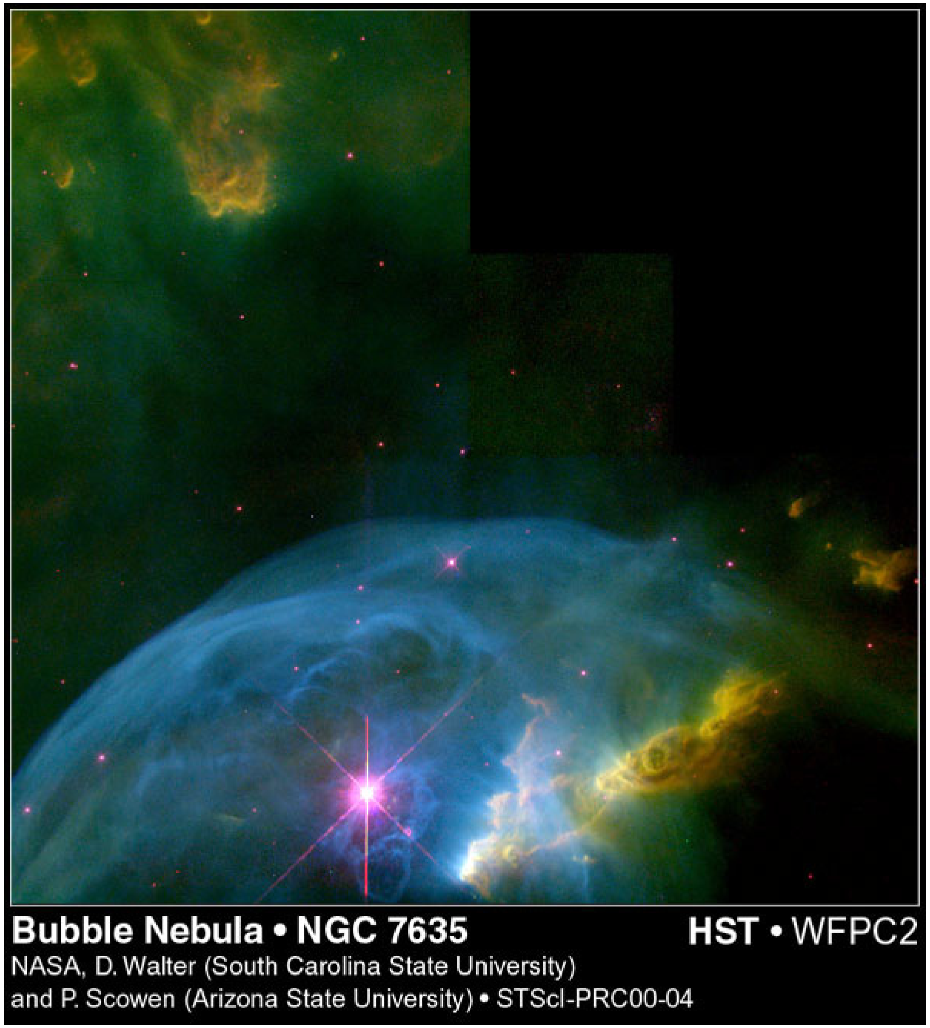
Bubbles

References

Castor, McCray and Weaver, 1975, ApJ, L107

Weaver et al., 1977, ApJ, 218, 377

1 Things that blow bubbles



Credits: NASA, Donald Walter (South Carolina State University), Paul Scowen and Brian Moore (Arizona State University)
Research Team: Donald Walter (South Carolina State University), Paul Scowen, Jeff Hester, Brian Moore (Arizona State University), Reggie Dufour, Patrick Hartigan and Brent Buckalew (Rice University).
Funding: STSCI, NASA MUSPIN and NASA URC.

Publicity release accompanying above picture

Astronomers, using the Wide Field Planetary Camera 2 on board NASA's Hubble Space Telescope in October and November 1997 and April 1999, imaged the Bubble Nebula (NGC 7635) with unprecedented clarity. For the first time, they are able to understand the geometry and dynamics of this very complicated system. Earlier pictures taken of the nebula with the Wide Field Planetary Camera 1 left many issues unanswered, as the data could not be fully calibrated for scientific use. In addition, those data never imaged the enigmatic inner structure presented here.

