Magnetohydrodynamics

MHD
References

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T.G. Cowling: *Magnetohydrodynamics*

E. Parker: *Magnetic Fields*

B. Rossi and S. Olbert: *Introduction to the Physics of Space*

T.J.M. Boyd and J.J Sanderson *The Physics of Plasmas*

General physical references

J.D. Jackson: *Classical Electrodynamics*

L.D. Landau & E.M. Lifshitz: *The Electrodynamics of Continuous Media*

E.M. Lifshitz & L.P. Pitaevskii: *Physical Kinetics*

K. Huang: *Statistical Mechanics*
Maxwell’s equations (cgs Gaussian units)

Gauss’s law of electrostatics
\[ \nabla \cdot \mathbf{E} = 4\pi \rho_e \]
\[ \nabla \cdot \mathbf{B} = 0 \]

No magnetic monopoles

Ampere’s law
\[ \nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_e + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \]

Faraday’s law of induction
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]

Displacement current

Electric current

Particle equations of motion
\[ m \frac{d\mathbf{v}}{dt} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - m \nabla \phi \]

Lorentz force

Gravitational force
Cartesian form of Maxwell’s equations

\[
\frac{\partial E_j}{\partial x_j} = 4\pi \rho_e \\
\frac{\partial B_j}{\partial x_j} = 0 \\
\epsilon_{ijk} \frac{\partial B_k}{\partial x_j} = \frac{4\pi}{c} J_i + \frac{1}{c} \frac{\partial E_i}{\partial t} \\
\epsilon_{ijk} \frac{\partial E_k}{\partial x_j} + \frac{1}{c} \frac{\partial B_i}{\partial t} = 0
\]

Equations of motion of a charged particle

\[
m \frac{d\mathbf{v}_i}{dt} = q \left( E_i + \epsilon_{ijk} \frac{v_j}{c} B_k \right) - m \frac{\partial \phi}{\partial x_i}
\]
Energy density, Poynting flux and Maxwell stress tensor

$$\epsilon_{\text{EM}} = \frac{E^2 + B^2}{8\pi} = \text{Electromagnetic energy density}$$

$$S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j B_k = \text{Poynting flux}$$

$$\Pi_{i}^{\text{EM}} = \frac{S_i}{c^2} = \text{Electromagnetic momentum density}$$

$$M_{ij}^{\text{B}} = \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right) = \text{Magnetic component of Maxwell stress tensor}$$

$$M_{ij}^{\text{E}} = \frac{1}{4\pi} \left( E_i E_j - \frac{1}{2} E^2 \delta_{ij} \right) = \text{Electric component of Maxwell stress tensor}$$
Relationships between electromagnetic energy, flux and momentum

The following relations can be derived from Maxwell’s equations:

\[ \frac{\partial \epsilon_{\text{EM}}}{\partial t} + \frac{\partial S_i}{\partial x_i} = -J_i E_i \]

\[ \frac{\partial \Pi_i}{\partial t} - \frac{\partial M_{ij}}{\partial x_j} = - \left( \rho_e E_i + \epsilon_{ijk} \frac{J_j}{c} B_k \right) \]
Momentum equations

Consider the electromagnetic force acting on a particle

\[ F^\alpha_i = q^\alpha \left[ E_i + \epsilon_{ijk} \frac{v^\alpha_j}{c} B_k \right] \]

\( \alpha \) refers to specific particle

Consider a unit volume of gas and the electromagnetic force acting on this volume

\[ F^\text{em}_i = \left[ \sum_\alpha q^\alpha \right] E_i + \epsilon_{ijk} \left[ \sum_\alpha q^\alpha \frac{v^\alpha_j}{c} \right] B_k \]

where the sum is over all particles within the unit volume. N.B. The velocity here is the particle velocity not the fluid velocity.
Momentum (cont’d)

\[ F_{i}^{\text{em}} = \left[ \sum_{\alpha} q^\alpha \right] E_i + \epsilon_{ijk} \left[ \sum_{\alpha} q^\alpha v_j^\alpha \right] B_k \]

