Exercises in Astrophysical Gas Dynamics

1. *Tensors and non-tensors.*
   In the following u_i and v_i are vectors.
   Show that:

   (a) i. $u_i v_i$ is a scalar
      ii. $u_i v_j$ is a second rank tensor.
      iii. $\partial u_i / \partial x_j$ is a second rank tensor.
      iv. $\partial u_i / \partial x_i$ is a scalar.

   (b) i. Show that $\sum_{i=1}^{3} u_i$ and $\sum_{i=1}^{3} |u_i|$ are *not* scalars.
      ii. What are the implications of this for the definition of the magnitude of a vector?

2. *Irreducible components of a tensor.*
   A tensor T_{ij} is symmetric if $T_{ij} = T_{ji}$ antisymmetric if $T_{ij} = -T_{ji}$ and traceless if $T_{ii} = 0$. Show that an arbitrary second rank tensor may be expressed as a sum of three tensors: a symmetric traceless tensor, a tensor proportional to the Kronecker delta and an antisymmetric tensor. These are known as the irreducible components of T_{ij}.

3. *Energy, momentum and stress in electrodynamics.*
   Derive the following identities from Maxwells equations:

$$\frac{\partial \epsilon_{\rm EM}}{\partial t} + \frac{\partial S_i}{\partial x_i} = -j_{e,i} E_i \qquad (1)$$

$$\frac{\partial \Pi_i^{\rm EM}}{\partial t} - \frac{\partial M_{ij}}{\partial x_j} = -\left(\rho_e E_i + \epsilon_{ijk} \frac{j_{e,j}}{c} B_k \right) \qquad (2)$$

   where E_i is the electric field, B_i is the magnetic field, $j_{e,i}$ is the electric current and ρ_e is the electric charge density and

$$\begin{aligned}
\epsilon_{EM} &= \frac{E^2 + B^2}{8\pi} = \text{Electromagnetic energy density} \\
S_i &= \frac{c}{4\pi} \epsilon_{ijk} E_j B_k = \text{Poynting flux} \\
\Pi_i^{EM} &= \frac{S_i}{c^2} = \text{Electromagnetic momentum density} \\
M_{ij} &= \left(\frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right) + \left(\frac{E_i E_j}{4\pi} - \frac{E^2}{8\pi} \delta_{ij} \right) = \text{Maxwell stress tensor}
\end{aligned}$$

4. *Conductivity of a plasma.*

   (a) Using order of magnitude estimates (following the approximate derivation for the Coulomb mean free path given in lectures) show that the electron-ion collision frequency in a plasma with ion density n_i and temperature T is:

$$\nu_c \approx 62 n_i T^{-3/2} \text{ Hz} \qquad (3)$$

   The value derived from transport theory is $\nu_c \approx 50 n_i T^{-3/2}$. Note that the order of magnitude estimate is close to the correct value and has the correct dependence on density and temperature.

(b) Using the exact value for ν_c show that the conductivity is given by

$$\sigma \approx 6 \times 10^6 \, T^{3/2} \, \mathrm{s}^{-1}$$

(c) Estimate the magnetic diffusion time scale in the environments given in Table 1.

Table 1: **Gas and magnetic parameters in different environments**

Environment	$n_e(\mathrm{cm}^{-3})$	$T(^\circ K)$	B(Gauss)	Typical Scale Size
Solar Corona	10^{4-8}	10^{3-6}	$10^{-5} - 1$	10^{10} cm
Solar chromosphere	10^{12}	10^4	10^3	10^8 cm
HII region (warm ISM)	10^{2-3}	10^{3-4}	10^{-6}	10^{1-2} pc
Hot ISM	10^{-2}	10^6	10^{-6}	10^3 pc

5. *Pressure tensor.*
Show that the pressure tensor corresponding to an isotropic distribution is isotropic, *i.e.*

$$P_{ij} = P\delta_{ij} \tag{4}$$

$$\text{with} \quad P = \frac{4\pi}{3m} \int_0^\infty p^4 f(p) \, dp \tag{5}$$

6. *Pressure of a thermal gas.*
The Maxwell-Boltzmann distribution of a thermal gas is given by

$$f_{\mathrm{MB}}(\mathbf{v}) = n \left(\frac{m}{2\pi kT} \right)^{3/2} \exp - \frac{mv^2}{2kT} \, d^3v$$

where m is the mass of each particle, n is the density and T is the temperature. Show that the pressure is given by

$$p = nkT$$

7. *Energy density and energy flux in a sound wave.*
Show that the energy density $\epsilon^{SW} = \frac{1}{2}\rho_0 \, v'^2 + \frac{1}{2} \frac{c_s^2}{\rho_0} \rho'^2$ and energy flux $F_i^{SW} = c_s^2 \, \rho' \, v'_i$ of a sound wave satisfy the continuity equation

$$\frac{\partial \epsilon^{SW}}{\partial t} + \frac{\partial F_i^{SW}}{\partial x_i} = 0$$

What does this mean, physically?

8. *The velocity of sound in a moving medium.*
Consider a gas moving at a constant speed $\mathbf{u}$. Make a transformation to a frame in which the medium is at rest and express the solution for a plane sound wave in the rest frame in terms of the wave vector $\mathbf{k}$ and rest frame frequency ω. Now make a transformation back to the moving frame and show from the expression for the phase that the frequency in the moving frame is given by

$$\omega = c_s \, k + \mathbf{u} \cdot \mathbf{k}$$

Show that the speed of sound in the moving medium is given by

$$\mathbf{v}_s = c_s \, \frac{\mathbf{k}}{k} + \mathbf{u}$$

9. *Doppler Effect.*
Consider a source of sound moving at a velocity $\mathbf{u}$ with respect to an observer. By transforming to a medium where the source is at rest (see the above question) show that the frequency received by the observer is given by:

$$\omega = \frac{\omega_0}{1 - u/c_s \cos\theta}$$

where ω_0 is the frequency emitted by the source (that is, the frequency in its own rest frame, θ is the angle between the velocity $\mathbf{u}$ and the direction of the wave-vector $\mathbf{k}$.