The remarkably spherical “Bubble” marks the boundary between an intense wind of particles from the star and the more quiescent interior of the nebula. The central star of the nebula is

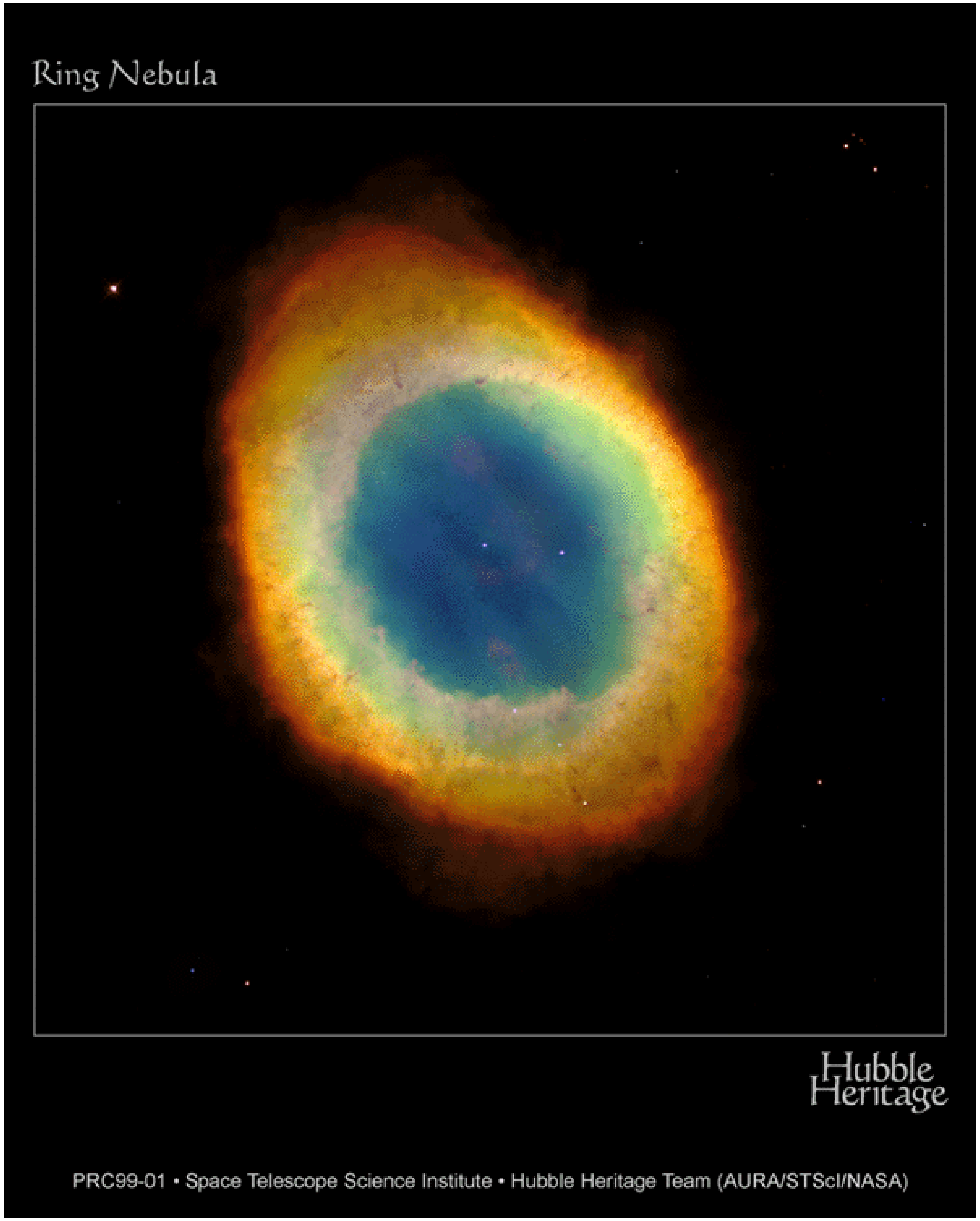
40 times more massive than the Sun and is responsible for a stellar wind moving at 2,000 kilometres per second (4 million miles per hour or 7 million kilometres per hour) which propels particles off the surface of the star. The bubble surface actually marks the leading edge of this wind's gust front, which is slowing as it ploughs into the denser surrounding material. The surface of the bubble is not uniform because as the shell expands outward it encounters regions of the cold gas, which are of different density and therefore arrest the expansion by differing amounts, resulting in the rippled appearance. It is this gradient of background material that the wind is encountering that places the central star off centre in the bubble. There is more material to the northeast of the nebula than to the

southwest, so that the wind progresses less in that direction, offsetting the central star from the geometric centre of the bubble. At a distance of 7,100 light-years from Earth, the Bubble Nebula is located in the constellation Cassiopeia and has a diameter of 6 light-years.

