Characterisation of Magnetic Forces

1 Introduction

The momentum equation

$$\rho \frac{dv_i}{dt} = -\rho \frac{\partial \phi}{\partial x_i} - \frac{\partial p}{\partial x_i} + \frac{\partial M_{ij}}{\partial x_j}$$

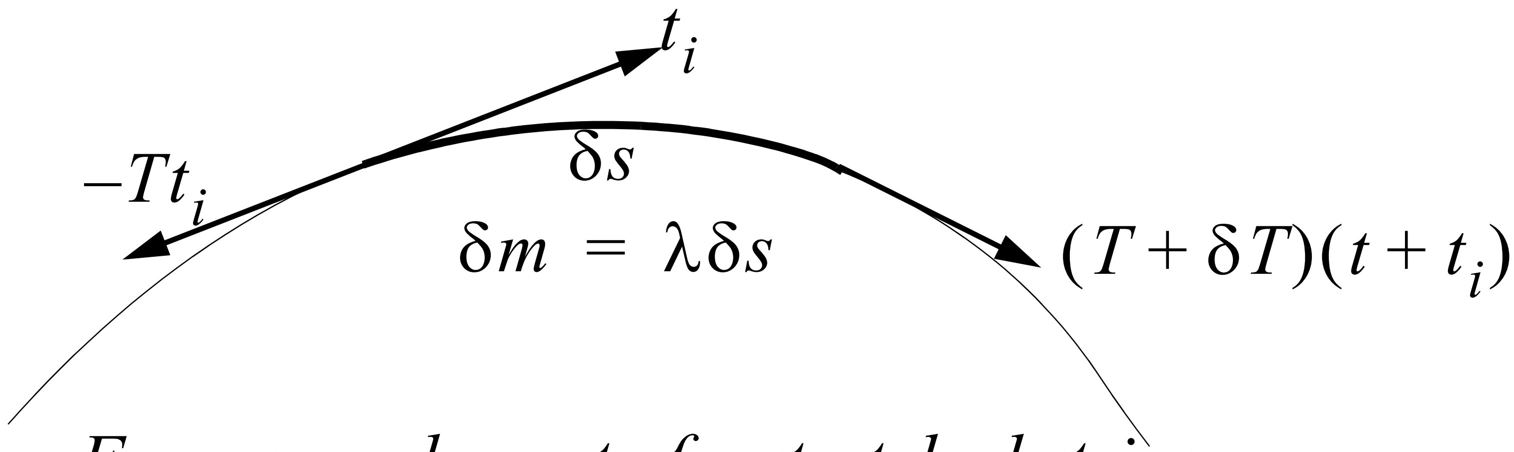
$$M_{ij} = \left[\frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right] \tag{1}$$

contains pressure gradient terms and gravitational force terms that we are familiar with together with the divergence of the term M_{ij} that we have referred to as "magnetic stresses". The purpose of the following is to come to a better physical understanding of what this term represents physically and what effect it can have on magnetised gas.

2 *Aside: the forces on a stretched string*

Before going further it is helpful to consider the forces acting on a stretched string. This analogy is useful for one part of the magnetic force.

Take the tension in a stretched string to be T. This is the force exerted over a cross-section of the string by the rest of the string.



Forces on element of a stretched string

Take t_i to be the unit tangent to the string, s to be the arc-length along the string, the mass per unit length to be λ so that the mass of the element is $\delta m = \lambda \delta s$ The force on an element of the string as shown in the diagram is

$$\begin{aligned} \delta F_i &= (T(s) + \delta T)(t_i + \delta t_i) - T(s)t_i \\ &= T(s)t_i + \delta T t_i + T(s)\delta t_i - T(s)t_i \\ &= \delta T t_i + T(s)\delta t_i \\ &= \left[\frac{dT}{ds}t_i + T(s)\frac{dt_i}{ds}\right]\delta s \end{aligned} \qquad (2)$$

Now the Frenet-Serret relations for a curve tell us that

$$\frac{dt_i}{ds} = \kappa n_i \qquad (3)$$

where κ is the curvature and n_i is the unit normal. Hence the equation of motion of the mass element is

$$\lambda \frac{dv_i}{dt} = \frac{dT}{ds} t_i + T\kappa n_i \qquad (4)$$

i.e. there is a force along the string equal to the rate of the change of the tension with arc-length and there is a force in the direction of curvature proportional to the curvature times the tension.

3 Decomposition of the magnetic forces

We can write

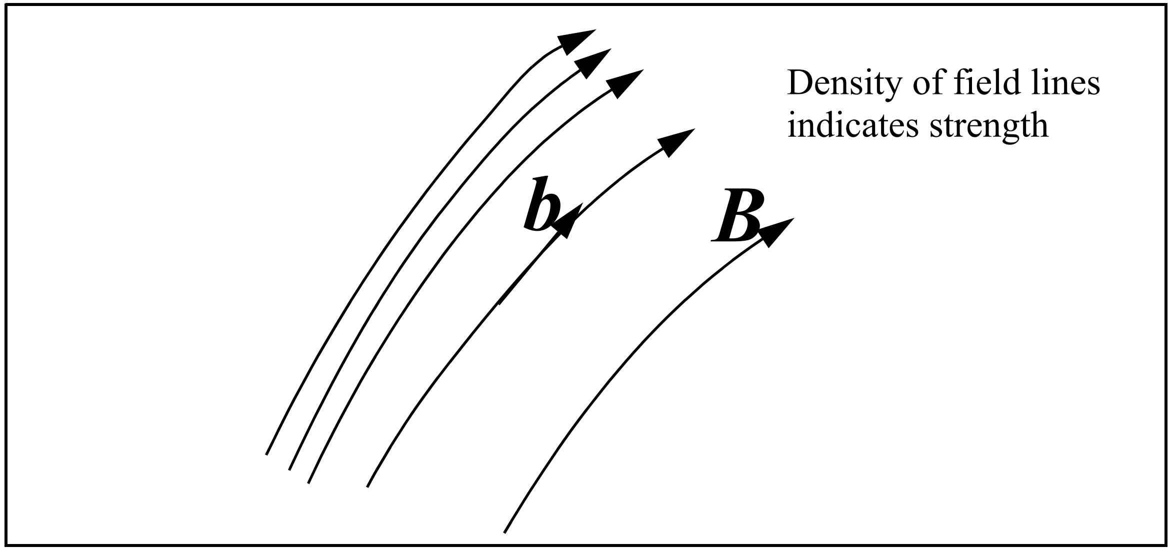
$$\frac{\partial M_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{B_i B_j}{4\pi}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)$$
$$= \left(\frac{B_j}{4\pi}\frac{\partial B_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) \qquad (5)$$