10. *Pressure fluctuations in a sound wave.*

    (a) Show that the magnitude of the time-averaged energy flux in a plane sound wave is given by

    $$\langle F_E \rangle = \frac{c_0 \langle p'^2 \rangle}{\rho_0}$$

    where $\langle p'^2 \rangle$ is the time-averaged square of the pressure fluctuations, c_0 is the sound speed and ρ_0 is the density.

    (b) Calculate the root-mean-square pressure fluctuations ($< p'^2 >^{1/2} / p_0$) of a sound wave 1 metre in air from an isotropic source of sound radiating an average power of 10 W. Make reasonable assumptions for whatever parameters you need.

11. *The Jeans mass at recombination.*
The Universe expanded from almost infinite density and temperature to the average sparse structure we see today. Various epochs are often described in terms of their redshift, z. The density of the Universe in atoms per cubic centimetre is given by:

$$n \approx 10^{-5} \Omega_B h^2 (1+z)^3 \text{ cm}^{-3}$$

In this equation, Ω_B is the baryon density parameter, and h is the Hubble constant in units of $100 \text{ km s}^{-1} \text{ Mpc}^{-1}$. The Universe started to recombine from its previously totally ionised state when $z \approx 1300$ and the temperature dropped to approximately $3,500\,\text{K}$. Present constraints on the density parameter are $\Omega_B h^2 \approx 0.01$.

    (a) Show that the density of the Universe at recombination is given by:

    $$\rho_{\text{rec}} \approx 4.3 \times 10^{-22} \text{ gm cm}^{-3}$$

    (b) Determine the Jeans mass in the recombination era. (This mass is often associated with globular clusters, the oldest known stellar concentrations in the Universe.)

12. *Timescale for gravitational collapse.*

    (a) Using the theoretical treatment of the Jeans mass, identify a timescale for the growth of gravitationally unstable perturbations. This Jeans timescale is important; it tells us how quickly a perturbation is likely to grow to interesting densities.

    (b) Evaluate the Jeans timescale for (i) typical ISM densities and temperatures (ii) recombination densities and temperatures. Express the answers in years.

13. *1-D flow.*
Show that the mass flux density in 1-D flow is given by:

$$j = \rho\, v = \left(\frac{p}{p_0}\right)^{\frac{1}{\gamma}} \left\{ \frac{2\gamma}{\gamma - 1}\, p_0\, \rho_0 \left[1 - \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]\right\}^{1/2}$$

14. *De Laval nozzle.*
Determine the value of p/p_0 at the minimum area of a de Laval nozzle as a function of the polytropic index, γ. Calculate p/p_0 for $\gamma = 7/5$ and $5/3$.

15. *Topology of isothermal wind solution.*
Determine the topology of the critical point for an isothermal wind.

16. *Numerical solution for isothermal wind.*
Write a computer program to exhibit the topology of curves in the $V - r$ plane for an isothermal wind. You should compute the curves which pass through the critical point as well as several typical curves which do not pass through the critical point, including some which correspond to a breeze solution. The main result of your program should be plots of the density, velocity and Mach number.

    Use any computer language you like, but preferably C, Fortran or Python.

17. *Non-linear Development of a Sound Wave.*
Consider a simple wave in which the initial ($t = 0$) velocity corresponds to that of a linear sound wave, that is

$$v = v_0\, \cos k\, x$$

    where k is the wave number. In this case, however, we allow v_0 to be arbitrary, that is the restriction $v_0/c_0 << 1$ is unnecessary.

    Show that the non-linear evolution of the simple wave is given by

$$x = t\left[c_0 + \frac{\gamma+1}{2}\, v\right] + \frac{1}{k}\, \cos^{-1}\left(\frac{v}{v_0}\right)$$

    where the meanings of the symbols is as in lectures. Show that a series of shocks will form from this simple wave after a time

$$t_{\rm shock} = \frac{1}{\pi(\gamma+1)}\, \frac{\lambda}{v_0}$$

    where λ is the wavelength of the sound wave.

    What does this result say about the validity of the linear sound wave approximation for long times?

18. Thermodynamic relation for weak shocks.
Show, for a polytropic equation of state, that

$$\frac{\partial^2 \tau}{\partial P^2} = \frac{\gamma+1}{\gamma^2}\, \tau\, P^{-2}$$

    where $\tau = 1/\rho$ is the specific volume.

19. *Pre- and post-shock Mach numbers.*
Using the relationships for a polytropic gas, show that the pre- and post-shock Mach numbers (in the frame of a normal shock) satisfy $M_1 > 1$ and $M_2 < 1$.

20. *Rankine-Hugoniot relations.*
Starting from the Rankine-Hugoniot relations derived in lectures, verify the following relations for a normal shock in a polytropic gas:

$$
\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{v_1}{v_2} = \frac{(\gamma+1)\,M_1^2}{(\gamma-1)\,M_1^2+2} \\
\frac{p_2}{p_1} &= \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1} \\
\frac{T_2}{T_1} &= \frac{[2\gamma M_1^2-(\gamma-1)][(\gamma-1)M_1^2+2]}{(\gamma+1)^2 M_1^2} \\
M_2^2 &= \frac{2+(\gamma-1)\,M_1^2}{2\,\gamma M_1^2-(\gamma-1)}
\end{aligned}
$$

where $M_1 = v_1/c_{s,1}$ is the Mach number of the pre-shock gas in the frame of the shock.