To the right of the central star is a ridge of much denser gas. The lower left portion of this ridge is closest to the star and so is brightest. It is experiencing the most intense ultraviolet radiation as well as the strong wind and is therefore being photoevaporated the fastest. The ridge forms a V-shape in the image, with two segments that are aligned at the brightest edge. The upper of these two segments is viewed quite obliquely as it trails off into the back of the nebula. The lower segment comes both toward the observer and off to the side. This lower ridge appears to lie within the sphere described by the bubble but is not actually "inside" the shocked region of gas. Instead it is being pushed up against the bubble like a hand being pushed against the outside of a party balloon. While the edge of the hand appears to be inside the balloon, it is not. As the bubble moves up but not through the ridge, bright blue arcs form where the supersonic wind strikes the ridge to form an apparent series of nested shock fronts.

The region between the star and ridge reveals several loops and arcs which have never been seen before. The high resolution capabilities of Hubble make it possible to examine these features in detail in a way that is not possible from the ground. The origin of this bubble-within-a-bubble” is unknown at this time. It may be due to a collision of two distinct winds. The stellar wind may be colliding with material streaming off the ridge as it is photo-evaporated by the star's radiation.

Located at the top of the picture are dense clumps or fingers of molecular gas which have not yet encountered the expanding shell. These structures are similar in form to the columns in the Eagle Nebula, except that they are not being eroded as energetically as they are in that nebula. As in the Eagle, the clumps are seen to emit light because they are being illuminated by the strong ultraviolet radiation from the central star, which travels much faster than the shell and has reached the outer knots long before the expanding rim will.

