Magnetohydrodynamic Waves

1 Perturbation of the MHD equations

1.1 General considerations

In any theory, the properties of waves are initially determined, through the use of first order perturbations to the equations of the theory. In MHD we perturb the equations of motion that we have determined from the continuum approach.

1.2 Perturbations of the MHD equations of motion

$$\frac{\partial\rho}{\partial t} + \nabla\bullet(\rho\boldsymbol{v}) = 0$$

$$\rho\frac{\partial}{\partial t}(\boldsymbol{v}) + \rho\boldsymbol{v}\bullet\nabla\boldsymbol{v} = -\nabla P - \rho\nabla\phi + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \qquad (1)$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + \nabla\times(\boldsymbol{B}\times\boldsymbol{v}) = 0$$

$$\nabla\bullet\boldsymbol{B} = 0$$

In the following we neglect gravity and take for the first order perturbations to a gas initially at rest:

$$\begin{aligned}
\rho &= \rho_0 + \rho_1 \\
\boldsymbol{v} &= \boldsymbol{v}_0 + \boldsymbol{v}_1 \qquad \boldsymbol{v}_0 = 0 \\
p &= p_0 + p_1 = p_0 + c_0^2 \rho_1 \\
\boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{B}_1
\end{aligned} \tag{2}$$

The expression for the perturbation in the pressure results from

$$\delta p = \left.\frac{\partial p}{\partial \rho}\right|_s \delta\rho = c_s^2 \delta\rho = c_0^2 \delta\rho \tag{3}$$

where c_s, (zero point value c_0) is the adiabatic speed of sound. The perturbations we are considering are adiabatic.

We neglect quadratic terms in the subscript 1 variables. The perturbation equations are:

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \bullet \boldsymbol{v}_1 &= 0 \\
\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} + c_0^2 \nabla \rho_1 - \frac{1}{4\pi}(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 &= 0 \\
\frac{\partial \boldsymbol{B}_1}{\partial t} + \nabla \times (\boldsymbol{B}_0 \times \boldsymbol{v}_1) &= 0 \\
\nabla \bullet \boldsymbol{B}_1 &= 0
\end{aligned}
\qquad (4)
$$

Since the coefficients in this equation are all constant, we can use Fourier analysis to determine the behaviour of the solutions. Therefore we put all quantities proportional to

$$\chi = \exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)] \tag{5}$$

remembering that

$$\nabla\chi = i\boldsymbol{k}\chi \qquad \frac{\partial\chi}{\partial t} = -i\omega\chi \tag{6}$$

The perturbation equations become:

$$
\begin{aligned}
-i\omega\rho_1 + i(k \cdot \rho_0 \boldsymbol{v}_1) &= 0 \\
-i\omega\rho_0 \boldsymbol{v}_1 + c_0^2 i\boldsymbol{k}\rho_1 - \frac{1}{4\pi}(i\boldsymbol{k} \times \boldsymbol{B}_1) \times \boldsymbol{B}_0 &= \mathbf{0} \\
-i\omega \boldsymbol{B}_1 + i\boldsymbol{k} \times (\boldsymbol{B}_0 \times \boldsymbol{v}_1) &= \mathbf{0} \\
i(\boldsymbol{k} \cdot \boldsymbol{B}_1) &= 0
\end{aligned}
\qquad (7)
$$

On dividing these equations through by -*i* and expanding out the vector products, we obtain

$$
\begin{aligned}
\omega\rho_1 - \rho_0(\boldsymbol{k}\cdot\boldsymbol{v}_1) &= 0 \\
\omega\rho_0\boldsymbol{v}_1 - c_0^2\rho_1\boldsymbol{k} - \frac{1}{4\pi}[(\boldsymbol{B}_0\cdot\boldsymbol{B}_1)\boldsymbol{k} - (\boldsymbol{B}_0\cdot\boldsymbol{k})\boldsymbol{B}_1] &= \boldsymbol{0} \\
\omega\boldsymbol{B}_1 - (\boldsymbol{k}\cdot\boldsymbol{v}_1)\boldsymbol{B}_0 + (\boldsymbol{k}\cdot\boldsymbol{B}_0)\boldsymbol{v}_1 &= \boldsymbol{0} \\
\boldsymbol{k}\cdot\boldsymbol{B}_1 &= 0
\end{aligned}
\tag{8}
$$

Note that the last equation derived from $\nabla\bullet\boldsymbol{B} = 0$ is redundant since it also follows from taking the scalar product of the third set with $\boldsymbol{k}$. Therefore, the above set of equations constitute 7 homogeneous equations in 7 unknowns.

