i[(\mathbf{k} \cdot \mathbf{x}) - \omega t]$$
$$\frac{\partial}{\partial x_i} \delta \rho = ik_i A \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t]$$
(17)

Substitute into perturbation equation for velocity:

$$\rho_0 \frac{\partial}{\partial t} \delta v_i + c_s^2 \frac{\partial}{\partial x_i} \delta \rho = 0$$
(18)

$$-i\omega\rho_0 A_i + c_s^2 i k_i A = 0$$

$$\Rightarrow A_i = c_s^2 \frac{k_i A}{\omega \rho_0} = c_s^2 \frac{k_i k A}{k \omega \rho_0} \quad (19)$$

$$= \pm c_s \frac{k_i A}{k \rho_0}$$

The amplitude of the velocity perturbation

$$|A_i| = c_s \left(\frac{A}{\rho_0} \right) \quad (20)$$

i.e.

$$\text{Amplitude of velocity perturbation} = c_s \times \text{Relative amplitude density perturbation} \quad (21)$$

What does small mean?

$$\frac{\delta\rho}{\rho} \ll 1 \Rightarrow |\delta v_i| \ll c_s \quad (22)$$

i.e perturbation velocities much less than the speed of sound.

Nature of sound wave

Since

$$A_i = \pm c_s \frac{k_i A}{k \rho_0} \quad (23)$$

then the velocity of the oscillating elements of gas are in the direction of the wave-vector, i.e. the wave is *longitudinal*.

1.3 Numerical value of the speed of sound

Take the equation of state

$$\begin{aligned} p &= K(s)\rho^\gamma \\ \Rightarrow c_s^2 &= \left. \frac{\partial p}{\partial \rho} \right|_s \\ &= K(s)\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho} \\ &= \frac{\gamma kT}{\mu m} \end{aligned} \tag{24}$$

Therefore:

$$c_s = \sqrt{\frac{\gamma k T}{\mu m}} \quad (25)$$

N.B. The speed of sound in an ideal gas only depends upon temperature.

e.g. air at sea level on a warm day

$$\gamma = 1.4, T = 300 \text{ K}, \mu = 28.8 \Rightarrow c_s = 340 \text{ m/s} \quad (26)$$

$T = 10^7 \text{ K}$ atmosphere in an elliptical galaxy

$$T = 10^7 \text{ K}, \mu = 0.62, \gamma = \frac{5}{3}$$

$$\Rightarrow c_s = 480 \text{ km/s} \quad (27)$$

2 Subsonic and supersonic flow

Disturbances in a gas travel at the speed of sound relative to the fluid. i.e. information in the gas propagates at the sound speed

Sound wave



A wave travelling at a velocity of $v = c_s$ wrt to the fluid cannot send a signal backwards to warn of an impending obstacle. This leads to the formation of shocks (to be considered later). Therefore the speed $V = c_s$ is a critical one in gas dynamics.

$$\begin{array}{ll} \text{Supersonic Flow} & v > c_s \\ \text{Subsonic Flow} & v < c_s \end{array} \quad (28)$$

$$\text{Mach Number} \quad M = \frac{v}{c_s} \quad (29)$$

Nobody ever heard the bullet that killed him -Th. von Karman

3 Energy & momentum in sound waves

3.1 Expressions for energy density and energy flux

Energy density

$$\varepsilon_{\text{tot}} = \frac{1}{2}\rho v^2 + \varepsilon \quad (30)$$

Expand out the quantities in this equation to second order

$$\begin{aligned} \varepsilon(\rho, s) = & \varepsilon_0 + \left. \frac{\partial \varepsilon}{\partial \rho} \right|_s \delta \rho + \left. \frac{\partial \varepsilon}{\partial s} \right|_{\rho} \delta s \\ & + \frac{1}{2} \frac{\partial^2}{\partial \rho^2} (\varepsilon (\delta \rho)^2) + \frac{\partial^2}{\partial \rho \partial s} \varepsilon (\delta \rho \delta s) + \frac{1}{2} \frac{\partial^2 \varepsilon}{\partial s^2} (\delta s)^2 \end{aligned} \quad (31)$$

