Solutions to Exercises in Astrophysical Gas Dynamics

1. (a) i. Since u_1, v_i are vectors then, under an orthogonal transformation,

$$u'_i = a_{ij}u_j \quad v'_i = a_{ik}u_k$$

Therefore,

$$u'_i v'_i = a_{ij}a_{ik}u_j v_k = \delta_{jk}u_j v_k = u_j v_j$$

Hence. $u_i v_i$ is invariant under transformation and is a scalar.

ii.

$$u'_i v'_j = a_{ik}u_k a_{jl}v_l = a_{ik}a_{jl}u_k v_l$$

and this is the transformation law for a second rank tensor.

iii.

$$\frac{\partial u'_i}{\partial x'_j} = \frac{\partial (a_{ik}u_k)}{\partial x_l}\frac{\partial x_l}{\partial x'_j}$$

using the chain rule for partial differentiation. Now, since

$$\begin{aligned} x'_j &= a_{jk}x_k \Rightarrow x_l = a_{kl}x'_k \\ \Rightarrow \frac{\partial x_l}{\partial x'_j} &= a_{kl}\frac{\partial x'_k}{\partial x'_j} \\ &= a_{kl}\delta_{kj} = a_{jl} \end{aligned}$$

then

$$\frac{\partial u'_i}{\partial x'_j} = a_{ik}a_{jl}\frac{\partial u_k}{\partial x_l}$$

and is therefore a second rank tensor.

(b) i.

$$\sum_{i=1}^{3} u'_i = \sum_{i=1}^{3} a_{ij}u_j = \left[\sum_{i=1}^{3} a_{ij}\right] u_j$$

For the terms on the right hand side to equal $\sum_{j=1}^{3} u_j$ we require $\sum_{i=1}^{3} a_{ij} = 1$ for each j, i.e. each row of the transformation matrix should sum to unity. This is an over-restriction on the properties of the orthogonal matrix.

ii. The proof is similar.

Since these quantities are not invariant, they are unsuitable choices for the magnitude of a vector. The quantity $\sqrt{u_i u_i}$ is invariant and is therefore a suitable choice for the magnitude of u_i.

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2. Write

$$T_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

The tensor

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

is symmetric and the tensor

$$A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

is antisymmetric. We can further write

$$S_{ij} = \frac{1}{3}S_{kk}\delta_{ij} + \left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right) = \frac{1}{3}T_{kk}\delta_{ij} + \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}T_{kk}\delta_{ij}\right)$$

The tensor $\frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}T_{kk}\delta_{ij}\right)$ is traceless. Hence,

$$T_{ij} = \frac{1}{3}T_{kk}\delta_{ij} + \frac{1}{2}\left(T_{ij} + T_{ji} - \frac{2}{3}T_{kk}\delta_{ij}\right) + \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

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3. *Energy density and energy flux in a sound wave.* Mass momentum and energy flux

   Use the perturbation equations derived in lectures for a sound wave, viz,

$$\begin{aligned} \frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'_i}{\partial x_i} &= 0 \\ \rho_0 \frac{\partial v'_i}{\partial t} + c_0^2 \frac{\partial \rho'}{\partial x_i} &= 0 \end{aligned}$$

   Thus

$$\frac{\partial \epsilon_{SW}}{\partial t} = \frac{1}{2}\rho_0 \frac{\partial (v'_i v'_i)}{\partial t} + \frac{c_0^2}{\rho_0}\rho' \frac{\partial \rho'}{\partial t}$$

   Now,

$$\begin{aligned} \rho' \frac{\partial \rho'}{\partial t} &= -\rho_0 \rho' \frac{\partial v'_i}{\partial x_i} \\ \text{and} \qquad \rho_0 v'_i \frac{\partial v'_i}{\partial t} &= -c_0^2 v'_i \frac{\partial \rho'}{\partial x_i} \end{aligned}$$