21. *Strong shocks.*
Derive the following relations for a *strong* shock (defined by $p_2 >> p_1$ or equivalently $M_1 >> 1$):

$$
\begin{aligned}
\frac{\tau_2}{\tau_1} &= \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1} \\
p_2 &= \frac{2}{\gamma+1}\,\rho_1\,v_1^2 \\
v_1 &= \left[\frac{1}{2}(\gamma+1)p_2\tau_1\right]^{1/2} \\
v_2 &= \left[\frac{1}{2}\frac{(\gamma-1)^2 p_2\tau_1}{\gamma+1}\right]^{1/2} \\
\frac{k\,T_2}{\mu\,m} &= 2\,\frac{(\gamma-1)}{(\gamma+1)^2}\,v_1^2
\end{aligned}
$$

22. *Temperatures in strong shocks.* Calculate the temperatures resulting from strong $200\,\mathrm{km\,s^{-1}}$ and $400\,\mathrm{km\,s^{-1}}$ shocks. (In astrophysics, the *speed* of a shock refers to the velocity of the shock with respect to the gas in front of it, that is, v_1. The flux of radiation resulting from a shock with a speed greater than $\sim 200\,\mathrm{km\,s^{-1}}$ is capable of ionising the surrounding plasma because of the high temperature. Such shocks have been invoked to explain the excitation of emission line regions in some active galaxies.)

23. *Critical velocity.*

The *critical velocity* c_* , in one-dimensional flow, is defined as the velocity at which the Mach number is unity. Using the Bernoulli equation, show that:

$$\frac{1}{2}V^2 + \frac{c_s^2}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}\,c_*^2$$

Now, using the fact that the energy flux is conserved, show that for a normal shock

$$v_1\,v_2 = c_*^2$$

24. *Strong shock in a stationary medium.*
All of the equations derived in lectures and in the above exercises have referred to quantities in

the frame of the shock. However, in many cases this is not the natural frame. Consider a strong shock moving into a stationary medium. Show that the the velocity of the fluid after the shock (that is, the velocity to which the shock accelerates the previously stationary pre-shock gas) is

$$v_{\text{post-shock}} = \left(\frac{2\,p_2/\rho_1}{\gamma+1} \right)^{1/2}$$

that the velocity of the shock with respect to the medium in front of the shock is

$$v_{\text{shock}} = \left(\frac{\gamma+1}{2} \frac{p_2}{\rho_1} \right)^{1/2}$$

and that consequently,

$$v_{\text{post-shock}} = \frac{2}{\gamma+1} \, v_{\text{shock}} =$$

25. *Shock tube.*

Two gases, with the same polytropic index, γ, different densities, ρ_1 and ρ_2 (in general) and different pressures p_1 and p_2 with $p_1 > p_2$ are brought into contact at $x = 0, t = 0$. The pressure profile develops as shown in figure 1. (The higher pressured fluid is taken to be the one on the left.) A shock wave propagates to the right into the low pressure gas and a rarefaction wave propagates to the left into the high pressure gas. The two gases remain separated by a contact discontinuity at the same pressure. Using the following steps, develop a method for solving for the flow in each region.

(a) Show that a similarity solution in the variable $\xi = x/t$ exists for the rarefaction region, 3, with the following properties:

- The velocity v is given by:

$$v = \frac{2}{\gamma+1} \left(c_{\text{s},1} + \xi \right)$$

- The sound speed

$$c_{\text{s}} = \frac{2}{\gamma+1} \, c_{\text{s},1} - \frac{\gamma-1}{\gamma+1} \xi$$

where the subscript 1 refers to conditions in region 1.

(b) *Contact discontinuity pressure in shock tube.*
Show that the pressure $P_5 = P_4$ of the gas on either side of the contact discontinuity is given by the solution of:

$$\left[1 - \left(\frac{p_5}{p_1} \right)^{(\gamma-1)/2\gamma} \right] = \frac{\gamma-1}{(2\gamma)^{1/2}} \left(\frac{c_2}{c_1} \right) \frac{p_5/p_2 - 1}{\left[(\gamma-1) + (\gamma+1) p_5/p_2 \right]^{1/2}}$$

(c) Denote the values of ξ between regions i and j by ξ_{ij}. Show that

$$\begin{aligned}
\xi_{13} &= -c_{\text{s},1} \\
\xi_{34} &= \frac{c_{\text{s},1}}{\gamma-1} \left[2 - (\gamma+1) \left(\frac{P_5}{p_1} \right)^{(\gamma-1)/2\gamma} \right] \\
\xi_{45} &= \frac{2c_{\text{s},1}}{\gamma-1} \left[1 - \left(\frac{P_5}{p_1} \right)^{(\gamma-1)/2\gamma} \right] \\
\xi_{52} &= \frac{c_{\text{s},2}}{(2\gamma)^{1/2}} \left[(\gamma-1) + (\gamma+1) \frac{P_5}{p_2} \right]^{1/2}
\end{aligned}$$