Since $\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0$, then one immediate consequence of these equations is that the component of the perturbed magnetic field in the direction of propagation is zero, that is, the magnetic field is *transverse* to the direction of propagation.

We define

$$v_w = \frac{\omega}{k} \qquad \text{and} \qquad \boldsymbol{n} = \frac{\boldsymbol{k}}{k} \tag{9}$$

so that $\boldsymbol{n}$ is a unit vector in the direction of the wave vector. On dividing through by k,

$$\begin{aligned} v_w\rho_1 - \rho_0(\boldsymbol{v}_1\cdot\boldsymbol{n}) &= 0 \\ \rho_0 v_w \boldsymbol{v}_1 - c_0^2\rho_1\boldsymbol{n} - \frac{1}{4\pi}(\boldsymbol{B}_0\cdot\boldsymbol{B}_1)\boldsymbol{n} + \frac{1}{4\pi}(\boldsymbol{B}_0\cdot\boldsymbol{n})\boldsymbol{B}_1 &= \boldsymbol{0} \\ v_w\boldsymbol{B}_1 - (\boldsymbol{v}_1\cdot\boldsymbol{n})\boldsymbol{B}_0 + (\boldsymbol{B}_0\cdot\boldsymbol{n})\boldsymbol{v}_1 &= \boldsymbol{0} \\ \boldsymbol{B}_1\cdot\boldsymbol{n} &= 0 \end{aligned} \qquad (10)$$

The aim of the following is to find the values of v_w which are consistent with the above equations. The condition for there to be a solution is that the determinant of the system be zero. However, rather than charge in and calculate the determinant we shall do things slightly differently.

We take scalar products of the vector equations with 3 independent vectors, $\boldsymbol{n}$, $\boldsymbol{B}_0$ and $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{B}_0$. This is the most elegant way to write out fully the homogeneous equations and this also gives us some insight into the properties of the different wave modes. The result is the following set of 6 equations (6 because one of the variables is eliminated by virtue of $\boldsymbol{B}_1 \bullet \boldsymbol{n} = 0$):

$$
\begin{aligned}
v_w \rho_1 - \rho_0 (\boldsymbol{v}_1 \cdot \boldsymbol{n}) &= 0 \\
\rho_0 v_w (\boldsymbol{v}_1 \cdot n) - c_0^2 \rho_1 - \frac{1}{4\pi} (\boldsymbol{B}_1 \cdot \boldsymbol{B}_0) &= 0 \\
\rho_0 v_w (\boldsymbol{v}_1 \cdot \boldsymbol{B}_0) - c_0^2 \rho_1 (B_0 \cdot \boldsymbol{n}) - \frac{1}{4\pi} (\boldsymbol{B}_1 \cdot \boldsymbol{B}_0)(\boldsymbol{B}_0 \cdot \boldsymbol{n}) & \\
+ \cancel{\frac{1}{4\pi} (\boldsymbol{B}_0 \cdot \boldsymbol{n})(\boldsymbol{B}_1 \cdot \boldsymbol{B}_0)} &= 0 \qquad (11) \\
\rho_0 v_w (\boldsymbol{v}_1 \cdot \boldsymbol{m}) + \frac{1}{4\pi} (\boldsymbol{B}_0 \cdot \boldsymbol{n})(\boldsymbol{B}_1 \cdot \boldsymbol{m}) &= 0 \\
v_w (\boldsymbol{B}_1 \cdot \boldsymbol{B}_0) - (\boldsymbol{v}_1 \cdot \boldsymbol{n}) B_0^2 + (\boldsymbol{B}_0 \cdot \boldsymbol{n})(\boldsymbol{v}_1 \cdot \boldsymbol{B}_0) &= 0 \\
v_w (\boldsymbol{B}_1 \cdot \boldsymbol{m}) + (\boldsymbol{B}_0 \cdot \boldsymbol{n})(\boldsymbol{v}_1 \cdot \boldsymbol{m}) &= 0 \\
\boldsymbol{B}_1 \cdot \boldsymbol{n} &= 0
\end{aligned}
$$