We now write

$$B_i = B b_i \qquad (6)$$

where b_i is a unit vector in the direction of the magnetic field and is therefore tangent to the magnetic field lines. If $x_i = x_i(s)$ are the coordinates of a field line with arclength s, then

$$\frac{dx_i}{ds} = b_i \tag{7}$$



We can therefore write the magnetic force terms as:

$$\frac{\partial M_{ij}}{\partial x_j} = \frac{1}{4\pi} B b_j \frac{\partial}{\partial x_j}(B b_i) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)$$

$$= \frac{B b_j b_i}{4\pi} \frac{\partial B}{\partial x_j} + \frac{B^2}{4\pi} b_j \frac{\partial b_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) \qquad (8)$$

$$= b_i b_j \frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\left(b_j \frac{\partial b_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)$$

The first and third terms can be combined in the form:

$$-P_{ij} \frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) \qquad (9)$$

where the projection operator

$$P_{ij} = \delta_{ij} - b_i b_j \tag{10}$$

projects vectors into the space normal to the magnetic field. That is, suppose we have a vector U_i, then $P_{ij}U_j$ is normal to the magnetic field, since,

$$b_i P_{ij} U_j = b_i(\delta_{ij} - b_i b_j)U_j = (b_j - b_j)U_j = 0 \tag{11}$$

The operator P_{ij} therefore projects the gradient operator $\dfrac{\partial}{\partial x_j}$ perpendicular to the magnetic field, ie. the operator $P_{ij}\dfrac{\partial}{\partial x_j}$ is the component of the gradient perpendicular to $\boldsymbol{B}$.

The second term in

$$\frac{\partial M_{ij}}{\partial x_j} = b_i b_j \frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\left(b_j \frac{\partial b_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) \qquad (12)$$

can be written:

$$\frac{B^2}{4\pi}\left(b_j \frac{\partial b_i}{\partial x_j}\right) = \frac{B^2}{4\pi} \times \frac{\partial b_i}{\partial x_j}\frac{dx_j}{ds} = \frac{B^2}{4\pi} \times \frac{db_i}{ds} \qquad (13)$$

We use the Frenet-Serret relations for the magnetic field lines in the form:

$$\frac{db_i}{ds} = \kappa_B n_i \qquad (14)$$

where κ_B is the curvature of the field line and n_i is the normal to the field line.

Hence, we express the divergence of the stress tensor in the form:

$$\frac{\partial M_{ij}}{\partial x_j} = -P_{ij}\frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) + \left(\frac{B^2}{4\pi}\right)\kappa_B n_i \qquad (15)$$

ie, the sum of a pressure term defined by the gradientof the magnetic energy densitybut also perpendicular to the magnetic field plus a term proportional to the curvature of the magnetic field lines.

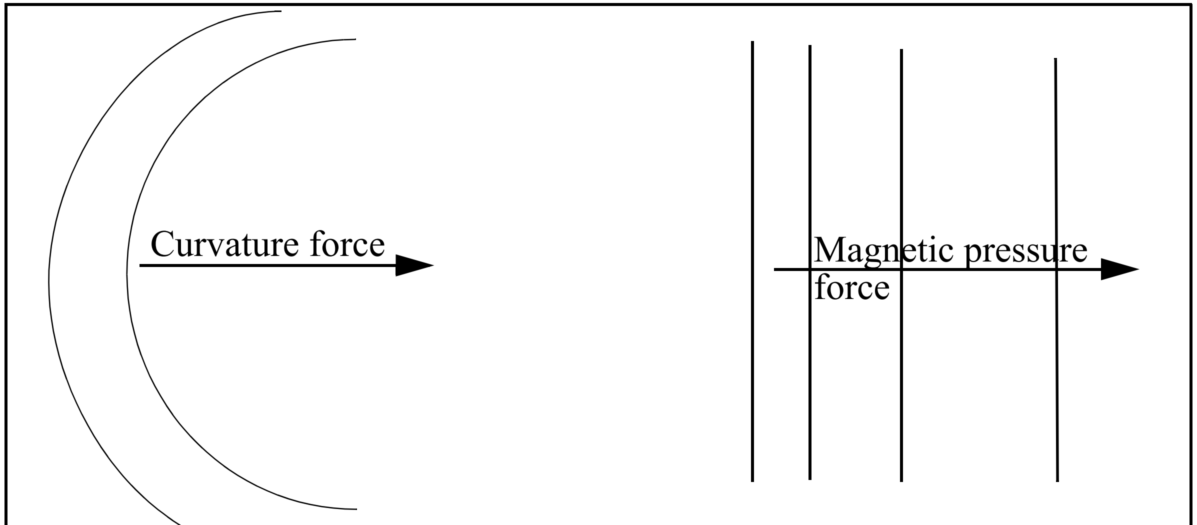


Illustration of components of magnetic force

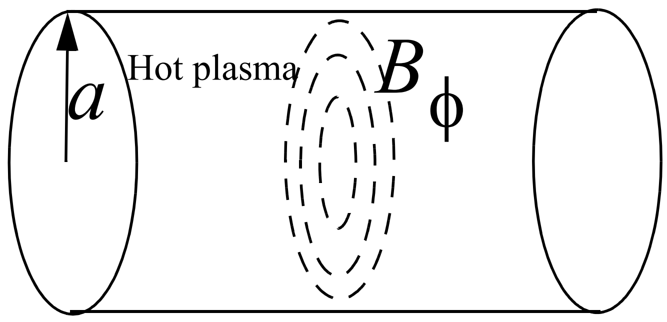
It is the last term, in particular that distinguishes magnetic forces from pure hydrostatic forces. Note also that the component of magnetic force along the field lines is zero:

$$B_i \frac{\partial M_{ij}}{\partial x_j} = 0 \qquad (16)$$

i.e. both the pressure force and the curvature force are perpendicular to the magnetic field.

4 The magnetic pinch

The confinement of a plasma by a toroidal magnetic field is an example of the different forces provided by a magnetic field. We can also analyse the stability of this configuration using the physical concepts derived above.



