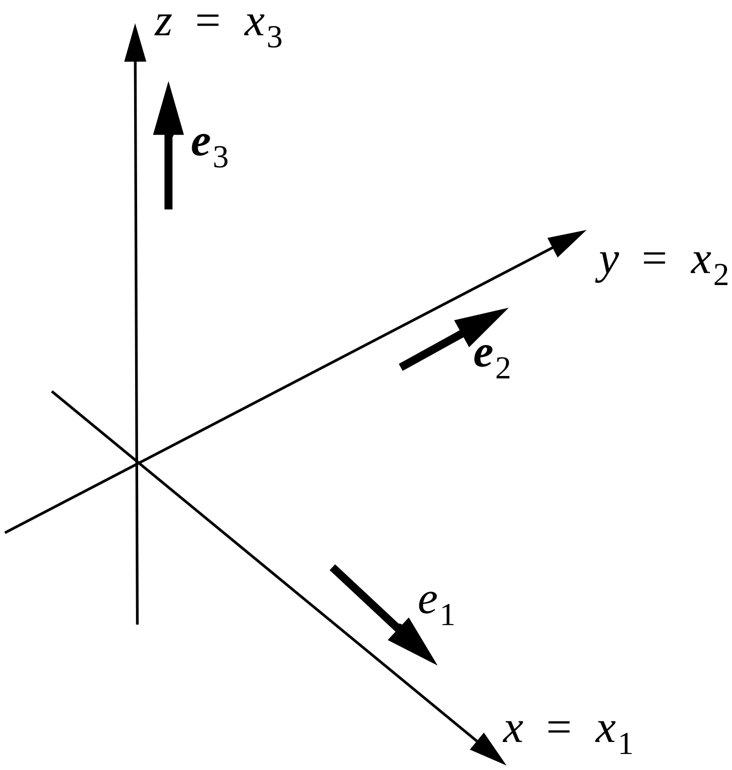
Cartesian Tensors

Reference: Jeffreys *Cartesian Tensors*

1 Coordinates and Vectors



Coordinates x_i, $i = 1, 2, 3$

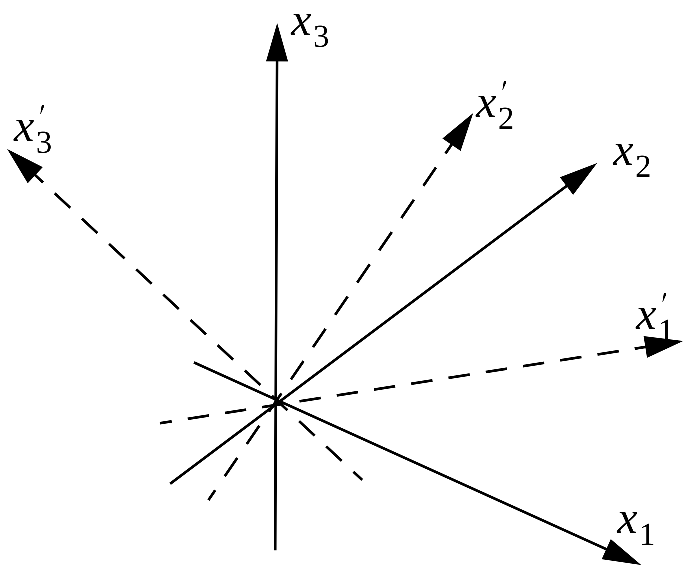
Unit vectors: $\boldsymbol{e}_i$, $i = 1, 2, 3$

General vector (formal definition to follow) denoted by components e.g. $\boldsymbol{u} = u_i$

Summation convention (Einstein) repeated index means summation:

$$u_i v_i = \sum_{i=1}^{3} u_i v_i \qquad u_{ii} = \sum_{i=1}^{3} u_{ii} \tag{1}$$

2 Orthogonal Transformations of Coordinates



$$x_i' = a_{ij} x_j \tag{2}$$

$$a_{ij} = \text{Transformation Matrix} \tag{3}$$

Position vector

$$\begin{aligned} \boldsymbol{r} &= x_i \boldsymbol{e}_i = x_j' \boldsymbol{e}_j' \\ \Rightarrow a_{ji} x_i \boldsymbol{e}_j' &= x_i \boldsymbol{e}_i \\ x_i (a_{ji} \boldsymbol{e}_j') &= x_i \boldsymbol{e}_i \\ \Rightarrow \boldsymbol{e}_i &= a_{ji} \boldsymbol{e}_j' \end{aligned} \tag{4}$$

i.e. the transformation of coordinates from the unprimed to the primed frame implies the reverse transformation from the primed to the unprimed frame for the unit vectors.