Since we are only considering adiabatic fluctuations, then $\delta s = 0$ and

$$\varepsilon = \varepsilon(\rho, s) = \varepsilon_0 + \left. \frac{\partial \varepsilon}{\partial \rho} \right|_s \delta \rho + \frac{1}{2} \frac{\partial^2}{\partial \rho^2} \varepsilon (\delta \rho)^2 \quad (32)$$

We now evaluate these terms using thermodynamic identities. Since,

$$kT ds = \frac{1}{\rho} d\varepsilon - \frac{h}{\rho} d\rho \quad (33)$$

then

$$d\varepsilon = h d\rho + \rho kT ds \Rightarrow \left. \frac{\partial \varepsilon}{\partial \rho} \right|_s = h \quad (34)$$

and

$$\begin{aligned}\frac{\partial^2 \varepsilon}{\partial \rho^2} \Big|_s &= \frac{\partial h}{\partial \rho} \Big|_s = \frac{\partial}{\partial \rho} \left(\frac{\varepsilon + p}{\rho} \right) \\ &= -\frac{1}{\rho^2} (\varepsilon + p) + \frac{1}{\rho} \left(\frac{\partial \varepsilon}{\partial \rho} + \frac{\partial p}{\partial \rho} \right) \\ &= -\frac{h}{\rho} + \frac{h}{\rho} + \frac{c_s^2}{\rho} = \frac{c_s^2}{\rho}\end{aligned}\tag{35}$$

Hence the total (kinetic + internal) energy is to second order:

$$\varepsilon_{\text{tot}} = \varepsilon_0 + \frac{1}{2}\rho_0(\delta v)^2 + h_0\delta\rho + \frac{1}{2}\left(\frac{c_s^2}{\rho_0}\right)(\delta\rho)^2 \quad (36)$$

The term $h_0\delta\rho$ is associated with a change in energy in a given volume due to a change in mass and eventually disappears.

Energy flux

$$\begin{aligned} F_{E, i} &= \rho v_i \left(\frac{1}{2}v^2 + h \right) \\ &= \rho_0 h_0 \delta v_i + \rho_0 \delta h \delta v_i + h_0 \delta\rho \delta v_i + O_3 \end{aligned} \quad (37)$$

Note that the kinetic energy term is of third order.

We need to relate δh to the variations in other thermodynamic variables. Since

$$kTds = dh - \frac{1}{\rho}dp \quad (38)$$

then

$$dh = \frac{1}{\rho}dp + kTds \Rightarrow \left. \frac{\partial h}{\partial p} \right|_s = \frac{1}{\rho} \quad (39)$$

and

$$\delta h = \frac{1}{\rho}\delta p = \frac{c_s^2}{\rho}\delta\rho \quad (40)$$

Hence, using the previous expression for $F_{E, i}$:

$$\begin{aligned} F_{Ei} &= \rho_0 h_0 \delta v_i + h_0 \delta \rho \delta v_i + \rho_0 \delta h \delta v_i \\ &= \rho_0 h_0 \delta v_i + h_0 \delta \rho \delta v_i + c_s^2 \delta \rho \delta v_i \end{aligned} \quad (41)$$

We combine this with the expression for the energy density:

$$\varepsilon_{\text{tot}} = \varepsilon_0 + h_0 \delta \rho + \frac{1}{2} \rho_0 (\delta v)^2 + \frac{1}{2} \left(\frac{c_s^2}{\rho_0} \right) (\delta \rho)^2 \quad (42)$$

Because of the energy equation, we have the conservation law:

$$\frac{\partial \varepsilon_{\text{tot}}}{\partial t} + \frac{\partial F_{Ei}}{\partial x_i} = 0 \quad (43)$$

Some of the terms in these expressions can be simplified. First, the leading red term in ε , ε_0 is constant and is just the background energy density.

Now examine the continuity equation written out to second order:

$$\frac{\partial}{\partial t} \delta \rho + \frac{\partial}{\partial x_i} (\rho_0 \delta v_i + \delta \rho \delta v_i) = 0 \quad (44)$$

Multiplying this by h_0 gives:

$$\frac{\partial}{\partial t} (h_0 \delta \rho) + \frac{\partial}{\partial x_i} (h_0 \rho_0 \delta v_i + h_0 \delta \rho \delta v_i) = 0 \quad (45)$$

so that when we insert the red and blue terms in the energy conservation equation, they drop out and we are left with