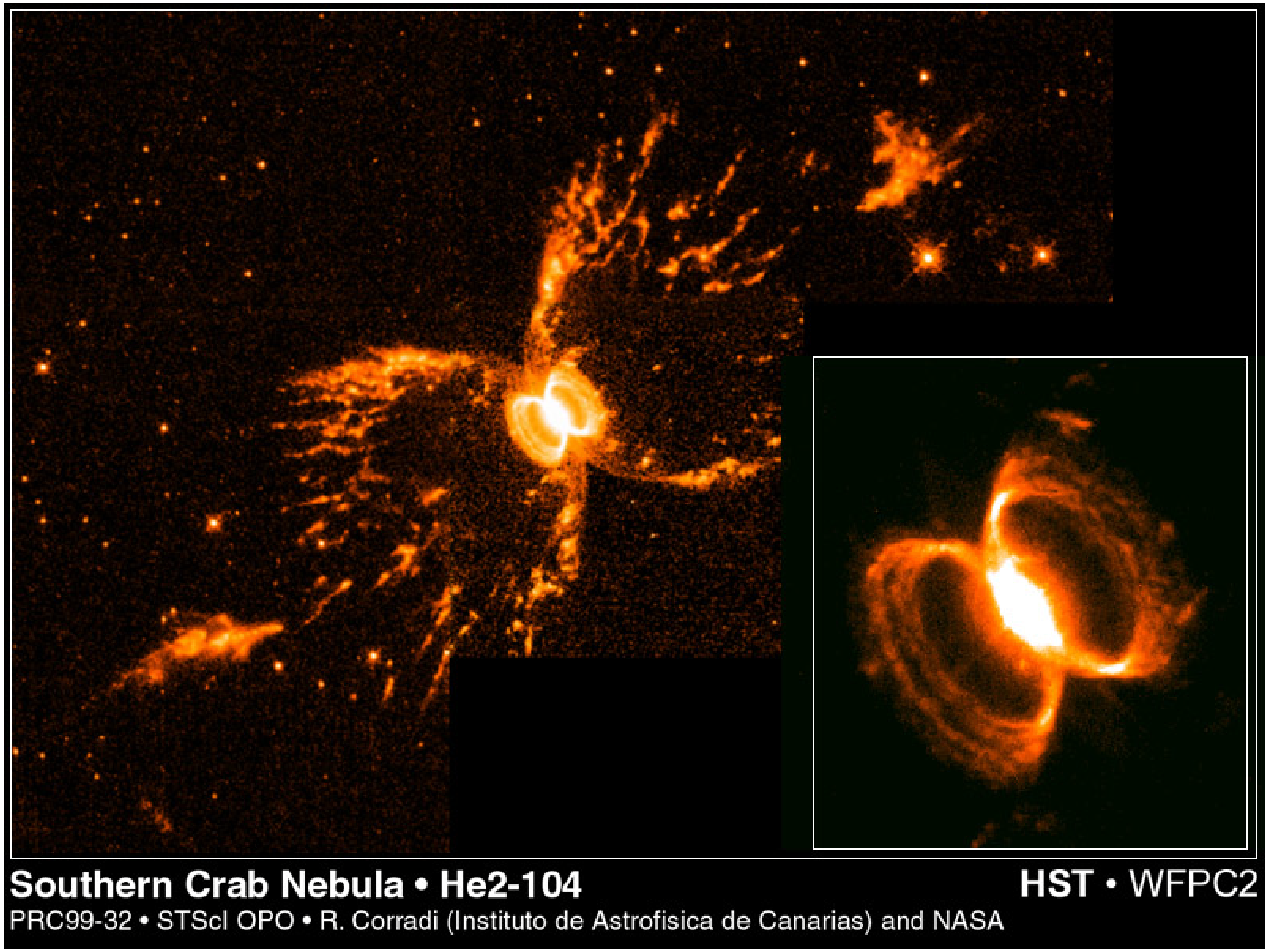


HST Image of the Ring Nebula

Publicity release on Ring Nebula

The NASA Hubble Space Telescope has captured the sharpest view yet of the most famous of all planetary nebulae: the Ring Nebula (M57). In this October 1998 image, the telescope has looked down a barrel of gas cast off by a dying star thousands of years ago. This photo reveals elongated dark clumps of material embedded in the gas at the edge of the nebula; the dying central star floating in a blue haze of hot gas. The nebula is about a light-year in diameter and is located some 2,000 light-years from Earth in the direction of the constellation Lyra.

The colours are approximately true colours. The colour image was assembled from three black-and-white photos taken through different colour filters with the Hubble telescope's Wide Field Planetary Camera 2. Blue isolates emission from very hot helium, which is located primarily close to the hot central star. Green represents ionized oxygen, which is located farther from the star. Red shows ionized nitrogen, which is radiated from the coolest gas, located farthest from the star. The gradations of colour illustrate how the gas glows because it is bathed in ultraviolet radiation from the remnant central star, whose surface temperature is a white-hot 216,000 degrees Fahrenheit (120,000 degrees Celsius).



HST image of the Southern Crab Nebula

The Southern Crab Nebula (NASA press release)

A tempestuous relationship between an unlikely pair of stars may have created an oddly shaped, gaseous nebula that resembles an hourglass nestled within an hourglass.

Images taken with Earth-based telescopes have shown the larger, hourglass-shaped nebula. But this picture, taken with NASA's Hubble Space Telescope, reveals a small, bright nebula embedded in the centre of the larger one (close-up of nebula in inset). Astronomers have dubbed the entire nebula the “Southern Crab Nebula” (He2-104), because, from ground-based telescopes, it looks like the body and legs of a crab. The nebula is several light-years long.

The possible creators of these shapes cannot be seen at all in this Wide Field and Planetary Camera 2 image. It's a pair of aging stars buried in the glow of the tiny, central nebula. One of them is a red giant, a bloated star that is exhausting its nuclear fuel and is shedding its outer layers in a powerful stellar wind. Its companion is a hot, white dwarf, a stellar zombie of a burned-out star. This odd duo of a red giant and a white dwarf is called a symbiotic system. The red giant is also a Mira Variable, a pulsating red giant, that is far away from its partner. It could take as much as 100 years for the two to orbit around each other.

Astronomers speculate that the interaction between these two stars may have sparked episodic outbursts of material, creating the gaseous bubbles that form the nebula. They interact by playing a celestial game of “catch”: as the red giant throws off its bulk in a powerful stellar wind, the white dwarf catches some of it. As a result, an accretion disk of material forms around the white dwarf and spirals onto its hot surface. Gas continues to build up on the surface until it sparks an eruption, blowing material into space.

This explosive event may have happened twice in the “Southern Crab.” Astronomers speculate that the hourglass-shaped nebulae represent two separate outbursts that occurred several thousand years apart. The jets of material in the lower left and upper right corners may have been accelerated by the white dwarf's accretion disk and probably are part of the older eruption.

The nebula, located in the Southern Hemisphere constellation of Centaurus, is a few thousand light-years from Earth.