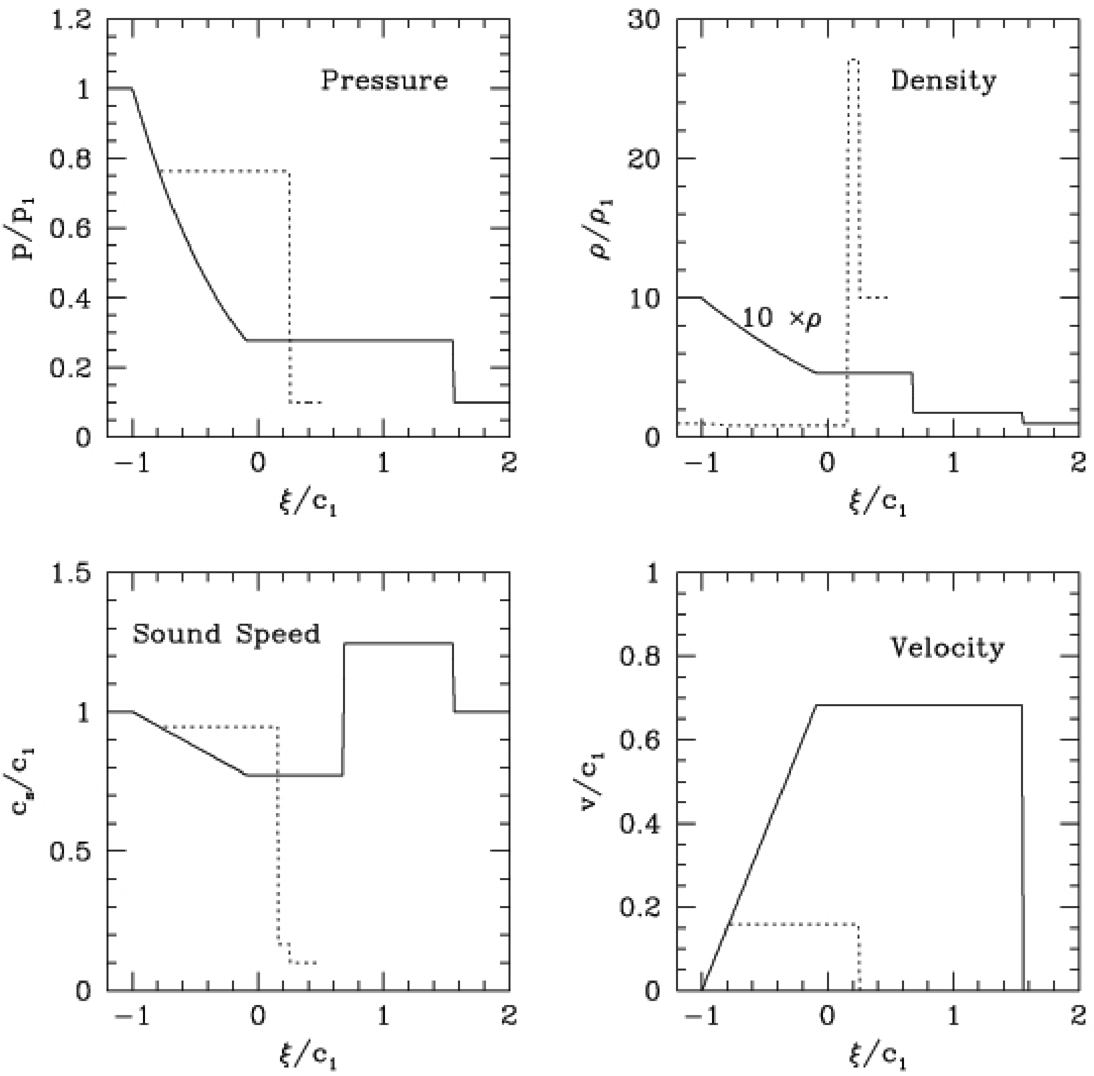

Fig. 1.— The pressure, density, sound speed and velocity in a shock tube.

26. *Limiting case for large pressure ratio.*
Referring to the preceding question show when $p_1/p_2 \gg 1$ and $\rho_1 \ll \rho_2$, that $P_5 \approx p_1$ and that the velocity of gas on either side of the contact discontinuity is given by

$$v_4 = v_5 \approx \left(\frac{2}{\gamma+1}\right)^{1/2} \left(\frac{p_1}{\rho_2}\right)^{1/2}$$

27. *Bernoulli's equation.*
(i) Derive the following equation from the momentum equations:

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + v_j \frac{\partial}{\partial x_j}\left(\frac{v^2}{2}\right) = kTv_i \frac{\partial s}{\partial x_i} - v_i \frac{\partial h}{\partial x_i} - v_i \frac{\partial \phi}{\partial x_i}$$

(ii) When the flow is time-independent and adiabatic show that:

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + h + \phi\right) = 0$$

along a streamline and that consequently

$$\frac{1}{2}v^2 + h + \phi = \text{A streamline constant.}$$

The last equation is Bernoulli's equation.

28. *Stagnation pressure of a cloud shock.* The idea of this exercise is to calculate the pressure on a cloud, which is moving supersonically with velocity v_1 through a medium of density ρ_1. The stagnation pressure is important since it is this pressure which drives a shock into the cloud, potentially making it visible in optical line emission.

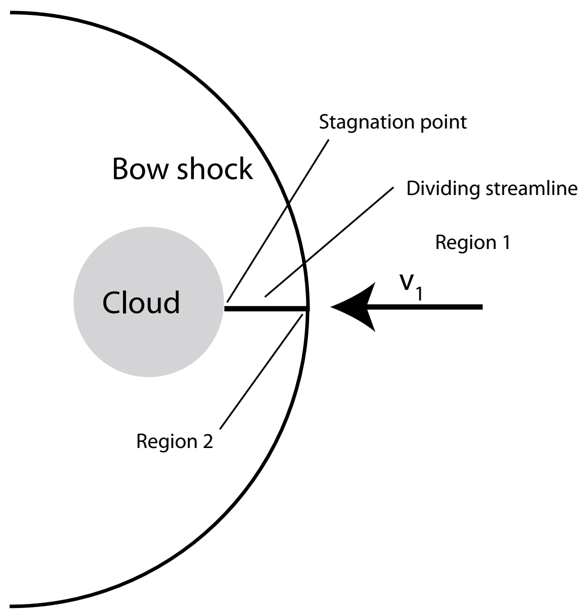

Fig. 2.— Cloud moving supersonically through a medium with velocity v_1, viewed in a frame in which the cloud is at rest.

(i) Referring to Figure 2, let the region to the right of the bow-shock be denoted by 1 and the region immediately following the bow shock on the dividing streamline be denoted with a 2. Let M_2 be the Mach number on the dividing streamline in region 2. Show that the specific enthalpy at the stagnation point is given by

$$h_s = h_2 \left[1 + \frac{\gamma - 1}{2} M_2^2\right]$$

(ii) Suppose that v_1 is highly supersonic with respect to the surrounding gas. Show that

$$\frac{h_s}{h_2} = \frac{(\gamma+1)^2}{4\gamma}$$

(iii) Show that the stagnation pressure is given by

$$p_s = \frac{2}{\gamma+1}\rho_1 v_1^2 \left[\frac{(\gamma+1)^2}{4\gamma}\right]^{\gamma/(\gamma-1)}$$

(iv) For $\gamma = 5/3$, show that

$$p_s \approx 0.88 \rho_1 v_1^2$$

29. *Numerical solution for shock tube.*
Numerically evaluate the above rarefaction-shock solution for the cases when

$$\begin{aligned} \frac{p_1}{p_2} &= 10; \quad \frac{\rho_1}{\rho_2} = 10 \\ \frac{p_1}{p_2} &= 10; \quad \frac{\rho_1}{\rho_2} = 0.1 \end{aligned}$$

Present your results as plots of the variables, pressure, density, sound speed and velocity, all normalised to the corresponding values in region 1 plotted against $\xi/c_{s,1}$.