Confinement of a plasma column by a magnetic field

4.1 Magnetostatic equilibrium

From the above diagram, once can see that it is feasible that the "curvature force" associated with the magnetic "tension" can plausibly confine a hot plasma. To see if this is possible, we analyse the magnetostatic configuration using the momentum equations:

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla p + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{0} \qquad (17)$$

We analyse this situation in cylindrical polars, and take

$$\boldsymbol{B} = (B_r, B_\phi, B_z) = (0, B_\phi, 0) \qquad (18)$$

so that

$$\nabla\times\boldsymbol{B} = \left[\frac{1}{r}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial B_r}{\partial\phi}\right]\hat{\boldsymbol{z}} \tag{19}$$

$$= -\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{z}}$$

and

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{B_\phi}{r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{r}} - B_\phi\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{z}} \tag{20}$$

We now take B_ϕ to be independent of z and

$$\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{r}} \qquad (21)$$

and the force on the plasma is in the inward radial direction if rB_ϕ increases outwards.

Radial magnetostatic equilibrium

Because of the limitations we have imposed, we only have to consider the radial force balance which is expressed by the equation:

$$-\frac{\partial p}{\partial r} - \frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi) = 0 \qquad (22)$$

There is a wide variety of magnetostatic equilibria that we could envisage. For the sake of simplicity, we consider one in which the current density in the plasma is uniform. Ampere's law becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \frac{4\pi}{c}j_z = \text{constant} \qquad (23)$$

The solution of this is

$$\begin{aligned} rB_\phi &= \frac{2\pi}{c}r^2 j_z + C \\ \Rightarrow B_\phi &= \frac{2\pi}{c}rj_z + \frac{C}{r} \qquad (24) \\ &= \frac{2\pi}{c}rj_z \end{aligned}$$

The constant C is put to zero so that the magnetic field is finite at $r = 0$.

The magnetostatic equilibrium equation becomes an equation for the pressure:

$$
\begin{aligned}
-\frac{\partial p}{\partial r} &= \frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi) \\
&= \frac{2\pi}{c^2} r j_z^2
\end{aligned}
\qquad (25)
$$

Hence

$$
P = A - \frac{\pi}{c^2} j_z^2 r^2 \qquad (26)
$$

where A is a constant which is determined by the condition that the plasma be confined to $r < a$, i.e.

$$p = 0 \qquad \text{at } r = a \qquad \Rightarrow p = \frac{\pi j_z^2}{c^2}(a^2 - r^2) \tag{27}$$

Note that, with this solution,

$$p + \frac{B_\phi^2}{4\pi} = \text{constant} \tag{28}$$

Magnetic field outside $r = a$

The region outside $r = a$ is envisaged as a vacuum, so that Ampere's law in this region becomes:

$$
\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) &= 0 \\
\Rightarrow rB_\phi &= \text{constant} \qquad (29) \\
B_\phi &= \frac{C}{r}
\end{aligned}
$$

This constant of integration is determined by continuity at $r = a$. Hence,

$$B_\phi = \frac{C}{a} = \frac{2\pi}{c} a j_z$$

$$\Rightarrow C = \frac{2\pi}{c} a^2 j_z \qquad (30)$$

$$\Rightarrow B_\phi = \frac{2\pi}{c} \frac{a^2}{r} j_z$$

We can also easily derive this form of the solution from the integral form of Ampere's law, viz,

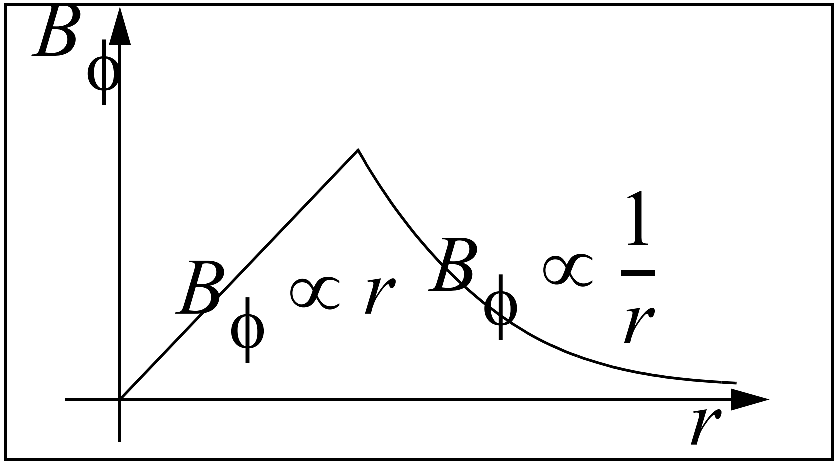
$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \frac{4\pi}{c} \int_A (j \cdot n) ds \qquad (31)$$

where C encloses the area A. Here we just take C to be a circle of radius $2\pi r$ outside the plasma column, so that the above integral formulation reads:

$$B \times 2\pi r = \frac{4\pi}{c} j_z \times \pi a^2 \Rightarrow B_\phi = \frac{2\pi}{c} \frac{a^2}{r} j_z \qquad (32)$$

as before.

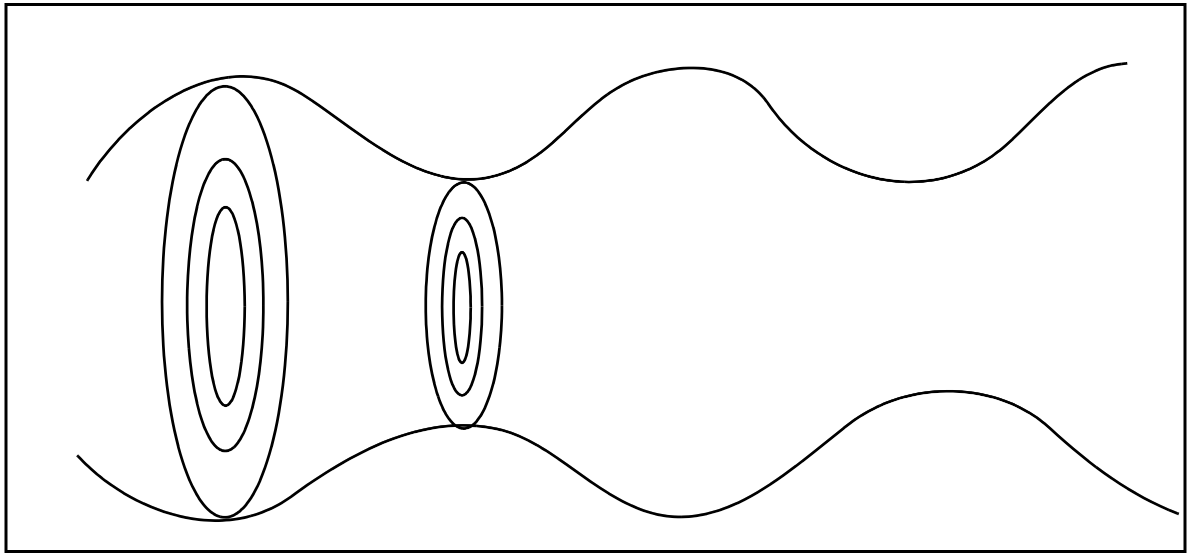
The radial profile of the toroidal field therefore looks like the following diagram:



4.2 Stability of the magnetic pinch

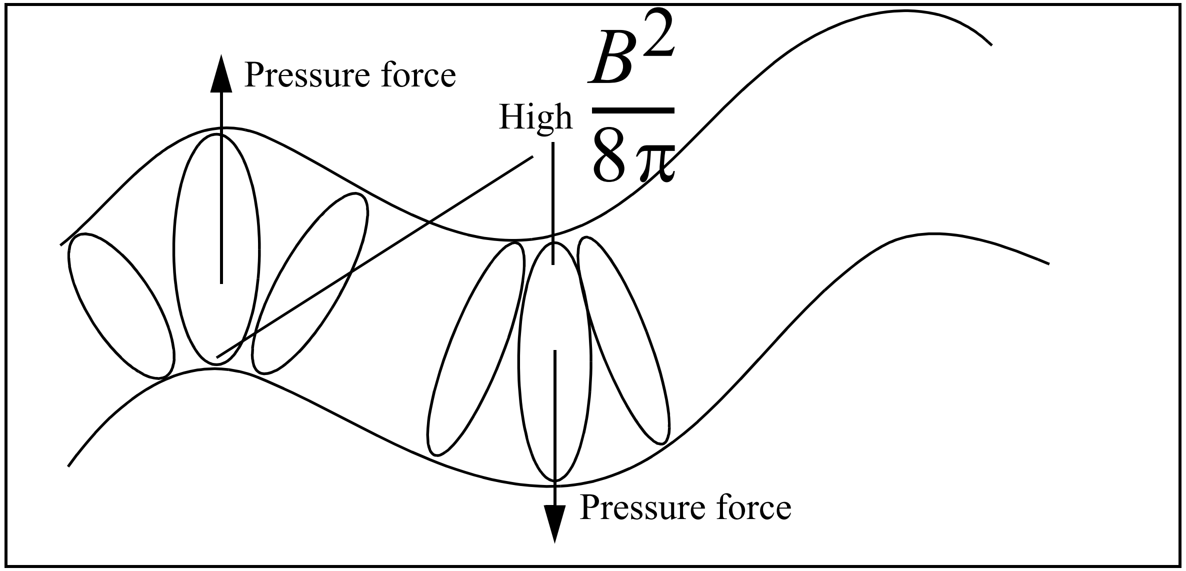
The magnetic pinch is subject to two well-known instabilities – the “sausage” or “pinch” instability and the “firehose” instability. With our knowledge of the nature of magnetic forces, we can analyse these instabilities as follows.

The "sausage" instability (pinch instability)



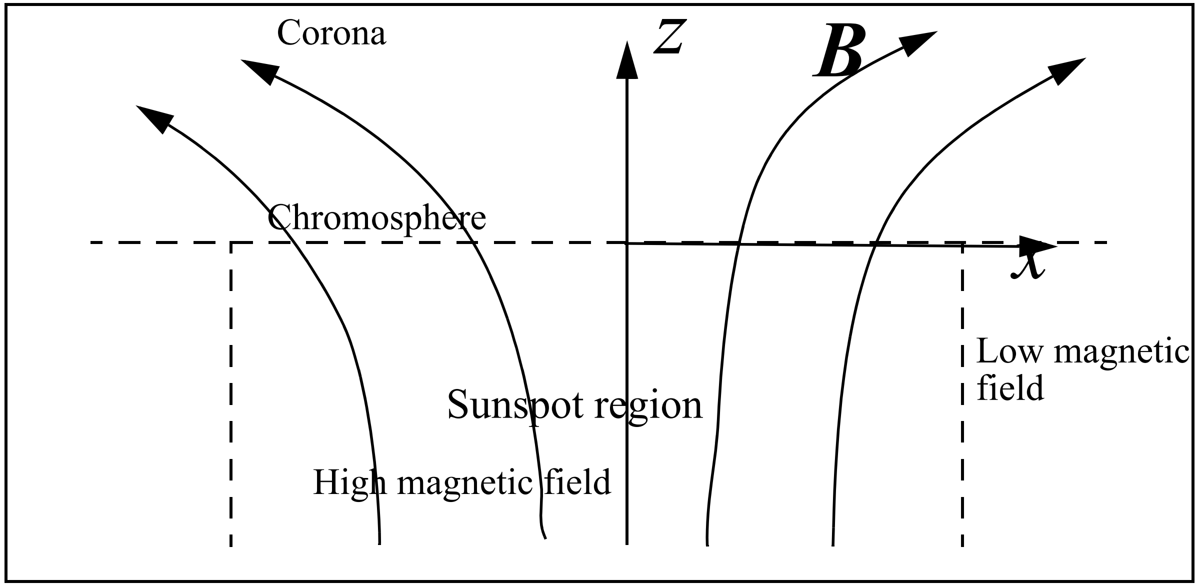
Consider an equilibrium plasma column which is perturbed by being "squeezed" as indicated. Since the field lines follow the motion they are squeezed as well. Hence the curvature force increases by virtue of the increased value of $\dfrac{B^2}{4\pi}$ and because of the higher curvature of the field lines. The pinching effect of the field is greater so that the toroidal magnetic field pinches the plasma even further. The end result is a sequence of blobs.

The "firehose" instability



Now consider a toroidally confined plasma column which is perturbed in an oscillatory fashion. Again because the field lines follow the motion of the plasma, the resulting perturbation to the field is as shown. The bunching up of field lines causing a magnetic pressure gradient as shown and the direction of this is to enhance the perturbation. The perturbation therefore grows in the manner of a hose with water flowing through it – hence the name *firehose instability*.