This image, taken in May 1999, captures the glow of nitrogen gas energized by the white dwarf's intense radiation.

These results were presented at the “Asymmetrical Planetary Nebulae II: From Origins to Microstructures” conference, which took place at the Massachusetts Institute of Technology, August 3-6, 1999.

Credits: Romano Corradi, Instituto de Astrofisica de Canarias, Tenerife, Spain; Mario Livio, Space Telescope Science Institute, Baltimore, Md.; Ulisse Munari, Osservatorio Astronomico di Padova-Asiago, Italy; Hugo Schwarz, Nordic Optical Telescope, Canarias, Spain; and NASA

2 Parameters for O & B type stars, White dwarfs

Velocities of winds:

$$v \approx 1500 - 3000 \text{ km/s}$$

Mass-loss rate

$$\dot{M} \approx 10^{-6} M_{\text{sun}} \text{ y}^{-1}$$

2.1 Introduction to ram pressure

The flux of momentum is

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j \tag{1}$$

i-th component of force on a surface with unit normal n_j is

$$\Pi_{ij} n_j = p n_i + \rho v_i (v_j n_j) \tag{2}$$

The part $\rho v_i(v_j n_j)$ is called the "ram pressure" the component of the force on the surface due to the motion of the fluid.

If the surface is oriented perpendicular to the flow of gas, i.e. $n_j \parallel v_j$, then the force on the surface is $\rho v^2 n_i$. We often simply refer to ρv^2 as the "ram pressure".

Consider the ram-pressure (dynamic pressure) of a radially flowing wind:

$$\begin{aligned}\rho v^2 &= \frac{\dot{M}}{4\pi v r^2} v^2 \\ &= \frac{\dot{M} v}{4\pi r^2} \\ &= 5.2\times 10^{-12}\ \mathrm{N/m^2} \left(\frac{\dot{M}}{10^{-6} M_O\ \mathrm{y}^{-1}}\right)\left(\frac{v}{10^3\,\mathrm{km/s}}\right)\left(\frac{r}{\mathrm{pc}}\right)^{-2}\end{aligned} \qquad (3)$$

The pressure of the ISM into which the wind is blowing is

$$p = nkT = 1.4\times 10^{-13} \times \frac{n}{10^6\ \mathrm{m}^{-3}} \times \frac{T}{10^4\ \mathrm{K}}\ \mathrm{N\ m^{-2}} \qquad (4)$$

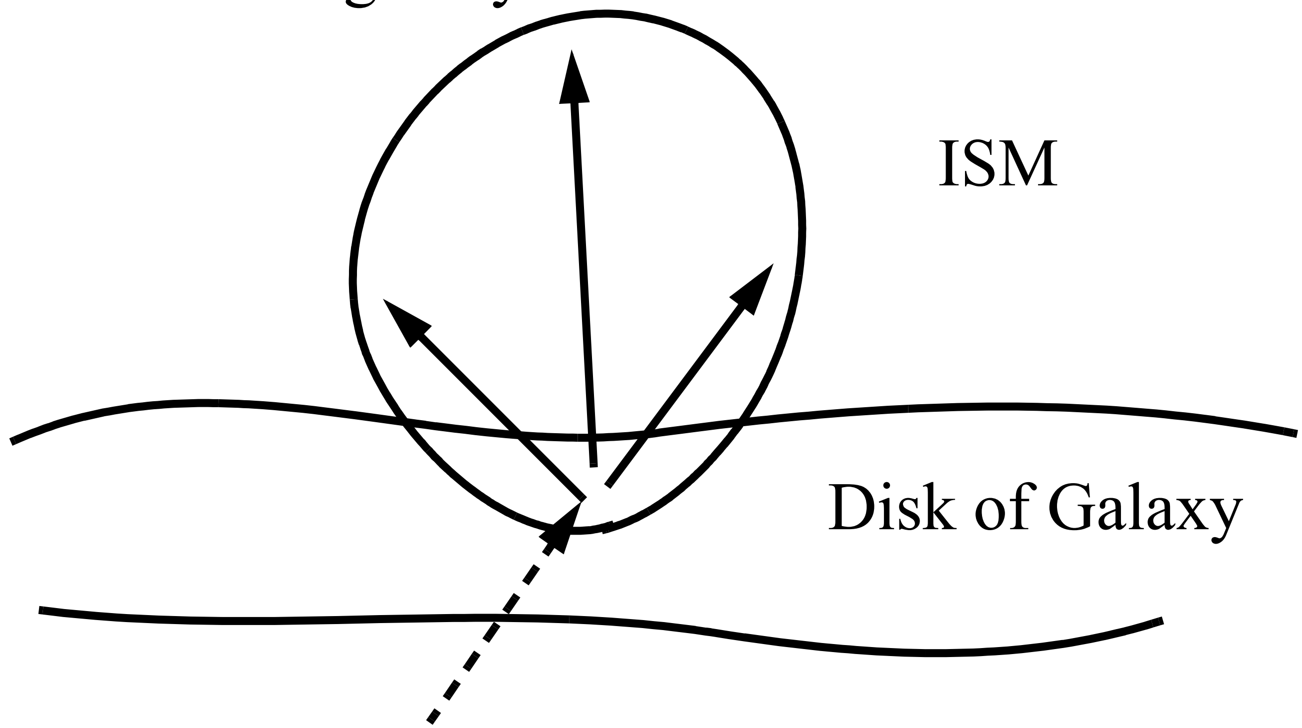
The ram pressure balances the ISM pressure when