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 (\delta v)^2 + \frac{1}{2} \left(\frac{c_s^2}{\rho_0} \right) (\delta \rho)^2 \right) + \frac{\partial}{\partial x_i} (c_s^2 \delta \rho \delta v_i) = 0 \quad (46)$$

Hence, we take for the energy density and its associated energy flux

$$\varepsilon^{\text{SW}} = \frac{1}{2} \rho_0 (\delta v)^2 + \frac{1}{2} \frac{c_s^2}{\rho_0} (\delta \rho)^2 \quad (47)$$

$$F_{Ei}^{\text{SW}} = c_s^2 \delta \rho \delta v_i = \delta p \delta v_i$$

3.2 Energy density and energy flux in a plane wave

Express $\delta\rho$ and δv_i in terms of travelling waves and take the real part:

$$\delta\rho = A \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t] \rightarrow A \cos[\mathbf{k} \cdot \mathbf{x} - \omega t]$$

$$\delta v_i = c_s \frac{k_i}{k} \frac{A}{\rho_0} \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t] \quad (48)$$

$$\rightarrow c_s \frac{k_i}{k} \frac{A}{\rho_0} \cos[\mathbf{k} \cdot \mathbf{x} - \omega t]$$

Hence,

$$\varepsilon^{\text{SW}} = \frac{1}{2}\rho_0(\delta v)^2 + \frac{1}{2}\frac{c_s^2}{\rho_0}(\delta\rho)^2 = \frac{c_s^2}{\rho_0}A^2\cos^2[\mathbf{k} \cdot \mathbf{x} - \omega t]$$
$$F_{Ei}^{\text{SW}} = c_s^2\delta\rho\delta v_i = \frac{c_s^3}{\rho_0}A^2\frac{k_i}{k}\cos^2[\mathbf{k} \cdot \mathbf{x} - \omega t] = c_s\varepsilon^{\text{SW}}\frac{k_i}{k}$$
(49)

Average the energy density and flux over a period ($T = 2\pi/\omega$) of the wave using:

$$\frac{1}{T}\int_0^T \cos^2[\mathbf{k} \cdot \mathbf{x} - \omega t] dt = \frac{1}{2}$$
(50)

so that the average energy density and flux are:

$$\langle \varepsilon^{\text{SW}} \rangle = \frac{1}{2} \frac{c_s^2}{\rho_0} A^2$$

(51)

$$\langle F_{Ei} \rangle = \frac{1}{2} \frac{c_s^3}{\rho_0} A^2 \frac{k_i}{k} = c_s \langle \varepsilon^{\text{SW}} \rangle \frac{k_i}{k}$$

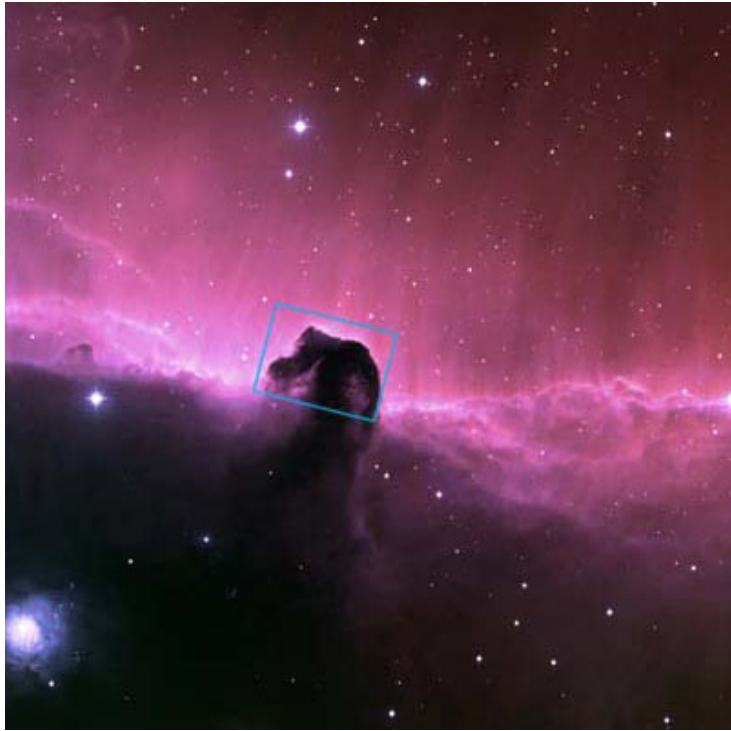
Notes

- The energy flux is in the direction of the wave.
- and is equal to the sound speed times the energy density.