5 Sunspots



Sunspots are a classic example of the simple application of magnetostatics and provide us with an example of the importance of magnetic pressure. They are regions of the solar photosphere which are much cooler than average. ($T_{\rm sunspot} \approx 3800$ K as opposed to $T \approx 5780$ K for the rest of the sun's photosphere.) Consider a model of a sunspot as indicated above. Equilibrium in the horizontal (x) direction implies for $\boldsymbol{B} = B_x\boldsymbol{i} + B_z\boldsymbol{k}$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\frac{B_x^2}{4\pi}\right) + \frac{\partial}{\partial z}\left(\frac{B_x B_z}{4\pi}\right) - \frac{\partial}{\partial x}\left(\frac{B_x^2 + B_z^2}{8\pi}\right) \qquad (33)$$

In this model we neglect B_x in the region below the sunspot so that

$$p + \frac{B_z^2}{8\pi} = \text{constant}. \qquad (34)$$

Therefore if we envisage the sunspot as having a high magnetic field inside and comparatively negligible field outside, then

$$\left(p + \frac{B_z^2}{8\pi}\right)\Bigg|_{\text{sunspot}} = p_{\text{photosphere}} \tag{35}$$

Since, in our model, we assume that the magnetic field is independent of height, then

$$\left.\frac{\partial p}{\partial z}\right|_{\text{sunspot}} = \left.\frac{\partial p}{\partial z}\right|_{\text{photopshere}} \tag{36}$$

Now consider the vertical equilibrium. This is expressed by the equation:

$$0 = -\frac{\partial p}{\partial z} + \rho g_z + \frac{\partial}{\partial x}\left(\frac{B_x B_z}{4\pi}\right) + \frac{\partial}{\partial z}\left(\frac{B_z^2}{4\pi}\right) - \frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right) \qquad (37)$$

where g_z is the local acceleration due to gravity. Since $B_x \ll B_z$ below the photosphere boundary and there is no dependence of B_z on height (why?), then

$$\frac{\partial p}{\partial z} = \rho g_z \qquad (38)$$

in both sunspot and the surrounding photosphere. Since we have shown that the pressure gradient is the same in both, then the density must be the same in both regions. Hence the equation for horizontal equilibrium becomes

$$\begin{aligned} nkT_{\text{sunspot}} + \frac{B^2_{\text{sunspot}}}{8\pi} &= nkT_{\text{photosphere}} \\ \Rightarrow \frac{B^2_{\text{sunspot}}}{8\pi} &= (nk)(T_{\text{photosphere}} - T_{\text{sunspot}}) \end{aligned} \qquad (39)$$

Typical parameters

$$
\begin{aligned}
n &\sim 10^{17}\ \ \mathrm{cm}^{-3} \\
T_{\mathrm{photosphere}} &\approx 5780\ \ \mathrm{K} \\
T_{\mathrm{sunspot}} &\approx 3800 \\
\Rightarrow B &\approx 830\ \mathrm{Gauss}
\end{aligned}
\qquad (40)
$$

This is typical of the magnetic field that is observed from Zeeman measurements of the sun's photosphere.

6 The β parameter

Magnetic forces are important when the magnetic field is "large". What does large mean? We parameterize the relative importance of magnetic and thermal forces via the β parameter, defined by:

$$\beta = \frac{p_{\mathrm{gas}}}{p_{\mathrm{magnetic}}} = \frac{nkT}{B^2/(8\pi)} \tag{41}$$

Thus "low β" means a stong magnetic field.

7 Evolution of the Magnetic Field

7.1 Development of evolution equation

So far we have only considered the effect of the magnetic field on the fluid. We also need to know how the magnetic field evolves with time as a result of the motion of the fluid. In principle this is given by Maxwell's equations. However, there are some simplifications in the MHD approximation which have some interesting consequences.

Let us first examine the consequences of the high conductivity of magnetised gases, without assuming that the conductivity is infinite. We still assume an Ohm's type law for the conduction current:

$$J_i' = \sigma E_i' \tag{42}$$

where σ is the conductivity and the prime refers to the rest frame of the gas. Remembering the behaviour of electric fields under a Galilean transformation, we have

$$E_i' = E_i + \varepsilon_{ijk}\frac{v_j}{c}B_k \qquad (43)$$

where v_j is the velocity of the gas in the lab frame. Hence,

$$J_i = \sigma\left(E_i + \varepsilon_{ijk}\frac{v_j}{c}B_k\right) \qquad (44)$$

We now use the following two of Maxwell's equations, neglecting the displacement current in the first,

$$\epsilon_{ijk}\frac{\partial B_k}{\partial x_j} = \frac{4\pi}{c}J_i + \frac{1}{c}\frac{\partial E_i}{\partial t} = \frac{4\pi}{c}J_i$$

$$\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} + \frac{1}{c}\frac{\partial B_i}{\partial t} = 0 \qquad (45)$$

Using the expression for the current to solve for the magnetic field, gives

$$E_i = \sigma^{-1}J_i - \epsilon_{ijk}\frac{v_j}{c}B_k \qquad (46)$$

We also neglect the displacement current so that we can substitute $\frac{c}{4\pi}\nabla\times\boldsymbol{B}$ for the current to obtain

$$E_i = \frac{c}{4\pi\sigma}\varepsilon_{ijk}B_{k,j} - \varepsilon_{ijk}\frac{v_j}{c}B_k$$

$$\boldsymbol{E} = \frac{c}{4\pi\sigma}\nabla\times\boldsymbol{B} - \frac{\boldsymbol{v}}{c}\times\boldsymbol{B} \tag{47}$$

and we substitute this into Faraday's law to obtain

$$\varepsilon_{ijk}\left[\frac{c}{4\pi\sigma}\varepsilon_{jlm}B_{m,l} - \varepsilon_{jlm}\frac{v_l}{c}B_m\right] = -\frac{1}{c}\frac{\partial B_i}{\partial t}$$

$$\nabla\times\left[\frac{c}{4\pi\sigma}\nabla\times\boldsymbol{B} - \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right] = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t} \tag{48}$$

Solving for $\frac{\partial \boldsymbol{B}}{\partial t}$,

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}B_l v_m) = -\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{c^2}{4\pi\sigma}\varepsilon_{klm}B_{m,l}\right) \tag{49}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times(\boldsymbol{B}\times\boldsymbol{v}) = -\nabla\times\left(\frac{c^2}{4\pi\sigma}\nabla\times\boldsymbol{B}\right)$$

We denote the electrical resistivity by

$$\eta = \frac{c^2}{4\pi\sigma} \tag{50}$$

The "curl curl" term on the right can be simplified as follows:

$$\varepsilon_{ijk}\varepsilon_{klm}\frac{\partial B_{m,l}}{\partial x_j} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial B_{m,l}}{\partial x_j} = \frac{\partial^2 B_j}{\partial x_j \partial x_i} - \frac{\partial^2 B_i}{\partial x_j \partial x_j} \tag{51}$$

$$= -\frac{\partial^2 B_i}{\partial x_j \partial x_j}$$

that is,

$$\nabla\times\nabla\times\boldsymbol{B} = -\nabla^2\boldsymbol{B} \tag{52}$$

since $\nabla\bullet\boldsymbol{B} = 0$.