   Thus

$$\begin{aligned} \frac{\partial \epsilon^{SW}}{\partial t} &= -c_0^2 v'_i \frac{\partial \rho'}{\partial x_i} - c_0^2 \rho' \frac{\partial v'_i}{\partial x_i} \\ &= -c_0^2 \frac{\partial (\rho' v'_i)}{\partial x_i} \end{aligned}$$

   Since $F_i^{SW} = c_0^2 \rho' v'_i$, then

$$\frac{\partial \epsilon^{SW}}{\partial t} + \frac{\partial F_i^{SW}}{\partial x_i} = 0$$

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4. *Velocity of sound in a moving medium.* Consider a plane wave in a reference frame (denoted by a prime) in which the gas is at rest. The variation of density in the wave is given by:

$$\rho' = A \exp i \left[k_i x'_i - \omega t\right]$$

   The transformation to the moving medium is given by:

$$x_i = x'_i + u_i\, t$$

   The equation for the density then transforms to:

$$\rho' = A \exp i \left[k_i x_i - (\omega' + k_i u_i) t\right]$$

   implying that

$$k_i = k'_i \qquad \omega = \omega' + k_i u_i = c_s\, k + k_i u_i$$

   since $\omega' = c_s k$ in the stationary medium.

   There are 2 ways to work out the velocity of this wave:

(a) A surface of constant phase is given by

$$k_i x_i - (c_s k + k_i u_i)t = \phi_0$$

This can be put in the form

$$\frac{k_i}{k} x_i = (c_s + \frac{k_i}{k} u_i)t + \frac{\phi_0}{k}$$

Let $n_i = k_i/k$ be the normal to the wave, then this equation implies that

$$n_i x_i = v_w t + \text{constant}$$

where

$$v_w = c_s + u_i n_i = \text{Sound speed} + \text{Component of } \mathbf{u} \text{ in direction of wave}$$

(b) The group velocity of waves given by this dispersion relation is:

$$c_i = \frac{\partial \omega}{\partial k_i} = c_s \frac{k_i}{k} + u_i$$

so that the wave speed in the direction of the wave is

$$c_i k_i = c_s + u_i k_i$$

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5. *Doppler Effect.* Consider the frequency of a wave in a reference frame in which the *source* is at rest. The medium is moving with velocity $-u_i$ in this frame. Thus, from the previous question, the relationship between the rest frequency (ω_0) and the frequency in the medium in which the source is moving (ω) is given by:

$$\begin{aligned} \omega_0 &= \omega - k_i u_i \\ &= \omega \left(1 - \frac{u}{c} \cos\theta\right) \end{aligned}$$

since in the stationary medium $k = \omega/c$. This equation then implies that

$$\omega = \frac{\omega_0}{1 - \frac{u}{c}\cos\theta}$$

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6. *Pressure fluctuations in a sound wave.* The mean energy flux of a sound wave is:

$$\langle F_{E,i} \rangle = p' v'_i$$

For a plane wave:

$$\begin{aligned} p' &= c_0^2 \, A \cos(k_j x_j - \omega t) \\ v'_i &= \frac{c_0}{\rho_0} n_i \, A \cos(k_j x_j - \omega t) \end{aligned}$$

where c_0 is the sound speed in the undisturbed medium, A is the amplitude of the density wave and n_i is the unit vector in the direction of propagation. Hence, the rms energy flux is given by

$$\begin{aligned} < p' v'_i > &= \frac{c_0^3}{\rho_0} A^2 \, n_i \, < \cos^2(k_j x_j - \omega t) > \\ &= \frac{c_0^3}{\rho_0} \frac{A^2}{2} n_i \\ \Rightarrow \langle F_E \rangle &= \frac{c_0^3 A^2}{2\rho_0} \end{aligned}$$

We can relate this to the mean square pressure fluctuation by:

$$\begin{aligned}\langle p'^2\rangle &= c_0^4 A^2 \langle \cos^2(k_j x_j - \omega t)\rangle \\ &= \frac{c_0^4 A^2}{2}\end{aligned}$$