4 Jeans mass - sound waves with self gravity

4.1 Physical motivation

There are numerous sites in the interstellar medium of our own galaxy that are the birthplaces of stars. These are cold dense clouds in which stars can form as a result of the process of gravitational collapse. Why is “cold” and “dense” important? Surprisingly, we can get some idea of this by looking at the propagation of sound waves in a molecular cloud.



The Horsehead nebula is a dark molecular cloud that is the site of ongoing star formation. The image at the left was obtained at the Kitt Peak National Observatory in Arizona

The image at the right was obtained using the Hubble Space Telescope.



4.2 *Mathematical treatment of gravitational instability*

Continuity, momentum and Poisson equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0$$

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\partial \phi}{\partial x_i} = 0 \quad (52)$$

$$\nabla^2 \phi = 4\pi G \rho$$

In order to model self-gravity of the gas we have included Poisson's equation for the gravitational potential.

Perturbations:

$$\rho = \rho_0 + \delta\rho \quad \rho_0 \neq \text{constant}$$

$$v_i = \delta v_i \quad (53)$$

$$\phi = \phi_0 + \delta\phi$$

Perturbation equations:

$$\frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial x_i}(\rho_0 \delta v_i) = \frac{\partial}{\partial t}\delta\rho + \rho_0 \frac{\partial}{\partial x_i}(\delta v_i) + \frac{\partial\rho_0}{\partial x_i} \delta v_i$$

$$\frac{\partial\delta v_i}{\partial t} + \frac{1}{\rho_0} \frac{\partial p_0}{\partial x_i} + \frac{1}{\rho_0} \frac{\partial}{\partial x_i} \delta p - \frac{\delta\rho}{\rho_0^2} \frac{\partial p_0}{\partial x_i} + \frac{\partial\phi_0}{\partial x_i} + \frac{\partial}{\partial x_i} \delta\phi = 0 \quad (54)$$

$$\nabla^2\phi_0 + \nabla^2\delta\phi = 4\pi G(\rho_0 + \delta\rho)$$

Zeroth order terms are shown in blue.

The term in red is associated with what is sometimes unkindly known as the *Jeans Swindle*. Jeans neglected this term in his derivation of the Jeans Mass.

When can we neglect the pressure gradient term?

Compare $\frac{1}{\rho_0} \frac{\partial}{\partial x_i} \delta p$ with $\frac{\delta \rho}{\rho_0^2} \frac{\partial p_0}{\partial x_i}$. Let us suppose that the back-

ground pressure gradient has a length scale L , and that the perturbations have a length scale l i.e.

$$\frac{1}{\rho_0} \frac{\partial}{\partial x_i} \delta p \sim \frac{\delta p}{\rho_0 l} \quad (55)$$

and

$$\frac{\delta \rho}{\rho_0^2} \frac{\partial p_0}{\partial x_i} \sim \frac{\delta \rho p_0}{\rho_0^2 L} \quad (56)$$

The pressure gradient term associated with the perturbations is larger than the background pressure gradient term if

$$\frac{\delta p}{\rho_0 l} \gg \frac{\delta \rho P_0}{\rho_0^2 L} \Rightarrow \frac{l}{L} \ll \frac{\delta p \rho_0}{\delta \rho P_0} = c_0^2 \times \frac{\gamma}{c_0^2} = \gamma \quad (57)$$

So, for perturbation length scales much less than the background length scale in the cloud, we can neglect the background pressure gradient.

Now use the zeroth order equations:

$$\frac{1}{\rho_0} \frac{\partial p_0}{\partial x_i} + \frac{\partial \phi_0}{\partial x_i} = 0 \quad (58)$$

$$\nabla^2 \phi_0 = 4\pi G \rho_0$$

This gives:

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \frac{\partial}{\partial x_i} (\delta v_i) + \frac{\partial \rho_0}{\partial x_i} \delta v_i = 0$$

$$\frac{\partial \delta V_i}{\partial t} + \frac{1}{\rho_0} \frac{\partial}{\partial x_i} \delta p - \frac{\delta \rho}{\rho_0^2} \frac{\partial p_0}{\partial x_i} + \frac{\partial}{\partial x_i} \delta \phi = 0 \quad (59)$$

$$\nabla^2 \delta \phi = 4\pi G \delta \rho$$

Simplest case: initial density constant and neglect variation of potential.