Hence for $\eta = \text{constant}$,

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk}(\varepsilon_{klm} V_l B_m) = \eta \frac{\partial^2 B_i}{\partial x_j \partial x_j} \tag{53}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = \eta \nabla^2 \boldsymbol{B}$$

7.2 Diffusion time scale

Obviously, if $\boldsymbol{v} = 0$, then we have a diffusion equation for $\boldsymbol{B}$, viz,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B} \tag{54}$$

The diffusion time scale, t_D, associated with this is determined by order of magnitude estimate of each side of this equation. Let the length scale of the magnetic field be L, then

$$\frac{B}{t_D} \sim \frac{\eta B}{L^2} \Rightarrow t_D \sim \frac{L^2}{\eta} \tag{55}$$

Normally, for astrophysical plasmas, the length scale is so long and the conductivity is so high that this time scale is very long. For many phenomena we are interested in, the timescales are much less than the characteristic time, t_D. (Estimates of times in various regions are given in the exercises.)

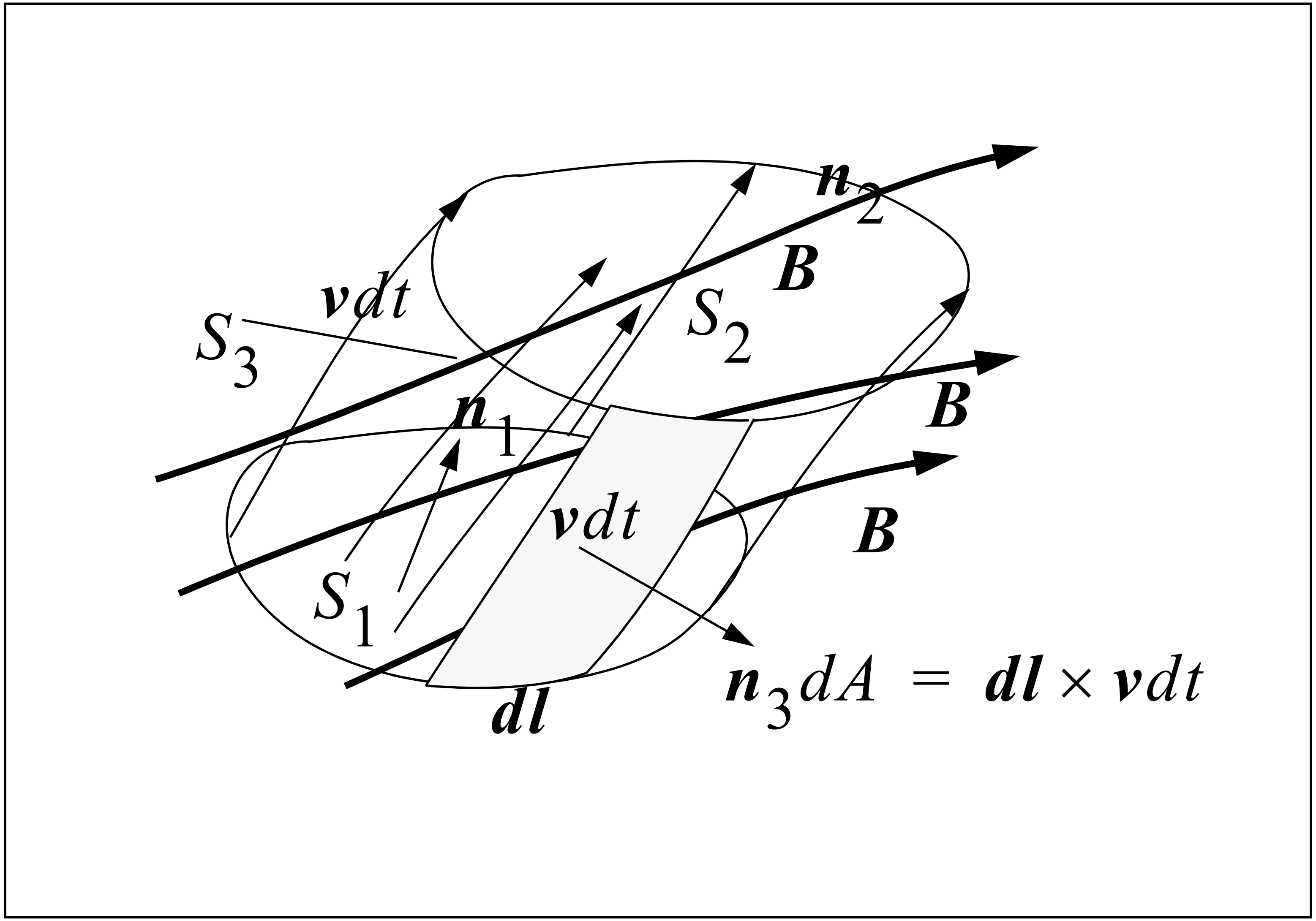
8 Alfven's flux-freezing theorem

When the conductivity is infinite, the equation for the evolution of the magnetic field becomes:

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}B_l v_m) = 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times(\boldsymbol{B}\times\boldsymbol{v}) = 0 \qquad (56)$$

The implications of this are extremely interesting: The flux through a comoving loop is conserved, as we now show.



Conservation of magnetic flux through a comvoving surfac

In the above diagram, the surface S_1 evolves to the surface S_2 in the element of time dt due to the motion of the fluid. The magnetic field is transported by the fluid according to the transport equation above. In the figure the magnetic field is shown at the time $t + dt$. The flux through the moving surface is given by

$$\Phi(t) = \int_{S_1} \boldsymbol{B}(\boldsymbol{r}, t) \bullet \boldsymbol{n} dA$$

$$\Phi(t + dt) = \int_{S_2} \boldsymbol{B}(\boldsymbol{r}, t + dt) \bullet \boldsymbol{n} dA$$

The directed area formed by the sides of the tube generated by the motion of the fluid is $\boldsymbol{n}_3 dA = d\boldsymbol{l} \times \boldsymbol{v} dt$. The flux through the sides of the volume generated by the moving surface is given to first order in dt by

$$\Phi_{\text{sides}} = \int_{S_3} \boldsymbol{B}(t) \bullet [d\boldsymbol{l} \times \boldsymbol{v} dt] = -dt \int_{S_3} [\boldsymbol{B}(t) \times \boldsymbol{v}] \bullet d\boldsymbol{l} \qquad (57)$$