Kronecker Delta

$$\begin{aligned} \delta_{ij} &= 1 \text{ if } i = j \\ &= 0 \text{ otherwise} \end{aligned} \tag{5}$$

2.1 Orthonormal Condition:

Now impose the condition that the primed reference is orthonormal

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \quad \text{and} \quad e'_i \cdot e'_j = \delta_{ij} \tag{6}$$

Use the transformation

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= a_{ki}\boldsymbol{e}'_k \cdot a_{lj}\boldsymbol{e}'_l \\ &= a_{ki}a_{lj}\boldsymbol{e}'_k \cdot \boldsymbol{e}'_l \\ &= a_{ki}a_{lj}\delta_{kl} \\ &= a_{ki}a_{kj} \end{aligned} \tag{7}$$

NB the last operation is an example of the substitution property of the Kronecker Delta.

Since $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, then the orthonormal condition on a_{ij} is

$$a_{ki}a_{kj} = \delta_{ij} \tag{8}$$

In matrix notation:

$$\boldsymbol{a}^T\boldsymbol{a} = \boldsymbol{I} \tag{9}$$

Also have

$$a_{ik}a_{jk} = \boldsymbol{a}\boldsymbol{a}^T = \delta_{ij} \tag{10}$$

2.2 Reverse transformations

$$x_i' = a_{ij}x_j \Rightarrow a_{ik}x_i' = a_{ik}a_{ij}x_j = \delta_{kj}x_j = x_k$$
$$\therefore x_k = a_{ik}x_i' \Rightarrow x_i = a_{ji}x_j' \tag{11}$$

i.e. the reverse transformation is simply given by the transpose.

Similarly,

$$\boldsymbol{e}_i' = a_{ij}\boldsymbol{e}_j \tag{12}$$

2.3 Interpretation of a_{ij}

Since

$$\boldsymbol{e}_i' = a_{ij}\boldsymbol{e}_j \tag{13}$$

then the a_{ij} are the components of e_i' wrt the unit vectors in the unprimed system.

3 Scalars, Vectors & Tensors

3.1 Scalar (f):

$$f(x'_i) = f(x_i) \tag{14}$$

Example of a scalar is $f = r^2 = x_i x_i$. Examples from fluid dynamics are the density and temperature.

3.2 Vector (u):

Prototype vector: x_i

General transformation law:

$$x_i' = a_{ij}x_j \Rightarrow u_i' = a_{ij}u_j \tag{15}$$

3.2.1 Gradient operator

Suppose that f is a scalar. Gradient defined by

$$(\text{grad } f)_i = (\nabla f)_i = \frac{\partial f}{\partial x_i} \tag{16}$$

Need to show this is a vector by its transformation properties.

$$\frac{\partial f}{\partial x_i'} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \tag{17}$$

Since,

$$x_j = a_{kj}x_k' \tag{18}$$

then

$$\frac{\partial x_j}{\partial x_i'} = a_{kj}\delta_{ki} = a_{ij}$$
$$\text{and} \quad \frac{\partial f}{\partial x_i'} = a_{ij}\frac{\partial f}{\partial x_j} \tag{19}$$

Hence the gradient operator satisfies our definition of a vector.

3.2.2 Scalar Product

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{20}$$

is the scalar product of the vectors u_i and v_i.

Exercise:

Show that $\boldsymbol{u} \cdot \boldsymbol{v}$ is a scalar.

3.3 Tensor

Prototype second rank tensor $x_i x_j$

General definition:

$$T'_{ij} = a_{ik} a_{jl} T_{kl} \qquad (21)$$

Exercise:

Show that $u_i v_j$ is a second rank tensor if u_i and v_j are vectors.