Note that we have used $\boldsymbol{B}_1 \cdot \boldsymbol{n} = 0$ and that the last two terms cancel in the third equation. These equations can be cast in the following matrix form:

$$
\begin{bmatrix}
v_w & -\rho_0 & 0 & 0 & 0 & 0 & 0 \\
-c_0^2 & \rho_0 v_w & 0 & -\dfrac{1}{4\pi} & 0 & 0 & 0 \\
-c_0^2(\boldsymbol{B}_0 \cdot \boldsymbol{n}) & 0 & \rho_0 v_w & 0 & 0 & 0 & 0 \\
0 & -B_0^2 & (\boldsymbol{B}_0 \cdot \boldsymbol{n}) & v_w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_0 v_w & \dfrac{\boldsymbol{B}_0 \cdot \boldsymbol{n}}{4\pi} & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{B}_0 \cdot \boldsymbol{n} & v_w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\boldsymbol{v}_1 \cdot \boldsymbol{n} \\
\boldsymbol{v}_1 \cdot \boldsymbol{B}_{\mathbf{0}} \\
\boldsymbol{B}_1 \cdot \boldsymbol{B}_{\mathbf{0}} \\
\boldsymbol{v}_1 \cdot \boldsymbol{m} \\
\boldsymbol{B}_1 \cdot \boldsymbol{m} \\
\boldsymbol{B}_1 \cdot \boldsymbol{n}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\tag{12}
$$

The condition for a nontrivial solution is that the determinant, Δ of the 7×7 matrix be zero. The seventh equatio, of course, resulting from the divergence free condition on the magnetic field, is trivial. The determinant splits into three subdeterminants,

$$\Delta = \Delta_2 \times \Delta_4 \times 1 \tag{13}$$

where

$$\begin{aligned} \Delta_2 &= \rho_0 v_w^2 - \frac{(\boldsymbol{B}_0 \cdot \boldsymbol{n})^2}{4\pi} \\ \Delta_4 &= -\rho_0^2 v_w^2 \left[v_w^2 - \frac{B_0^2}{4\pi\rho} - c_0^2 \right] - \rho_0 c_0^2 \frac{(\boldsymbol{B}_0 \cdot \boldsymbol{n})^2}{4\pi} \end{aligned} \tag{14}$$

This is a convenient point to introduce the *Alfven wave speed*, v_A, defined by:

$$v_A^2 = \frac{B^2}{4\pi\rho} \tag{15}$$

Hence,

$$\frac{(\boldsymbol{B}_0 \cdot \boldsymbol{n})^2}{4\pi\rho_0} = \frac{B_0^2}{4\pi\rho_0}\cos^2\psi = v_A^2\cos^2\psi \tag{16}$$

where ψ is the angle between the direction of propagation ($\boldsymbol{n}$) and $\boldsymbol{B}_0$

Alfven waves

With this defintion, the equation $\Delta_2 = 0$ has the solution

$$v_w^2 = v_A^2 \cos^2\psi \Rightarrow v_w = \pm v_A \cos\psi \qquad (17)$$

The waves satisfying this solution are known as *Alfven waves*.

Magnetoacoustic waves

Consider now the second determinant:

$$\Delta_4 = 0 \Rightarrow v_w^2[v_w^2 - v_A^2 - c_0^2] + c_0^2 v_A^2 \cos^2\psi = 0$$

$$\Rightarrow v_W^4 - (v_A^2 + c_0^2)v_w^2 + c_0^2 v_A^2 \cos^2\psi = 0 \qquad (18)$$

The solution of this quadratic equation in v_w^2 is

$$v_W^2 = \frac{(v_A^2 + c_0^2) \pm \sqrt{(v_A^2 + c_0^2)^2 - 4c_0^2 v_A^2 \cos^2\psi}}{2} \qquad (19)$$

The upper branch corresponds to *fast magnetoacoustic waves*; the lower branch represents *slow magnetoacoustic waves*. Sometimes, especially in the older textbooks, one also sees these waves referred to as *magnetosonic* waves.