Solutions of the 1D Euler equations such as these are used as analytical tests of numerical codes.

30. *Blast wave in power-law medium.*
Show that a suitable similarity variable for a point explosion of energy E_0 in a medium in which the density $\rho_0 = Ar^\alpha$ is:

$$\xi = \beta \left[\frac{E_0}{A}\right]^{-\frac{1}{5+\alpha}} r\, t^{-\frac{2}{5+\alpha}}$$

where β is dimensionless.

31. *Similarity equations for blast wave.*

Using the similarity variable,

$$\xi = C\, r\, t^\lambda,$$

where $\lambda = -2/(5+\alpha)$ and the substitutions

$$\begin{aligned} \rho &= A\, r^\alpha G(\xi) \\ V &= -\lambda\, \frac{r}{t}\, U(\xi) \\ c_s^2 &= \lambda^2\, \frac{r^2}{t^2}\, Z(\xi) \end{aligned}$$

show that the equations for spherically symmetric gas dynamics become

$$\begin{aligned} \frac{dU}{d\, ln\xi} - (1-U)\, \frac{d\ln G}{d\ln\xi} &= -(\alpha+3)\, V \\ (1-U)\frac{dU}{d\ln\xi} - Z\, (\alpha + \frac{d\ln G}{d\ln\xi}) &= \frac{U\, (1+\lambda U)}{\lambda} \\ \frac{d\ln Z}{d\ln\xi} - (\gamma-1)\, \frac{d\ln G}{d\ln\xi} &= \frac{2+\lambda U\, [2-\alpha(\gamma-1)]}{\lambda(1-U)} \end{aligned}$$

32. *Boundary conditions for similarity solution.*
Show that the junction conditions at the blast wave lead to the following boundary conditions on $G(\xi), U(\xi)$ and $Z(\xi)$:

$$\begin{aligned} G(1) &= \frac{\gamma+1}{\gamma-1} \\ U(1) &= \frac{2}{\gamma+1} \\ Z(1) &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \end{aligned}$$

33. *Conservation of energy in blast wave.*

(a) Show that the energy within a surface $\xi =$ constant is given by:

$$E(\xi) = 4\pi\,\lambda^2\,\beta^{-(\alpha+5)}\,E_0 \int_0^{\xi} \xi'^{\alpha+4}\left[\frac{1}{2}U^2(\xi') + \frac{Z(\xi')}{\gamma(\gamma-1)}\right]\,G(\xi')\,d\xi'$$

and is therefore independent of time.

(b) Show that the parameter β is determined by

$$\beta^{\alpha+5} = 4\pi\,\lambda^2 \int_0^1 \xi^{\alpha+4}\left[\frac{1}{2}U^2(\xi) + \frac{Z(\xi)}{\gamma(\gamma-1)}\right]\,G(\xi)\,d\xi$$

(c) Show that the conservation of energy implies that

$$Z(\xi) = \frac{\gamma\,(\gamma-1)}{2}\,\frac{U^2(\xi)\,(1-U(\xi))}{\gamma U(\xi)-1}$$

Note that this equation is independent of α.

34. *Supernova in red giant wind.*

(a) A star produces a stellar wind with a mass-loss rate $\dot{M}$ and a terminal velocity v_∞. Show that, far from the star, that is, well outside the sonic radius,

$$\rho = \frac{\dot{M}}{4\pi\,v_\infty\,r^2}$$

(b) The precursor to a supernova is often a red-giant. Such stars generally possess a stellar wind with a terminal velocity $v_\infty \approx 10\,\mathrm{km\,s^{-1}}$ and a mass-loss rate $\dot{M} \sim 10^{-6}\,M_\odot\,\mathrm{y}^{-1}$. After the star explodes as a supernova, show that the location of the blast wave, which propagates into the preexisting red-giant wind, is given by:

$$\begin{aligned} r_{BW} &= \beta^{-1}\left[\frac{4\pi v_\infty E_0}{\dot{M}}\right]^{1/3} t^{2/3} \\ &= 4.1\,\mathrm{pc}\;\beta^{-1}\left(\frac{v_\infty}{10\,\mathrm{km\,s^{-1}}}\right)^{1/3}\left(\frac{E_0}{10^{51}\,\mathrm{ergs}}\right)^{1/3}\left(\frac{\dot{M}}{10^{-6}\,M_\odot\,\mathrm{y}^{-1}}\right)^{-1/3}\left(\frac{t}{100\,\mathrm{yr}}\right)^{2/3} \end{aligned}$$

and that the velocity of the blast wave is:

$$\begin{aligned} v_{BW} &= \frac{2}{3}\,\beta^{-1}\left[\frac{4\pi v_\infty E_0}{\dot{M}}\right]^{1/3} t^{-1/3} \\ &= 2.7\times 10^4\,\mathrm{km\,s^{-1}}\,\beta^{-1}\left(\frac{v_\infty}{10\,\mathrm{km\,s^{-1}}}\right)^{1/3}\left(\frac{E_0}{10^{51}\,\mathrm{ergs}}\right)^{1/3}\left(\frac{\dot{M}}{10^{-6}\,M_\odot\,\mathrm{y}^{-1}}\right)^{-1/3}\left(\frac{t}{100\,\mathrm{yr}}\right)^{-1/3} \end{aligned}$$

(Given that $\beta \sim 1$ this gives a good estimate of the blast wave velocity in such an environment. Note also that the blast wave decelerates less rapidly in a preexisting stellar wind than it would in a homogeneous medium.)