$$r = 6.1\,\mathrm{pc}\left(\frac{\dot{M}}{10^{-6}M_{\mathrm{sun}}\ \mathrm{y}^{-1}}\right)^{1/2}\left(\frac{v}{10^{3}\mathrm{km/s}}\right)^{1/2} \times \left(\frac{n}{10^{6}\ \mathrm{m}^{-3}}\right)^{-1/2}\left(\frac{T_{ISM}}{10^{4}K}\right)^{-1/2} \quad (5)$$

This an estimate of the extent of the bubble blown by a star and it os clear that the extent of such a bubble can be considerable.

2.2 Bubbles blown by starbursts in other galaxies

In a starburst the combined effect of winds and supernovae from many stars is to blow a bubble in the local ISM



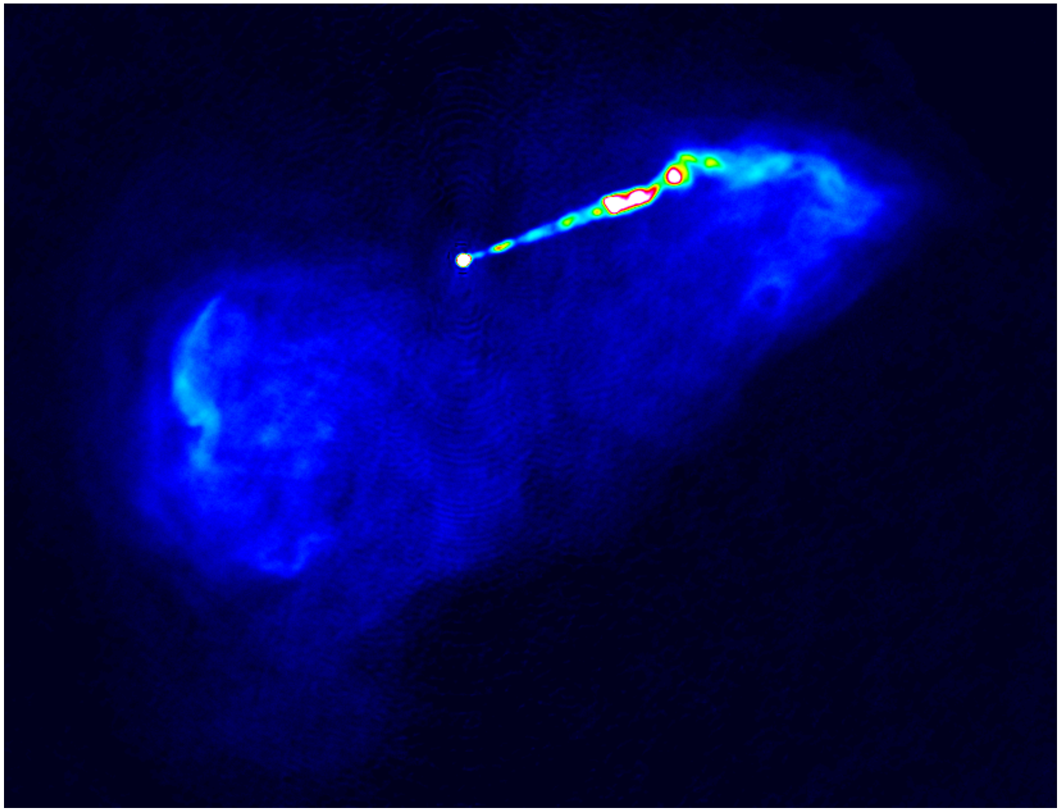
e.g. NGC 3079 (Veilleux et al., 1994, ApJ, **433**, 48 and references therein to general properties of superbubbles)