We can identify the following components

\[ \sum_{\alpha} q^\alpha = \rho_e = \text{Electric charge density} \]

\[ \sum_{\alpha} q^\alpha v_j^\alpha = J_i = \text{Electric current density} \]

so that the electromagnetic force can be written

\[ F_{i}^{\text{em}} = \rho_e E_i + \epsilon_{ijk} \frac{J_j}{c} B_k \]

i.e. \[ \mathbf{F} = \rho_e \mathbf{E} + \frac{1}{c} \mathbf{J} \times \mathbf{B} \]
Momentum (cont’d)

We add the body force to the momentum equations to obtain

\[
\rho \frac{dv_i}{dt} = - \frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \rho_e E_i + \epsilon_{ijk} \frac{J_j}{c} B_k
\]

Now use the equation for the conservation of electromagnetic momentum:

\[
\rho_e E_i + \frac{1}{c} \epsilon_{ijk} J_j B_k = - \frac{\partial \Pi_i}{\partial t} + \frac{\partial M_{ij}}{\partial x_j}
\]
Momentum (cont’d)

so that the momentum equations become:

\[
\frac{\partial}{\partial t} \left( \rho v_i + \Pi_i \right) + \frac{\partial}{\partial x_j} \left( \rho v_i v_j + p \delta_{ij} - M_{ij} \right) = -\rho \frac{\partial \phi}{\partial x_i}
\]

For non-relativistic motions and large conductivity some very useful approximations are possible.
Limit of infinite conductivity

In the plasma rest frame (denoted by primes), Ohm’s law is

\[ J'_i = \sigma E'_i \]

The conductivity of a plasma is very high so that for a finite current

\[ E'_i \approx 0 \]

This has implications for the lab-frame electric and magnetic field
Transformation of electric and magnetic fields

Lorentz transformation of electric and magnetic fields

\[ E' = \Gamma \left( E + \frac{v \times B}{c} \right) \]
\[ B' = \Gamma \left( B - \frac{v \times E}{c} \right) \]

\( \Gamma = \) Lorentz factor

If \( E' = 0 \)

\[ E + \frac{v \times B}{c} = 0 \]

\[ \Rightarrow E = -\frac{v \times B}{c} = O \left( \frac{vB}{c} \right) \]

This is the magnetohydrodynamic approximation.
Maxwell tensor

Since \[ \mathbf{E} = \mathcal{O} \left( \frac{vB}{c} \right) \]

then

\[ M_{ij}^E = \mathcal{O} \left( \frac{v^2}{c^2} \right) \times M_{ij}^B \]

Hence, we neglect the electric component of the Maxwell tensor. We can also neglect the displacement current, as we show later.
Electromagnetic momentum

We want to compare the electromagnetic momentum density with the matter momentum density, i.e. compare

\[
\Pi_{i}^{EM} = \frac{1}{4\pi c} \varepsilon_{ijk} E_j B_k
\]

with

\[
\rho \nu_i
\]

\[
\Pi_{i}^{EM} = \mathcal{O} \left( \frac{\nu T^2}{c^2} \right)
\]

\[
\rho \nu_i = \mathcal{O}(\rho \nu)
\]

\[
\frac{\Pi_{i}^{EM}}{\rho \nu_i} = \mathcal{O} \left( \frac{B^2}{4\pi \rho c^2} \right)
\]
Electromagnetic momentum

\[ \frac{\Pi_{i}^{EM}}{\rho v_{i}} = \mathcal{O} \left( \frac{B^2}{4\pi \rho c^2} \right) \]

As we shall discover later, the quantity

\[ \frac{B^2}{4\pi \rho} = v_{A}^2 = (\text{Alfven speed})^2 \]

where the Alfven speed is a characteristic wave speed within the plasma. We assume that the magnetic field is low enough and/or the density is high enough such that

\[ \frac{v_{A}^2}{c^2} \ll 1 \]

and we neglect the electromagnetic momentum density.
Final form of the momentum equation

Given the above simplifications, the final form of the momentum equations is:

\[
\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right]
\]