Now, since $\text{div}\boldsymbol{B} = \boldsymbol{0}$, the total flux through the correctly oriented surfaces S_1, S_2 and S_3 at a fixed time, is zero, since these surfaces enclose a fixed volume. Hence,

$$\int_{S_2} \boldsymbol{B}(r, t+dt) \bullet \boldsymbol{n} dA - \int_{S_1} \boldsymbol{B}(r, t+dt) \bullet \boldsymbol{n} dA - dt \int_{S_3} [\boldsymbol{B}(t) \times \boldsymbol{v}] \bullet dl \qquad (58)$$

$$= 0$$

The integral through S_1 can be expanded to first order in dt to

$$-\int_{S_1} \boldsymbol{B}(r, t+dt) \bullet \boldsymbol{n} dA = -\int_{S_1} \boldsymbol{B}(r, t) \bullet \boldsymbol{n} dA - dt \int_{S_1} \frac{\partial \boldsymbol{B}}{\partial t} \bullet \boldsymbol{n} dA \qquad (59)$$

$$(60)$$

and, using Green's theorem

$$\int_{S_3} [\boldsymbol{B}(t) \times \boldsymbol{v}] \bullet dl = \int_{S_1} \nabla\times(\boldsymbol{B} \times \boldsymbol{v}) \bullet \boldsymbol{n} dA$$

so that we end up with

$$\Phi(t+dt) - \Phi(t) - dt \int_{S_1} \left[\frac{\partial \boldsymbol{B}}{\partial t} + \nabla\times(\boldsymbol{B} \times \boldsymbol{v})\right] \bullet \boldsymbol{n} dA = 0$$

However, the integral over S_1 is zero because of the induction equation, so that

$$\Phi(t+dt) - \Phi(t) = 0$$

i.e.

$$\frac{d\Phi}{dt} = 0$$

where the time derivative refers to the time derivative following the motion of the loop. This elegant result is known as Alfven's flux-freezing theorem.

8.1 Motion of the field lines

There is another way to characterize the motion of the field lines when diffusion is negligible. We expand

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}B_l v_m) = 0 \tag{61}$$

to

$$\frac{\partial B_i}{\partial t} + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}(B_l v_m) = 0$$

$$\Rightarrow \frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j}(B_i v_j) - \frac{\partial}{\partial x_j}(B_j v_i) = 0, p \qquad (62)$$

$$\frac{\partial B_i}{\partial t} + v_j\frac{\partial B_i}{\partial x_j} + B_i\frac{\partial v_j}{\partial x_j} - v_i\frac{\partial B_j}{\partial x_j} - B_j\frac{\partial v_i}{\partial x_j} = 0$$

Using

$$\frac{\partial B_j}{\partial x_j} = 0 \qquad \text{and} \qquad \frac{\partial v_j}{\partial x_j} = -\frac{1}{\rho}\frac{d\rho}{dt} \qquad (63)$$

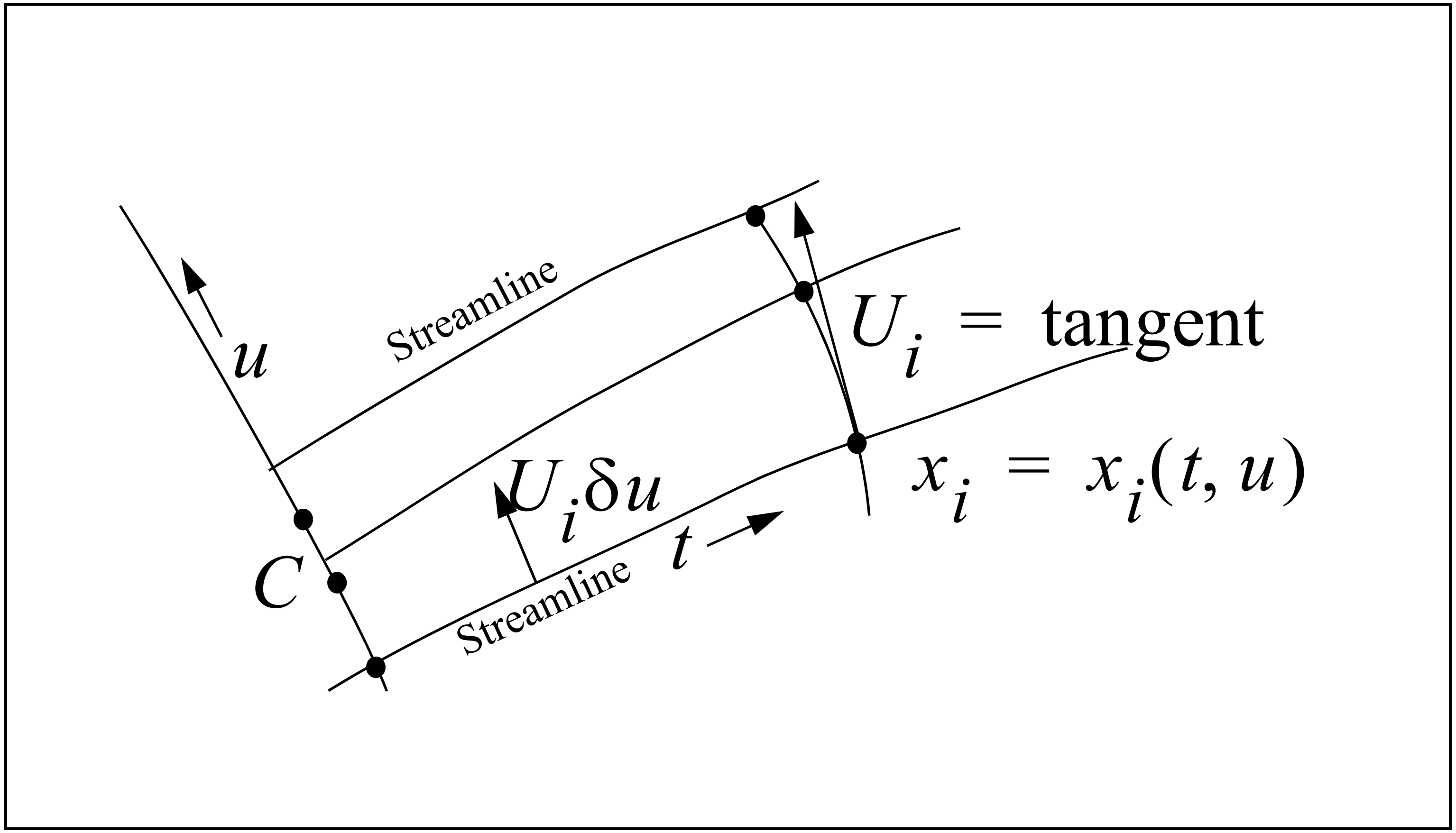
gives

$$\frac{dB_i}{dt} - \frac{B_i}{\rho}\frac{d\rho}{dt} = B_j\frac{\partial v_i}{\partial x_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{B_i}{\rho}\right) = \left(\frac{B_j}{\rho}\right)\frac{\partial v_i}{\partial x_j} \qquad (64)$$

To consider the implications of this equation, we need to make a slight diversion into the theory of two-dimensional congruences of curves.