35. *Inner radius in stellar wind bubble.*
Show that the ratio of the radius of the stellar wind shock to the radius of the bubble in a mass-loss bubble in which the shocked ISM forms a radiative shell is given by:

$$\begin{aligned}\frac{R_1}{R_s} &= \left(\frac{33}{20}\right)^{1/2} v_w^{-1/2} \left[\frac{5^3}{7\cdot 22\pi}\frac{L_w}{\rho_a}\right]^{1/10} t^{-1/5} \\ &= 0.20 \left(\frac{\dot{M}_w}{10^{-6} M_\odot\,\mathrm{y}^{-1}}\right)^{0.1} \left(\frac{v_w}{10^3\,\mathrm{km\,s^{-1}}}\right)^{-0.3} \left(\frac{n_a}{10^6\,\mathrm{m^{-3}}}\right)^{-0.1} t_6^{-0.2}\end{aligned}$$

where the symbols have the usual meanings. This justifies the assumption, $R_1^3 << R_s^3$ used to derive the bubble equations.

36. *Interacting winds in planetary nebulae.*
The "interacting wind" model of planetary nebulae is based on the idea that the white dwarf phase of stellar evolution is preceded by a red giant phase. A fast wind from the hot white dwarf overtakes the more slowly moving red giant wind and the region of interaction forms a shell of material which is driven outwards by the fast wind and photoionised by the white dwarf. The emission from this shell produces the observed planetary nebula. (See Kwok, ApJ, 258, 280).

Typical parameters for this situation are:

- Mass-loss rate in red giant phase: $\dot{M}_{RG} \sim 10^{-5} M_\odot\,\mathrm{y}^{-1}$
- Velocity of red giant wind: $v_{RG} \approx 5 - 20\,\mathrm{km\,s^{-1}}$.
- Mass-loss rate in white dwarf phase: $\dot{M}_{WD} \sim 10^{-6}\, M_\odot\,\mathrm{y}^{-1}$.
- Velocity of fast wind: $v_{WD} \approx 2000\,\mathrm{km\,s^{-1}}$.

(a) Assume that the fast wind switches on instantaneously at the end of the red giant phase at $t = 0$. Show that the mass of swept up material is given by:

$$\frac{dM_s}{dt} = A\,(v_s - v_{RG})$$

where $v_s = dR_s/dt$ is the velocity of the shell and

$$A = \frac{\dot{M}_{RG}}{v_{RG}} = 6.3\times 10^{13}\,\mathrm{kg\,m^{-1}} \left(\frac{\dot{M}_{RG}}{10^{-5} M_\odot\,\mathrm{y}^{-1}}\right) \left(\frac{v_{RG}}{10\,\mathrm{km\,s^{-1}}}\right)^{-1}$$

In the case where the velocity of expansion of the shell is much greater than the red giant wind velocity, show that the mass of the shell of swept-up red giant material is given by

$$M_s = AR_s$$

(b) Assuming that the evolution of the interior of the mass-loss bubble is adiabatic, show that the radius of the swept up shell and the pressure are given by

$$\begin{aligned}R_s(t) &= \left(\frac{2L_w}{3A}\right)^{1/3} t \\ P(t) &= \frac{A}{4\pi}\,t^{-2}\end{aligned}$$

where L_w is the mechanical luminosity of the fast wind.

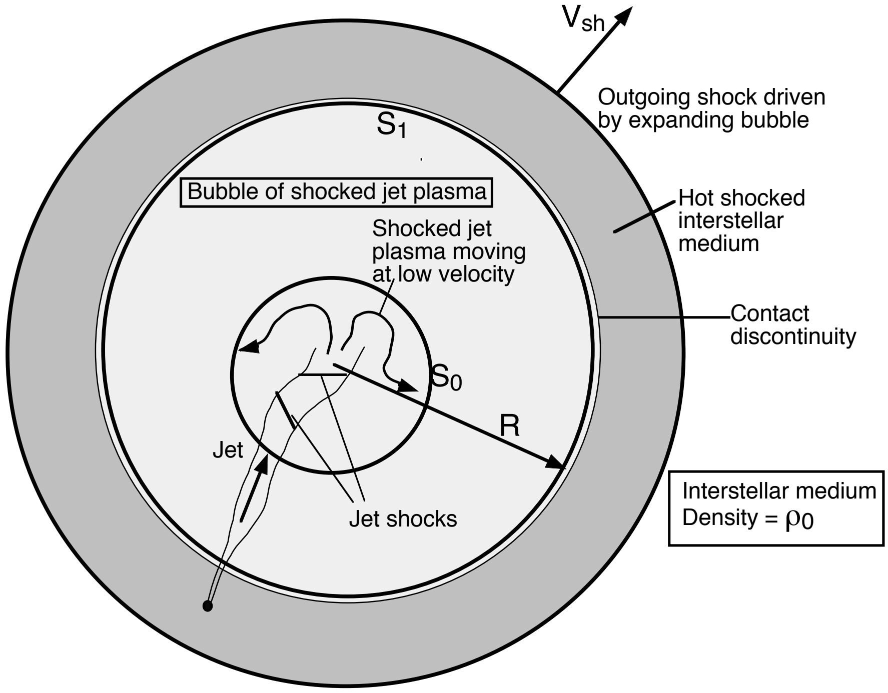

Fig. 3.— Schematic of a jet-inflated bubble.

(c) Estimate the radius and mass of the planetary nebula shell for typical parameters. (The lifetime of the planetary nebula phase is generally taken to be a few thousand years.)

37. *Jet-driven bubble.*

Background to this question: (See figure 3.) In the section of this course dealing with wind-driven bubbles, we considered the case where the ambient medium is so radiative that it formed a thin shell around the expanding bubble. The opposite limit occurs when the background medium is so hot and/or tenuous that the gas outside the bubble can be considered to be adiabatic. This is often the case for a jet-driven bubble expanding into the hot interstellar medium of the (elliptical) host galaxy. We assume that shock waves in the jet convert all of the jet's directed kinetic energy into internal energy of the relativistic plasma and that the pressure of this plasma drives the expansion of the bubble into the ambient interstellar medium. The expansion of the bubble is initially supersonic and, as a consequence, the bubble is preceded by a shock wave which produces a "sandwich" of hot, shocked interstellar medium in between the bubble and the external interstellar medium.