We also have

\[
\frac{1}{4\pi} \text{curl} \mathbf{B} \times \mathbf{B} = \text{div} \left[ \frac{\mathbf{B} \mathbf{B}}{4\pi} - \frac{B^2}{8\pi} \mathbf{I} \right]
\]

where \( \mathbf{I} \) is the unit tensor
Thus the momentum equations can be written:

\[ \rho \frac{d\mathbf{v}}{dt} = -\nabla p - \rho \nabla \phi + \frac{1}{4\pi} \text{curl} \mathbf{B} \times \mathbf{B} \]

Either form can be more useful dependent upon circumstances.
Displacement current

Let $L$ be a characteristic length, $T$ a characteristic time and $V = L/T$ a characteristic velocity in the system.

The equation for the current is:

$$J_i = \frac{c}{4\pi} \varepsilon_{ilm} \frac{\partial B_m}{\partial x_l} - \frac{1}{4\pi} \frac{\partial E_i}{\partial t}$$

$$\mathcal{O} \left( \frac{cB}{L} \right) \quad \mathcal{O} \left( \frac{E}{T} = \frac{vB}{cT} \right)$$

$$\Rightarrow \frac{\text{Displacement Current}}{\text{Curl B current}} = \mathcal{O} \left( \frac{vV}{c^2} \right) \ll 1$$
In the MHD approximation we always put

\[ J_i = \frac{c}{4\pi} \varepsilon_{ilm} \frac{\partial B_m}{\partial x_l} \]

i.e. \( \mathbf{J} = \frac{c}{4\pi} \text{curl} \mathbf{B} \)
Energy equation

The total electromagnetic energy density is

\[ E_{\text{EM}} = E^2 + B^2 \approx \frac{B^2}{8\pi} \]

since \[ E^2 = \mathcal{O} \left( \frac{v^2}{c^2} \right) B^2 \]

In order to derive the total energy equation for a magnetised gas, we add the electromagnetic energy to the total energy and the Poynting flux to the energy flux.
Energy equation (cont’d)

The final result is:

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho v^2 + \epsilon + \rho \phi + \frac{B^2}{8\pi} \right] + \frac{\partial}{\partial x_j} \left[ \rho \left( \frac{1}{2} v^2 + h + \phi \right) v_i + S_i \right] = \rho kT \frac{ds}{dt}
\]

\[
h = \frac{\epsilon + p}{\rho}
\]

\[
S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j B_k
\]

\[
= \frac{1}{4\pi} \left[ B^2 v_i - B_j v_j B_i \right] = \frac{B^2}{4\pi} v_i^\perp
\]

where \( v_i^\perp \) = Component of velocity perpendicular to magnetic field
The induction equation

The final equation to consider is the induction equation, which describes the evolution of the magnetic field.

We have the following two equations:

Faraday’s Law:  \( \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \)

Infinite conductivity:  \( \mathbf{E} = -\frac{\mathbf{v}}{c} \times \mathbf{B} \)

Together these imply the \textit{induction equation}

\[
\frac{\partial \mathbf{B}}{\partial t} = \text{curl}(\mathbf{v} \times \mathbf{B})
\]
Summary of MHD equations

Continuity:

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0 \]

Momentum:

\[ \rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right] \]
Summary (cont.)

Energy:

\[
\frac{\partial}{\partial t} \left[ \frac{1}{2} \rho v^2 + \epsilon + \rho \phi + \frac{B^2}{8\pi} \right] = \frac{\partial}{\partial x_j} \left[ \left( \frac{1}{2} v^2 + h + \phi \right) \rho v_i + S_i \right]
\]

\[= \rho kT \frac{ds}{dt} \]

where

\[
h = \frac{\epsilon + p}{\rho} \quad \quad S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j B_k = \frac{B^2}{4\pi} v_i \]

\[
\]
Summary (cont.)

Induction:

\[ \frac{\partial B_i}{\partial t} = [\text{curl}(\mathbf{v} \times \mathbf{B})]_i \]

\[ \frac{\partial B_i}{\partial t} = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} v_l B_m) \]