$t = \text{constant}$



We consider streamlines, originating from a curve, ∪, so that the *congruence* of streamlines defines a two dimensional space,

$$x_i = x_i(t, u) \tag{65}$$

with the velocity along each streamline being defined by

$$v_i = \frac{\partial}{\partial t} x_i(t, u) \tag{66}$$

We define the separation vector

$$U_i = \frac{\partial}{\partial u} x_i(t, u) \tag{67}$$

which is a tangent vector to the curves formed by $t = $ constant.

Sometimes it is intuitively easier to think in terms of the infinitesimal separation between two neighbouring streamlines. This is $U_i \delta u$ and motivates the use of the term separation vector for U_i.

Now consider the *rate of change* of the separation vector with respect to time, as we move along a trajectory. This is

$$\frac{\partial U_i}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial u} x_i(t, u)\right] = \frac{\partial}{\partial u}\left[\frac{\partial}{\partial t} x_i(t, u)\right] = \left.\frac{\partial v_i}{\partial u}\right|_t \tag{68}$$

Now the rate at which the components of the velocity, v_i, change at a fixed time, is given by their spatial derivatives, i.e.

$$\frac{\partial v_i}{\partial u} = \frac{\partial v_i}{\partial x_j}\frac{\partial}{\partial u}x_j(t, u) = \frac{\partial v_i}{\partial x_j}U_j \tag{69}$$

Hence,

$$\frac{\partial U_i}{\partial t} = \frac{\partial v_i}{\partial x_j}U_j \tag{70}$$

The operator $\frac{\partial}{\partial t}$ at fixed u represents differentiation following the motion, so that we have

$$\frac{dU_i}{dt} = v_{i,j}U_j \qquad (71)$$

In terms of the infinitesimal separation vector $\delta x_i = U_i \delta u$

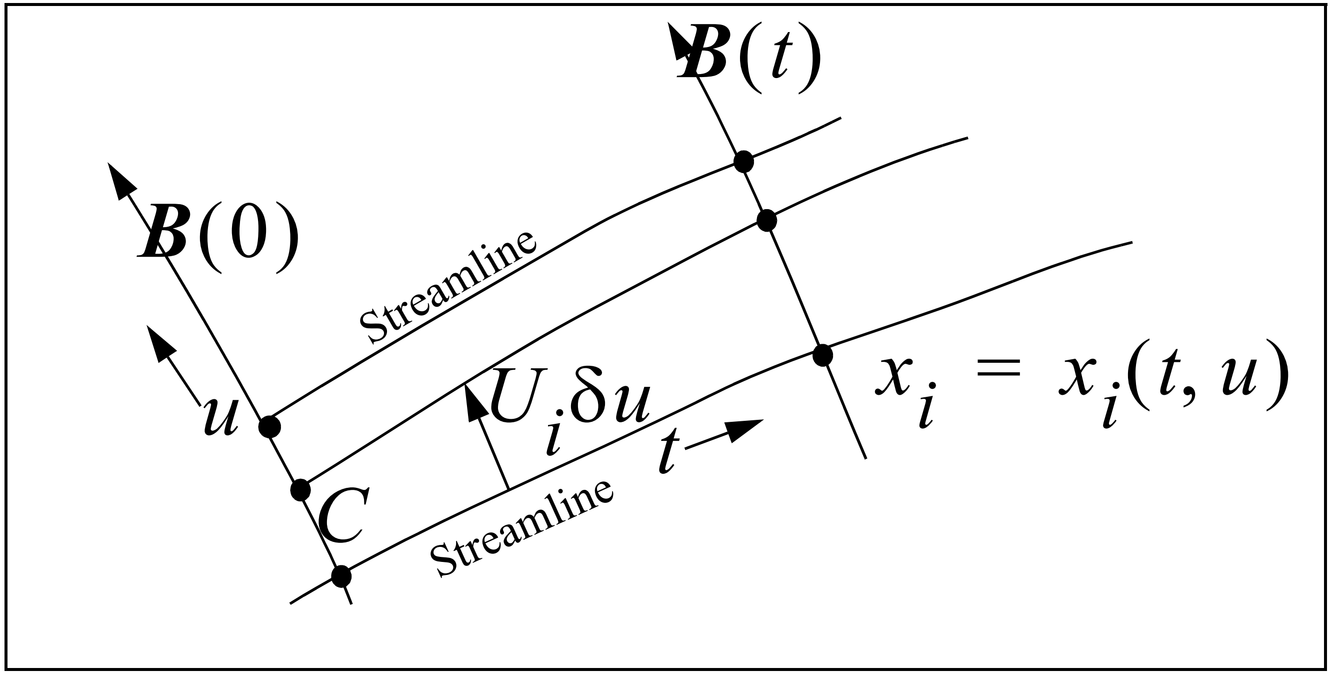
$$\frac{\partial}{\partial t}\delta x_i = v_{i,j}\delta x_j \qquad (72)$$

since δu is constant between neighbouring streamlines.

These equations show us that the way in which points on neighbouring streamlines separate is determined by the gradient of the velocity.

Note that the separation vector satisfies the same equation as the magnetic field divided by the density and this leads to the following interpretation. Consider the vector $U_i - \frac{B_i}{\rho}$. This satisfies the equation,

$$\frac{d}{dt}\left(U_i - \frac{B_i}{\rho}\right) = v_{i,j}\left(U_j - \frac{B_j}{\rho}\right) \tag{73}$$



If we now take our initial curve C such that it is tangent to a magnetic field line and choose the parametrization of that curve so that $U_i = \dfrac{B_i}{\rho}$, then from the above equation we can see that

$U_i - \frac{B_i}{\rho} = 0$ always. Therefore the line $t = \text{constant}$ formed by the evolution of fluid elements along the magnetic field will remain parallel to $\boldsymbol{B}$. In other words, the magnetic field remains parallel to the curve defined by the new positions of the fluid elements. Thus, the magnetic field is carried along by the fluid.