$$\rho_0 = \text{constant} \quad (60)$$

$$\phi_0 = \text{constant} \quad (61)$$

The assumption means that the red term above is neglected. The perturbation equations simplify to:

$$\frac{\partial}{\partial t} \delta \rho + \rho_0 \frac{\partial}{\partial x_i} (\delta v_i) = 0$$

$$\frac{\partial \delta v_i}{\partial t} + \frac{1}{\rho_0} \frac{\partial}{\partial x_i} \delta p + \frac{\partial}{\partial x_i} \delta \phi = 0 \quad (62)$$

$$\nabla^2 \delta \phi = 4\pi G \delta \rho$$

Plane wave solution:

$$\delta \rho = A \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t]$$

$$\delta v_i = A_i \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t] \quad (63)$$

$$\delta \phi = B \exp i[\mathbf{k} \cdot \mathbf{x} - \omega t]$$

Substitute into perturbation equations:

$$-i\omega A + \rho_0 i k_i A_i = 0$$

$$-i\omega A_i + \frac{c_s^2}{\rho_0} i k_i A + i k_i B = 0 \quad (64)$$

$$-k^2 B = 4\pi G A$$

We can arrange these equations into the following set of linear equations:

$$\omega A - \rho_0 k_i A_i = 0$$

$$\frac{c_s^2}{\rho_0} k_i A - \omega A_i + k_i B = 0 \quad (65)$$

$$4\pi G A + k^2 B = 0$$

There are a number of ways of dealing with this set of linear algebraic equations for the quantities, A , A_i and B . One of the best is simply to write them as a set of 5 equations as follows:

$$\begin{bmatrix}
 \omega & -\rho_0 k_1 & -\rho_0 k_2 & -\rho_0 k_3 & 0 \\
 \frac{c_s^2}{\rho_0} k_1 & -\omega & 0 & 0 & k_1 \\
 \frac{c_s^2}{\rho_0} k_2 & 0 & -\omega & 0 & k_2 \\
 \frac{c_s^2}{\rho_0} k_3 & 0 & 0 & -\omega & k_3 \\
 4\pi G & 0 & 0 & 0 & k^2
 \end{bmatrix}
 \begin{bmatrix}
 A \\
 A_1 \\
 A_2 \\
 A_3 \\
 B
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0
 \end{bmatrix}
 \quad (66)$$

This is a homogeneous set of equations that only has a non-trivial solution if the determinant is zero. It is possible to work the determinant out by hand, since it contains a number of zeroes. However, Maple/Mathematica also readily provides the following expression for the determinant:

$$\Delta = \omega^2 k^2 (-\omega^2 + c_s^2 k^2 - 4\pi G \rho_0) \quad (67)$$

Apart from the trivial solutions (ω or $k = 0$) to $\Delta = 0$, the only non-trivial solutions are described by:

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \quad (68)$$

giving

$$\omega = \pm \sqrt{c_s^2 k^2 - 4\pi G \rho_0} \quad (69)$$

This constitutes an interesting and fundamental difference from normal sound waves because of the additional term relating to self-gravity.

Define the Jeans wave number k_J by

$$\begin{aligned}c_s^2 k_J^2 &= 4\pi G \rho_0 \\ \Rightarrow \omega^2 &= c_s^2 (k^2 - k_J^2)\end{aligned}\tag{70}$$

Then if $k < k_J$

$$\omega^2 < 0 \Rightarrow i\omega = \pm \omega_g \text{ say}\tag{71}$$

For these imaginary solutions the perturbation in each variable is proportional to

$$\exp(i\omega t) = \exp\pm\omega_g t. \quad (72)$$

One of the imaginary roots corresponds to decaying modes, the other to growing modes, i.e. instability.

$$\text{Jeans Length} = \lambda_J = \frac{2\pi}{k_J} = 2\pi \sqrt{\frac{c_s^2}{4\pi G\rho_0}} \quad (73)$$