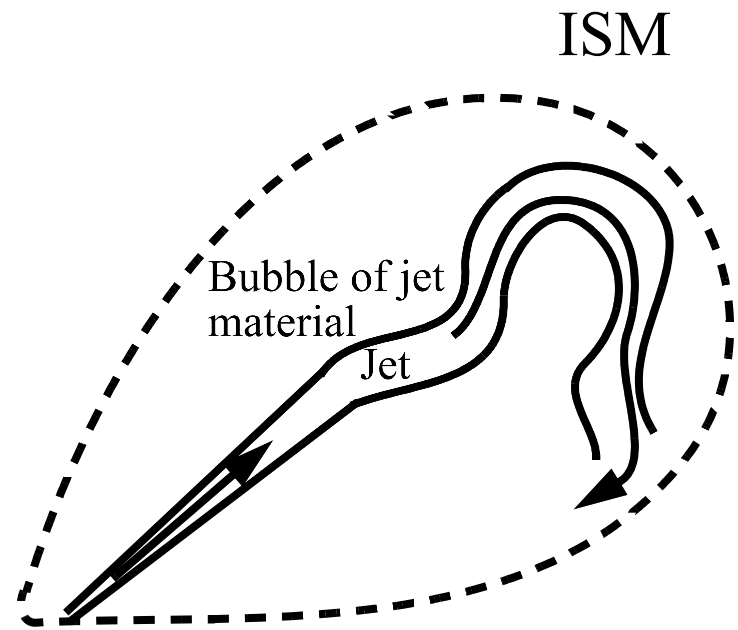
Velocity of superwind $v_w \approx 1000$ km/s

Mechanical Luminosity $L_w \approx 10^{34} - 10^{35} W$

Mass-loss $\dot{M}_w \sim 1 M_{\rm sun}\ {\rm y}^{-1}$

2.3 Bubbles blown by jets





Radio image of the M87 jet - Biretta et al.

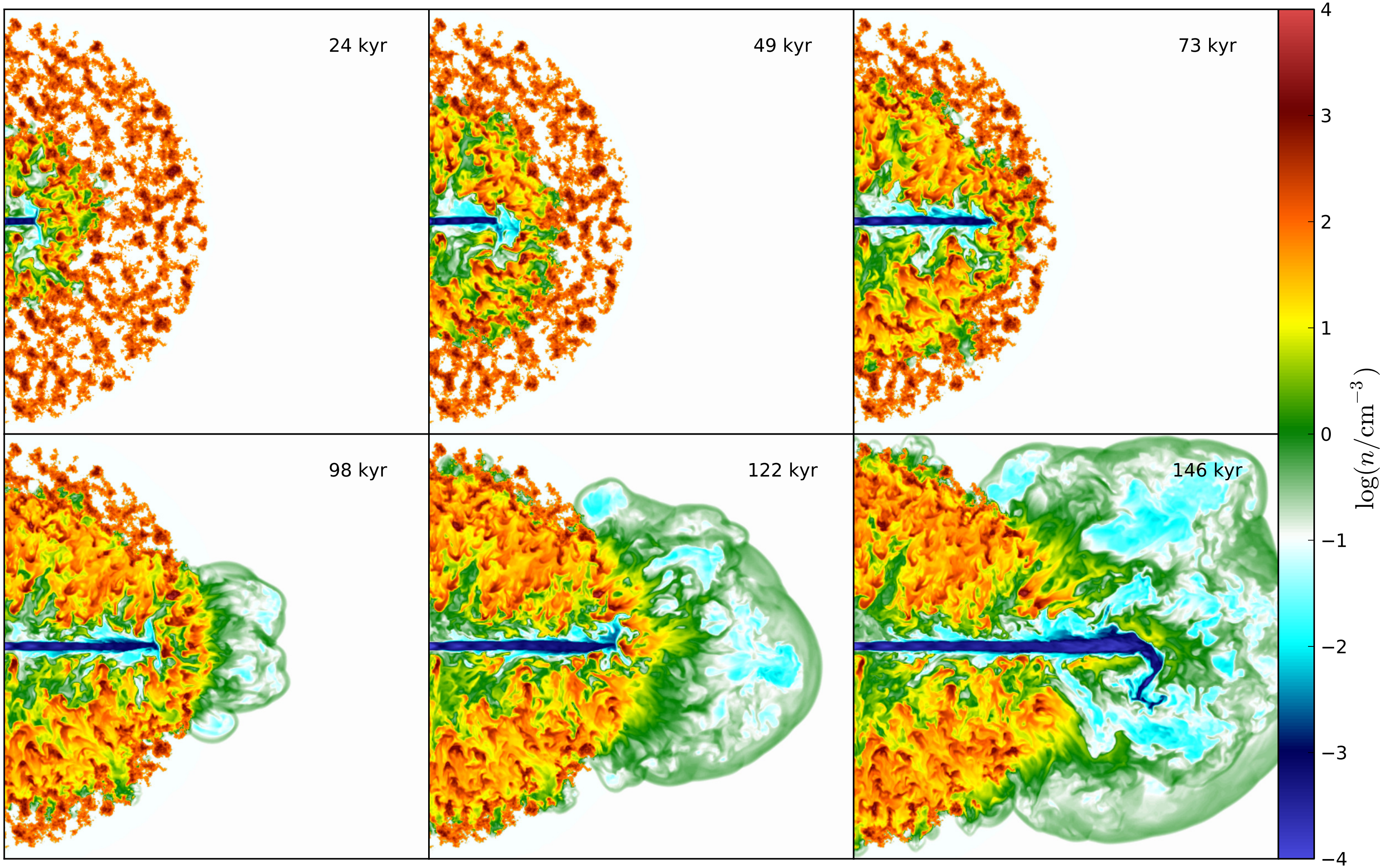
See Begelman and Cioffi, ApJ, **345**, L21.