The following components of this question are designed to guide you to relationships describing the radius and pressure inside the bubble as a function of time.

(a) Assume that the bubble is approximately spherical, with radius R and consider control surfaces S_0 and S_1 as shown in the above figure. Further assume that the bubble is expanding with a velocity U_i with $|U_i| = dR/dt$.

Show that the total energy content, $E_{\rm tot}$ of the bubble is given by:

$$\frac{\partial E_{\rm tot}}{\partial t} + \int_{S_1} p\, U_i n_i \, dS = F_{\rm E}$$

where the energy flux of the jet is given in terms of its density, $\rho_{\rm jet}$, velocity, $v_{\rm jet}$ and specific enthalpy, $h_{\rm jet}$, by:

$$F_{\rm E} = \int_{S_0} \rho_{\rm jet} \left(\frac{1}{2} v_{\rm jet}^2 + h_{\rm jet} \right) v_{\rm jet}\, dS$$

where it is assumed that, on S_0, the jet has slowed enough that its velocity can be considered as non-relativistic.

(b) Assuming that the energy density of the material in the bubble is dominated by internal energy of relativistic particles show that the pressure, p, and radius are related by:

$$R^3 \frac{dp}{dt} + 4pR^2 \frac{dR}{dt} = \frac{F_{\rm E}}{4\pi}$$

(c) Assume that the external interstellar medium is a non-relativistic, monatomic gas with density ρ_0. Further assume that the shocked external medium expands at the same velocity as the bubble. Hence, show that the velocity, $v_{\rm sh}$, of the strong shock external to the expanding bubble is given by:

$$v_{\rm ext} \approx \frac{4}{3} \frac{dR}{dt}$$

Show also that the pressure of the shocked interstellar medium is given by:

$$p_{\rm sh} \approx \frac{4}{3} \rho_0 \left(\frac{dR}{dt} \right)^2$$

(d) Assume that the shocked external medium and the bubble are in pressure equilibrium, i.e. the surface of the bubble is a contact discontinuity. Hence show that the equations describing the evolution of the bubble are

$$\begin{aligned} R^3 \frac{dp}{dt} + 4pR^2 \frac{dR}{dt} &= \frac{F_{\rm E}}{4\pi} \\ p &= \frac{4}{3} \rho_0 \left(\frac{dR}{dt} \right)^2 \end{aligned}$$

Show that a solution of these equations is:

$$\begin{aligned} R(t) &= At^{3/5} \\ p(t) &= Bt^{-4/5} \end{aligned}$$

where

$$\begin{aligned} A &= \left[\frac{5^3}{3 \times 2^7 \pi} \frac{F_E}{\rho_0} \right]^{1/5} \\ B &= \frac{12}{25} \rho_0 A^2 \end{aligned}$$

(e) The jet in the radio galaxy M87 is inflating a bubble in the interstellar medium. Assume that the bubble is approximately spherical and the energy flux through the jet is 10^{37} W and that the number density of the interstellar medium is 10^4 m^{-3}. Determine the radius of the bubble, in kiloparsecs after a time of 10^6 yrs.

38. *Magnetic field of a neutron star.*

(a) A star with an initial radius R_* and surface magnetic field B_* collapses to a neutron star with a radius $R_{\rm NS}$. Show that the magnetic field on the surface of the neutron star is given by

$$B_{\rm NS} \approx B_* \left(\frac{R_*}{R_{\rm NS}}\right)^2 \tag{6}$$

(b) Estimate the magnetic field of a neutron star which collapses from a star ten times the size of the sun with a magnetic field $B \sim 1$ G to a radius of 10 km. The solar radius is $R_. = 6.96 \times 10^{10}$ cm.

39. *Magnetic field in a jet.*
A jet is travelling in the z-direction; the plasma in the jet has speed $V(z)$ and radius $R(z)$. Assume that the jet plasma expands homologously, that is, the relative expansion of each element is independent of radial coordinate.

(a) By considering the equation for relative motion of streamlines, and assuming that the jet flow is steady, show that the relative distance between two fluid elements in the direction is given approximately by

$$\delta z \propto V(z) \tag{7}$$

.

(b) By considering the magnetic flux through surface elements perpendicular to each of the coordinate directions, show that the magnetic field components behave in the following way:

$$B_z \quad \propto \quad \frac{1}{R(z)^2} \tag{8}$$

$$(B_r, B_\phi) \quad \propto \quad \frac{1}{V(z)R(z)} \tag{9}$$

40. *Poynting Flux.*

Show that in a highly conducting gas the Poynting flux is given by:

$$\mathbf{S} = \frac{B^2}{4\pi}\left[\mathbf{V} - (\mathbf{V}\cdot\hat{\mathbf{B}})\hat{\mathbf{B}})\right] \tag{10}$$

that is, $B^2/4\pi$ times the component of velocity perpendicular to the magnetic field.

41. *Mass, momentum and energy flux.*
A completely ionised jet with a Mach number of 2 (relative to its internal sound speed) has a velocity of $20,000\ \mathrm{km\,s^{-1}}$ and an internal gas pressure of 10^{-11} dynes $\mathrm{cm^{-2}}$. The magnetic field is perpendicular to the jet and is in equipartition with the gas pressure. The jet pressure is dominated by thermal particles so that the ratio of specific heats is 5/3. The diameter of the jet is 20 pc.

Determine (a) the jet temperature in °K, (b) the number density in $\mathrm{cm^{-3}}$, (c) the mass flux in solar masses per year, (d) the momentum flux (in dynes) and (e) the energy flux (in $\mathrm{ergs\,s^{-1}}$).

42. *Equation of state for relativistic gas*

For a relativistic gas, let p be the pressure, n the baryon density, σ the entropy per baryon. Show that

$$p = K(\sigma)n^{4/3}$$

43. *Energy flux and mass flux in relativistic jets and winds*

(a) For a relativistic gas, show that the equations for the conservation of mass and energy can be written:

$$\begin{aligned}\frac{d}{dt}\int_V \rho_{\rm lab}\, c^2\, dV + \int_S \Gamma \rho c^2 v^i n_i\, dS &= 0 \\ \frac{d}{dt}\int_V \left[\Gamma^2(\rho c^2 + \epsilon + p) - p\right] dV + \int_S \Gamma^2 \left[\rho c^2 + (\epsilon + p)\right] v^i n_i\, dS &= 0\end{aligned}$$

where ρ is the rest-mass density and S is the surface bounding the volume V.