Hence

$$\begin{aligned}\frac{\langle p'^2\rangle}{F_E} &= \rho_0 c_0 \\ \Rightarrow \langle p'^2\rangle &= \rho_0 c_0 \langle F_E\rangle\end{aligned}$$

The background pressure can be expressed as $p_0 = \rho_0 c_0^2/\gamma$. Hence

$$\frac{\langle p'^2\rangle^{1/2}}{p_0} = \gamma \left(\frac{F_E}{\rho_0 c_0^3}\right)^{1/2}$$

Parameters for this problem are density of air, $\rho_0 \approx 1.225\,\mathrm{Kg\,m^{-3}}$, $c_0 \approx 330\mathrm{m\,s^{-1}}$, $\gamma = 1.4$ and $< F >= 10\mathrm{W}/(4\pi \times 1\,\mathrm{m}^2)$. This gives,

$$\frac{< p'^2 >^{1/2}}{p_0} \approx 1.8 \times 10^{-4}$$

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7. *Jeans mass at recombination.*

At recombination, the Universe consists mainly of H and He, with the abundance by mass of He, $Y \approx 0.2534$. The atomic masses of H and He are 1.0079 and 4.0026 respectively. Hence the ratio of the densities is given by

$$\frac{\rho_{\mathrm{He}}}{\rho_{\mathrm{H}}} = \frac{n_{\mathrm{He}} \times 4.0026}{n_{\mathrm{H}} \times 1.0079} = 3.97 \times \frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} = 0.2534$$

Hence

$$\frac{n_{\mathrm{He}}}{n_H} = 0.0638$$

The density of the Universe in terms of the number density of atoms (n_a), can be found from

$$\frac{\rho}{n_a} = \frac{n_H m_H (1+Y)}{n_H(1 + n_{\mathrm{He}}/n_{\mathrm{H}})} = 1.19\, m_H$$

Therefore,

$$\rho \approx 1.19 \times 1.0079 m \times n_a \approx 4.3 \times 10^{-22}\,\mathrm{gm\,cm^{-3}}$$

for the given parameters.

Let us define the Jeans mass as the mass within a sphere of diameter the Jeans length λ_J where

$$\lambda_J = 2\pi\sqrt{\frac{c_s^2}{4\pi G\rho_0}} = 2\pi\sqrt{\frac{\gamma n_a kT}{4\pi G\rho_0^2}} = 2\pi\sqrt{\frac{\gamma kT}{4\pi G\rho_0 \times 1.19 m_H}} = 2.10 \times 10^{20}\,\mathrm{cm}$$

Therefore the Jeans mass is

$$M_J = \frac{\pi}{6} \times \lambda_J^3 \times 4.3 \times 10^{-22}\,\mathrm{gm} = 2.1 \times 10^{59}\,\mathrm{gm} = 1.0 \times 10^6 M_\odot$$

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8. *Timescale for gravitational collapse.*

(a) When $k < k_J$, the growth rate according to the Jeans theory is given by:

$$\omega_g^2 = c_s^2(k_J^2 - k^2)$$

and the maximum growth rate $\omega_g = c_s k_J$ with associated growth timescale

$$\tau_g = \frac{1}{c_s k_J} = \frac{1}{\sqrt{4\pi G \rho_0}}$$

(b)
(i) For typical ISM densities, $n \sim 10^4\ \mathrm{cm}^{-3}$ and $\mu \sim 1$, the collapse timescale is of order 2.7×10^5 yrs. (ii) With $\rho_0 = 4.3 \times 10^{-22}\ \mathrm{gm\ cm}^{-3}$ from the previous question, this gives

$$\tau_g = 5.3 \times 10^{13}\ \mathrm{s} = 1.7 \times 10^6 \mathrm{yr}$$