2 Characteristics of Alfven waves

2.1 Components

Consider the matrix equation for the wave modes:

$$
\begin{bmatrix}
v_w & -\rho_0 & 0 & 0 & 0 & 0 & 0 \\
-c_0^2 & \rho_0 v_w & 0 & -\dfrac{1}{4\pi} & 0 & 0 & 0 \\
-c_0^2(\boldsymbol{B}_0\cdot\boldsymbol{n}) & 0 & \rho_0 v_w & 0 & 0 & 0 & 0 \\
0 & -B_0^2 & (\boldsymbol{B}_0\cdot\boldsymbol{n}) & v_w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_0 v_w & \dfrac{\boldsymbol{B}_0\cdot\boldsymbol{n}}{4\pi} & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{B}_0\cdot\boldsymbol{n} & v_w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\boldsymbol{v}_1\cdot\boldsymbol{n} \\
\boldsymbol{v}_1\cdot\boldsymbol{B}_0 \\
\boldsymbol{B}_1\cdot\boldsymbol{B}_0 \\
\boldsymbol{v}_1\cdot\boldsymbol{m} \\
\boldsymbol{B}_1\cdot\boldsymbol{m} \\
\boldsymbol{B}_1\cdot\boldsymbol{n}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
\qquad (20)
$$

Alfven waves correspond to

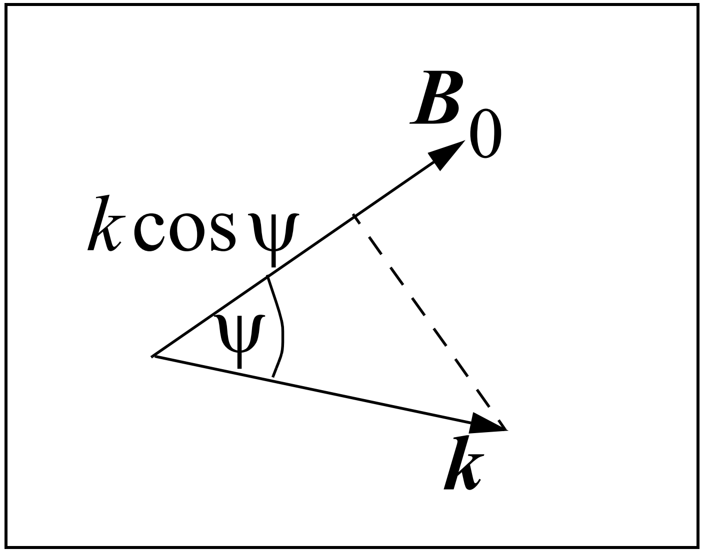
$$\begin{aligned} \boldsymbol{v}_1 \cdot \boldsymbol{m} &= \boldsymbol{v}_1 \cdot (\boldsymbol{n} \times \boldsymbol{B}_0) \neq 0 \\ \boldsymbol{B}_1 \cdot \boldsymbol{m} &= B_1 \cdot (\boldsymbol{n} \times \boldsymbol{B}_0) \neq 0 \end{aligned} \qquad (21)$$

but with

$$\rho_1 = \boldsymbol{v}_1 \cdot \boldsymbol{n} = v_1 \cdot \boldsymbol{B}_{\boldsymbol{0}} = \boldsymbol{B}_1 \cdot \boldsymbol{n} = \boldsymbol{B}_1 \cdot \boldsymbol{B}_0 = 0 \qquad (22)$$

i.e. Both the velocity and magnetic field in Alfven waves are transverse both to the direction of propagation and the magnetic field. They are also non-compressive ($\rho_1 = 0$).