Exercise:

$$u_{i,\,j} = \frac{\partial u_i}{\partial x_j} \qquad (22)$$

is a second rank tensor. (Introduces the comma notation for partial derivatives.) In dyadic form this is written as grad $\boldsymbol{u}$ or $\nabla \boldsymbol{u}$.

3.3.1 Divergence

Exercise:

Show that the quantity

$$\nabla \cdot v = \mathrm{div}\ v = \frac{\partial v_i}{\partial x_i} \qquad (23)$$

is a scalar.

4 Products and Contractions of Tensors

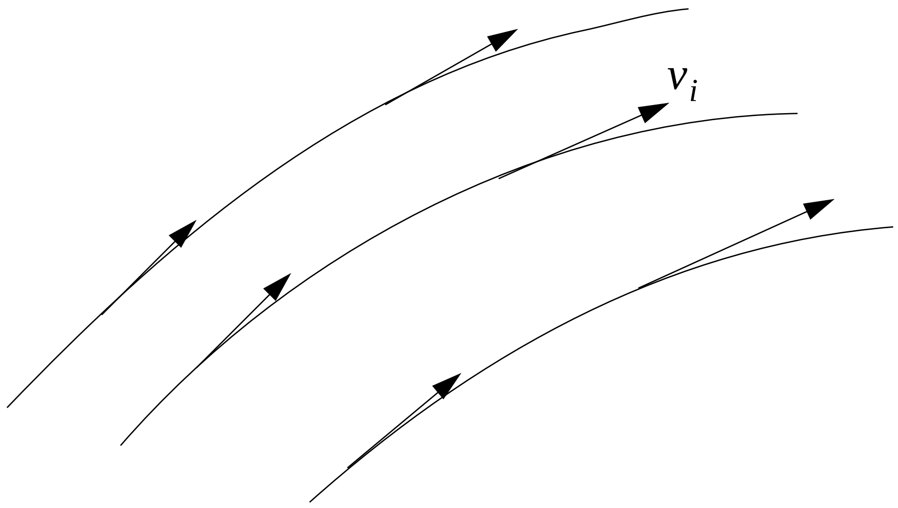
It is easy to form higher order tensors by multiplication of lower rank tensors, e.g. $T_{ijk} = T_{ij} u_k$ is a third rank tensor if T_{ij} is a second rank tensor and u_k is a vector (first rank tensor). It is straightforward to show that T_{ijk}has the relevant transformation properties.

Similarly, if T_{ijk} is a third rank tensor, then T_{ijj} is a vector. Again the relevant tr4ansformation properties are easy to prove.

5 Differentiation following the motion

This involves a common operator occurring in fluid dynamics. Suppose the coordinates of an element of fluid are given as a function of time by

$$x_i = x_i(t) \tag{24}$$



The velocities of elements of fluid at all spatial locations within a given region constitute a vector field, i.e. $v_i = v_i(x_j, t)$

The derivative of a function, $f(x_i, t)$ along the trajectoryof a parcel fluid is given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = \frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} \tag{25}$$

Derivative of velocity

If we follow the trajectory of an element of fluid, then on a particular trajectory $x_i = x_i(t)$. The acceleration of an element is then given by:

$$f_i = \frac{dv_i}{dt} = \frac{d}{dt}v_i(x_j(t), t) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} \qquad (26)$$

Exercise: Show that f_i is a vector.

6 The permutation tensor ε_{ijk}

$$\begin{aligned} \varepsilon_{ijk} &= 0 \quad \text{if any of } i, j, k \text{ are equal} \\ &= 1 \quad \text{if } i, j, k \text{ unequal and in cyclic order} \\ &= -1 \quad \text{if } i, j, k \text{ unequal and not in cyclic order} \end{aligned} \qquad (27)$$

e.g.

$$\varepsilon_{112} = 0 \qquad \varepsilon_{123} = 1 \qquad \varepsilon_{321} = -1 \qquad (28)$$

Is ε_{ijk} a tensor?

In order to show this we have to demonstrate that ε_{ijk}, when defined the same way in each coordinate system has the correct transformation properties.