Jeans length in a molecular cloud

$$n \sim 10^9 \text{ m}^{-3} \quad T \sim 10^\circ \text{ K} \quad \mu \sim 1$$

$$c_s = \left(\frac{\gamma k T}{\mu m} \right)^{1/2} \approx 0.29 \text{ km/s} \quad (74)$$

$$\lambda_J = \left(\frac{\pi \gamma k T}{\mu m G \rho_0} \right)^{1/2} = \frac{1}{\mu m} \left(\frac{\pi \gamma k T}{G n} \right)^{1/2} \approx 2 \text{ pc}$$

Jeans mass

The Jeans mass, M_J is defined to be the mass in the region defined by the reciprocal of the Jeans wave number:

$$\begin{aligned} M_J &= \rho_0 k_J^{-3} \\ &= \rho_0 \left(\frac{c_s^2}{4\pi G \rho_0} \right)^{3/2} \\ &= \left(\frac{kT}{4\pi G \mu m \rho_0^{1/3}} \right)^{3/2} \\ &\approx 0.4 \text{ solar masses} \end{aligned} \tag{75}$$

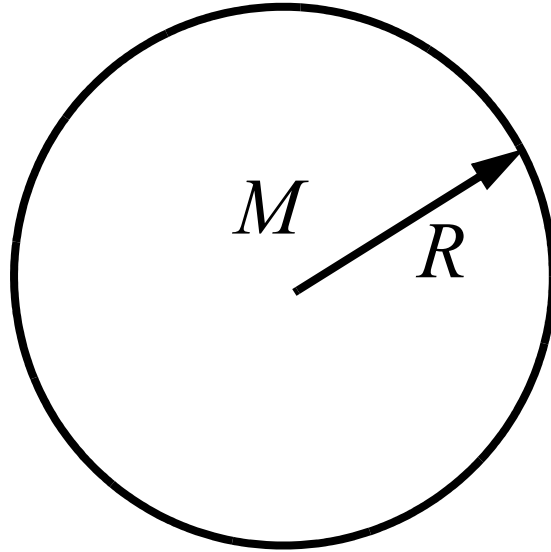
for the above parameters.

Star formation - the modern approach

The above discussion is the standard treatment for gravitational instability without the influence of a magnetic field and gives interesting sizes for the initial collapsing region and the mass of the collapsing object.

The current attitude to the Jeans mass is that the physics of star formation is more complicated and that magnetic fields and turbulence are involved. However, it is thought that the Jeans Mass is relevant to the masses of molecular cloud *cores*.

Physics of the Jean mass



Consider a gas cloud of mass M and radius R and neglect pressure forces. Then the equation of motion of the cloud is:

$$\frac{dv_i}{dt} = -\nabla\phi \approx -\frac{GM^x_i}{R^2 R} \quad (76)$$

Dimensionally, this equation is:

$$\frac{R}{t^2} = \frac{GM}{R^2} \Rightarrow \text{Free-fall time} = t_{\text{ff}} \sim \left(\frac{R^3}{GM} \right)^{1/2} \sim (G\rho)^{-1/2} \quad (77)$$

Now consider the sound crossing time:

$$t_s \sim \frac{R}{c_s} \quad (78)$$

If the sound-crossing time is less than the free-fall time, then the collapsing region is able to produce enough pressure to halt the collapse since signals have the time to go from one side of the region to the other. Thus the condition for collapse is

$$t_s > t_{\text{ff}} \Rightarrow \frac{R}{c_s} > (G\rho)^{-1/2} \Rightarrow R > \left(\frac{c_s^2}{G\rho} \right)^{1/2} \approx \lambda_J \quad (79)$$

Another criterion:

Euler's equations with pressure and gravity:

$$\frac{dv_i}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \quad (80)$$

Collapse will occur if gravitational forces overwhelm the pressure forces. Since

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x_i} = c_s^2 \frac{\partial \rho}{\partial x_i} \quad (81)$$

then Euler's equations are:

$$\frac{dv_i}{dt} = -\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \sim \frac{c_s^2}{\rho} \frac{\rho}{R} - \frac{GM}{R^2} \sim \frac{c_s^2}{R} - \frac{GM}{R^2} \quad (82)$$

For collapse the gravitational forces have to win, so that,

$$\frac{GM}{R^3} > \frac{c_s^2}{R^2} \Rightarrow R > \left(\frac{c_s^2}{G\rho} \right)^{1/2} \approx \lambda_J \quad (83)$$