(b) Now consider the case where an almost stationary region (V) is being fed by relativistic gas. Examples include the lobe of a radio galaxy, or the region of a bubble fed by a shocked relativistic wind. We can approximate the dynamical variables in region V by the non-relativistic limit, but the integrals over S may involve relativistic velocities.

Show, that in the slowly moving region, V:

$$\Gamma^2(\rho c^2 + \epsilon + p) - p \doteq \rho_{\rm lab}\, c^2 + \frac{1}{2}\rho_{\rm lab} v^2 + \epsilon$$

Hence show, using these approximations, that the evolution of kinetic plus internal energy in V, may be approximated by:

$$\frac{d}{dt}\int_V \left[\frac{1}{2}\rho_{\rm lab} v^2 + \epsilon\right] dV + \int_S \left[\Gamma^2(\epsilon + p) + \Gamma(\Gamma - 1)\rho c^2\right] v^i n_i\, dS = 0$$

The second term identifies the appropriate energy flux in these circumstances.

(c) Show that in the non-relativistic limit, the above energy flux becomes:

$$F_E = \int_S \left[\frac{1}{2}\rho v^2 + (\epsilon + p)\right] v^i n_i\, dS$$

(d) Consider a relativistic flow in which the flow variables may be taken to constant over the cross-section of the flow, which has area A (e.g., a relativistic jet or a relativistic wind). Show that the energy flux may be approximately written as

$$F_E = \Gamma^2(\epsilon + p)c\beta A + (\Gamma - 1)\dot{M}c^2$$

where A is the cross-sectional area of the flow and $\dot{M}$ is the mass flux.

44. Analysis of Cygnus A

The idea of this question is to estimate the contributions to the jet energy flux in the FR2 radiogalaxy Cygnus A.

Given the image on p.2 of the chapter on *Relativistic Applications*, and:

- The pressure of component J6 is approximately $10^{-7}\ {\rm N\,m^{-2}}$ (estimated from radio observations)
- The redshift to Cygnus A is $z = 0.056075$
- The Hubble constant is $70\ {\rm km\,s^{-1}\,Mpc^{-1}}$
- The Lorentz factor of the jet is approximately what we derived in lectures for the large scale jet, namely $\Gamma = 2.2$

- The accretion rate onto the black hole in Cygnus A is estimated at $(1.5-6)\times 10^{-2} M_\odot\,\mathrm{y}^{-1}$ (Tadhunter et al. 2003, MNRAS, 342, 861). Hence assume a fiducial $\dot{M}$ in the jet of $10^{-2} M_\odot\,\mathrm{y}^{-1}$.

Estimate the contributions to the energy flux (kinetic power) of the jet.

45. *Jet knots*

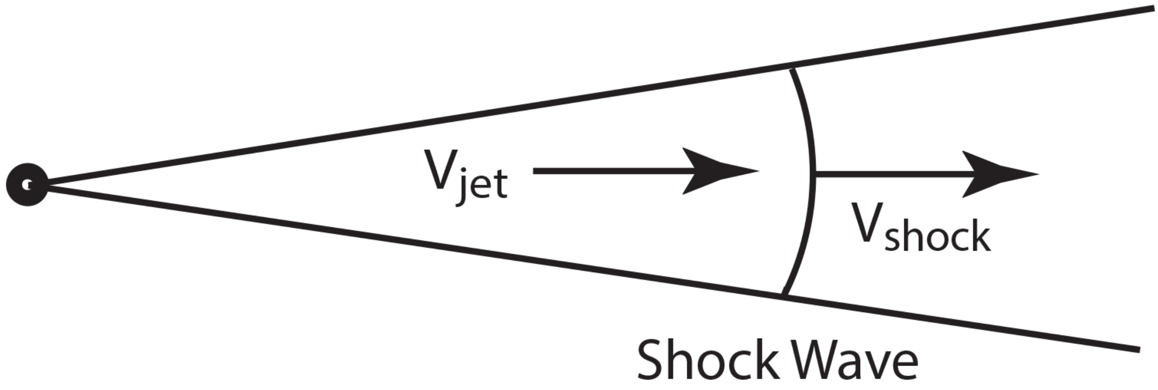

Fig. 4.— Schematic indication of a shock in a relativistic jet.

Figure 4 represents a shock propagating in a relativistic jet. One idea for the production of such shocks is that they are the result of faster moving gas catching up with slower moving gas. When astronomers measure the velocities of knots in jets, they are actually measuring the pattern speed of a relativistic shock, which is in general different from the velocity of the jet. There are arguments based on the radiation from such shocks to indicate that they may be weak.

The idea of this exercise is to see what the difference between the observed and measured velocities can be.

(i) Assume that the shock is weak, and that the jet is described by an ultra-relativistic equation of state, show that the jet velocity ($\beta_{\rm jet}$) and Lorentz factor ($\Gamma_{\rm jet}$) are related to the observed shock speed ($\beta_{\rm sh}$), and Lorentz factor ($\Gamma_{\rm sh}$) by the following equations:

$$\beta_{\rm jet} = \frac{\beta_{\rm sh} + 1/\sqrt{3}}{1+\beta_{\rm sh}/\sqrt{3}}$$

$$\text{and} \qquad \Gamma_{\rm jet} = \sqrt{\frac{3}{2}}\left(1+\frac{\beta_{\rm sh}}{\sqrt{3}}\right)\Gamma_{\rm sh}$$

(ii) Determine $\beta_{\rm jet}$ and $\Gamma_{\rm jet}$ for an observed shock velocity of 0.5 c.

(iii) Show that, for shock velocities approaching the speed of light,

$$\Gamma_{\rm jet} \approx 1.93\,\Gamma_{\rm sh}$$