Define

$$
\begin{aligned}
\varepsilon'_{ijk} &= \varepsilon_{lmn} a_{il} a_{jm} a_{kn} \\
&= \varepsilon_{123} a_{i1} a_{j2} a_{k3} + \varepsilon_{312} a_{i3} a_{j1} a_{k2} + \varepsilon_{231} a_{i2} a_{j1} a_{k2} \\
&\quad + \varepsilon_{213} a_{i2} a_{j1} a_{k3} + \varepsilon_{321} a_{i3} a_{j2} a_{k1} + \varepsilon_{132} a_{i1} a_{j3} a_{k2} \\
&= a_{i1}(a_{j2} a_{k3} - a_{j3} a_{k2}) - a_{i2}(a_{j1} a_{k3} - a_{j3} a_{k2}) \\
&\qquad + a_{i3}(a_{j1} a_{k2} - a_{j2} a_{k1}) \\
&= \begin{vmatrix} a_{i1} & a_{i2} & a_{i3} \\ a_{j1} & a_{j2} & a_{j3} \\ a_{k1} & a_{k2} & a_{k3} \end{vmatrix}
\end{aligned}
\qquad (29)
$$

In view of the interpretation of the a_{ij}, the rows of this determinant represent the components of the primed unit vectors in the unprimed system. Hence:

$$\varepsilon'_{ijk} = \boldsymbol{e}'_i \cdot \boldsymbol{e}'_j \times \boldsymbol{e}'_k \qquad (30)$$

This is zero if any 2 of i, j, k are equal, is +1 for a cyclic permutation of unequal indices and -1 for a non-cyclic permutation of unequal indices. This is just the definition of ε'_{ijk}. Thus ε_{ijk} transforms as a tensor.

6.1 Uses of the permutation tensor

6.1.1 Cross product

Define

$$c_i = \varepsilon_{ijk} a_j b_k \qquad (31)$$

then

$$\begin{aligned} c_1 &= \varepsilon_{123}a_2b_3 + \varepsilon_{132}a_3b_2 = a_2b_3 - a_3b_2 \\ c_2 &= \varepsilon_{231}a_3b_1 + \varepsilon_{213}a_1b_3 = a_3b_1 - a_1b_3 \\ c_3 &= \varepsilon_{312}a_1b_2 + \varepsilon_{321}a_2b_1 = a_1b_2 - a_2b_1 \end{aligned} \tag{32}$$

These are the components of $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$.

6.1.2 Triple Product

In dyadic notation the triple product of three vectors is:

$$t = \boldsymbol{u} \cdot \boldsymbol{v} \times \boldsymbol{w} \tag{33}$$

In tensor notation this is

$$t = u_i \varepsilon_{ijk} v_j w_k = \varepsilon_{ijk} u_i v_j w_k \tag{34}$$

6.1.3 Curl

$$(\text{curl } \boldsymbol{u})_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{35}$$

e.g.

$$(\text{curl } \boldsymbol{u})_1 = \varepsilon_{123} \frac{\partial u_3}{\partial x_2} + \varepsilon_{132} \frac{\partial u_2}{\partial x_3} = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \tag{36}$$

etc.

6.1.4 The tensor $\varepsilon_{iks}\varepsilon_{mps}$

Define

$$T_{ikmp} = \varepsilon_{iks}\varepsilon_{mps} \tag{37}$$

Properties:

- If $i = k$ or $m = p$ then $T_{ikmp = 0}$.

- If $i = m$ we only get a contribution from the terms $s \neq i$ and $k \neq i, s$. Consequently $k = p$. Thus $\varepsilon_{iks} = \pm 1$ and $\varepsilon_{mps} = \varepsilon_{iks} = \pm 1$ and the product $\varepsilon_{iks}\varepsilon_{iks} = (\pm 1)^2 = 1$.
- If $i = p$, similar argument tells us that we must have $s \neq i$ and $k = m \neq i$. Hence, $\varepsilon_{iks} = \pm 1, \quad \varepsilon_{mps} = \mp 1 \Rightarrow \varepsilon_{iks}\varepsilon_{mps} = -1$.