2.2 Phase & group velocity



Since

$$\omega = v_A k \cos \psi = \boldsymbol{v}_A \cdot \boldsymbol{k} \qquad (23)$$

where

$$\boldsymbol{v}_A = v_A \hat{\boldsymbol{B}} \tag{24}$$

then the phase velocity is proportional to the projection of the wave direction onto the magnetic field.

The ***group velocity*** of Alfven waves is given by:

$$\frac{\partial \omega}{\partial k_i} = v_{A,\,i} \tag{25}$$

i.e.

$$\boldsymbol{v}_{A,\,g} = \boldsymbol{v}_A \tag{26}$$

and the group velocity of a wave packet is along the magnetic field.

3 Characteristics of magnetoacoustic waves

3.1 General points

Consider again our wave equation for the wave modes:

$$
\begin{bmatrix}
v_w & -\rho_0 & 0 & 0 & 0 & 0 & 0 \\
-c_0^2 & \rho_0 v_w & 0 & -\dfrac{1}{4\pi} & 0 & 0 & 0 \\
-c_0^2(\boldsymbol{B}_0 \cdot \boldsymbol{n}) & 0 & \rho_0 v_w & 0 & 0 & 0 & 0 \\
0 & -B_0^2 & (\boldsymbol{B}_0 \cdot \boldsymbol{n}) & v_w & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \rho_0 v_w & \dfrac{\boldsymbol{B}_0 \cdot \boldsymbol{n}}{4\pi} & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{B}_0 \cdot \boldsymbol{n} & v_w & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\rho_1 \\
\boldsymbol{v}_1 \cdot \boldsymbol{n} \\
\boldsymbol{v}_1 \cdot \boldsymbol{B}_\mathbf{0} \\
\boldsymbol{B}_1 \cdot \boldsymbol{B}_\mathbf{0} \\
\boldsymbol{v}_1 \cdot \boldsymbol{m} \\
\boldsymbol{B}_1 \cdot \boldsymbol{m} \\
\boldsymbol{B}_1 \cdot \boldsymbol{n}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

The components corresponding to Alfven waves are zero, i.e.,

$$\boldsymbol{v}_1 \cdot \boldsymbol{m} = \boldsymbol{B}_1 \cdot \boldsymbol{m} = 0 \tag{27}$$

so that the velocity and magnetic field lie in the plane perpendicular to $\boldsymbol{m} = \boldsymbol{n} \times \boldsymbol{b}$, i.e. in the plane of the wave vector $\boldsymbol{k}$ and the magnetic field $\boldsymbol{B}_0$. In general, the components $\boldsymbol{v}_1 \cdot \boldsymbol{n}$, $\boldsymbol{v}_1 \cdot \boldsymbol{B}_\mathbf{0}$ and $\boldsymbol{B}_1 \cdot \boldsymbol{B}_0$ are nonzero. However, $\boldsymbol{B}_1 \cdot \boldsymbol{n} = 0$ so that the only component of the magnetic field is in the direction of the unperturbed magnetic field.

Note also that magnetoacoustic waves are compressive. In general $\rho_1 \neq 0$.

3.2 Special cases

The equation for the wave speed is:

$$v_W^2 = \frac{(v_A^2 + c_0^2) \pm \sqrt{(v_A^2 + c_0^2)^2 - 4c_0^2 v_A^2 \cos^2\psi}}{2} \qquad (28)$$

Pure hydrodynamics: Zero magnetic field

When there is no magnetic field, ($v_A = 0$), then the solutions collapse to

$$v_w = c_0 \qquad \text{or} \qquad v_w = 0 \qquad (29)$$

Propagation along the field

When $\psi = 0$, then

$$v_w^2 = v_A^2 \qquad \text{or} \qquad v_w^2 = c_0^2 \tag{30}$$

$$v_w = v_A \qquad \text{or} \qquad v_w = c_0$$

for the fast and slow modes. Which mode is fast or slow depends upon the relative magnitudes of the Alfven speed and the sound speed.

Propagation perpendicular to the field

When $\psi = \frac{\pi}{2}$, then

$$v_w^2 = v_A^2 + c_0^2 \qquad \text{and} \qquad v_w^2 = 0 \tag{31}$$

for the fast and slow modes respectively.