So,

$$\begin{aligned} i = m, k = p &\Rightarrow 1 \text{ unless } i = k \Rightarrow 0 \\ i = p, k = m &\Rightarrow -1 \text{ unless } i = k \Rightarrow 0 \end{aligned} \qquad (38)$$

These are the components of the tensor $\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$.

$$\therefore \varepsilon_{iks}\varepsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km} \qquad (39)$$

6.1.5 Application of $\varepsilon_{iks}\varepsilon_{mps}$

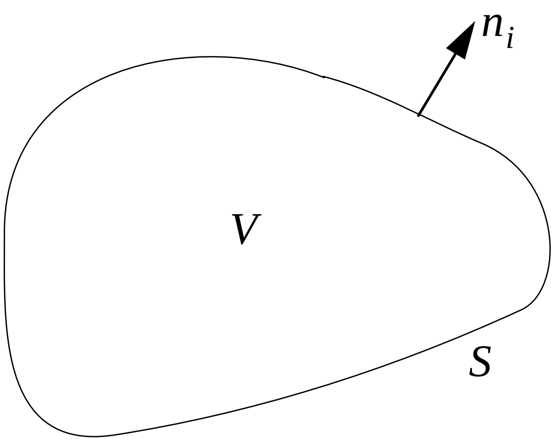
$$\begin{aligned} (\text{curl }(\boldsymbol{u} \times \boldsymbol{v}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}u_l v_m) = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_j}(u_l v_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\frac{\partial u_l}{\partial x_j}v_m + u_l\frac{\partial v_m}{\partial x_j}\right) \\ &= \frac{\partial u_i}{\partial x_m}v_m - v_i\frac{\partial u_j}{\partial x_j} + u_i\frac{\partial v_m}{\partial x_m} - u_j\frac{\partial v_i}{\partial x_j} \\ &= v_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial v_i}{\partial x_j} + u_i\frac{\partial v_j}{\partial x_j} - v_i\frac{\partial u_j}{\partial x_j} \\ &= (\boldsymbol{v} \cdot \nabla\boldsymbol{u} - \boldsymbol{u} \cdot \nabla\boldsymbol{v} + \boldsymbol{u}\nabla \cdot \boldsymbol{v} - \boldsymbol{v}\nabla \cdot \boldsymbol{u})_i \end{aligned} \qquad (40)$$

7 The Laplacean

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \frac{\partial^2\phi}{\partial x_i \partial x_i} \qquad (41)$$

8 Tensor Integrals

8.1 Green's Theorem



In dyadic form:

$$\int_V (\nabla \cdot \boldsymbol{u}) dV = \int_S (\boldsymbol{u} \cdot \boldsymbol{n}) dS \qquad (42)$$

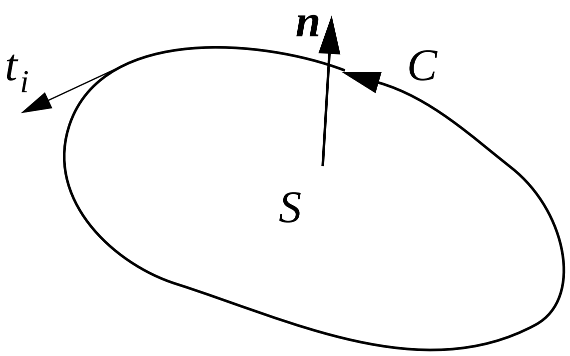
In tensor form:

$$\int_V \frac{\partial u_i}{\partial x_i} dV = \int_S u_i n_i dS = \text{Flux of } \boldsymbol{u} \text{ through } S \qquad (43)$$

Extend this to tensors:

$$\int_V \frac{\partial T_{ij}}{\partial x_j} dV = \int_S T_{ij} n_j dS = \text{Flux of } T_{ij} \text{ through } S \qquad (44)$$

8.2 Stoke's Theorem



In dyadic form:

$$\int_S (\text{curl } \boldsymbol{u}) \cdot \boldsymbol{n} dS = \int_C \boldsymbol{u} \cdot \boldsymbol{t} ds \qquad (45)$$

In tensor form:

$$\int_S \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} n_i dS = \int_C u_i t_i ds \qquad (46)$$