Bubble blow by the jet in M87: Bicknell and Begelman, ApJ, **467**, 597

M87 parameters:

$$\begin{aligned} \text{Energy flux of jet} &\approx 10^{36} - 10^{37}\ \text{W} \\ \text{Pressure in ISM} &\approx 5 \times 10^{-11}\ \text{N m}^{-2} \\ \text{Ambient nunber density} &\approx 10^{5}\ \text{m}^{-3} \end{aligned} \quad (6)$$

2.4 Simulations of jet-blown bubbles



3 Effect of cooling

3.1 Adiabatic fluid equations

It is useful here to summarise the entire set of fluid equations:

Continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \qquad (7)$$

Momentum

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} \equiv \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial\phi}{\partial x_i} \qquad (8)$$

Internal energy

$$\varepsilon = \text{Internal energy density} = \frac{1}{(\gamma - 1)} p$$

$$\frac{d\varepsilon}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} \equiv \frac{\partial \varepsilon}{\partial t} + v_j \frac{\partial \varepsilon}{\partial x_j} + p\frac{\partial v_i}{\partial x_i} = 0 \qquad (9)$$

Total energy

$$h = \text{Specific enthalpy} = \frac{\varepsilon + p}{\rho}$$

$$\frac{\partial\left[\frac{1}{2}\rho v^2 + \varepsilon\right]}{\partial t} + \frac{\partial\left[\left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right)v_i\right]}{\partial x_j} = 0$$

$$\text{Total energy density} = \frac{1}{2}\rho v^2 + \varepsilon$$

$$\text{Energy flux} = F_{E,\,i} = \left(\frac{1}{2}\rho v^2 + \rho h + \rho\phi\right)v_i$$

Equation of state

$$p = (\gamma - 1)\varepsilon$$

Substitute into internal energy equation =>

$$p = K(s)\rho^{\gamma} \qquad (10)$$

3.2 Fluid equations in the case of cooling

If we limit consideration to the case where the cooling is optically thin so that we do not have to consider the transfer of energy through the gas due to radiation, then the energy equation is fairly simple.

Let j represent the emissivity per unit volume due to radiation, then, the equation for the internal energy becomes:

$$\frac{d\varepsilon}{dt} - \frac{\varepsilon + p}{\rho}\frac{d\rho}{dt} = -j \qquad (11)$$

3.3 Effect of cooling on the continuity and momentum equations

Cooling has no effect on the continuity equation since the emission of photons has no effect on the number density of ions and electrons.

The momentum carried off by the photons is small ($j/c \ \mathrm{N\ cm^{-3}\ s^{-1}}$) and the momentum balance is therefore unaffected.

3.4 Thermal emission processes

Thermal emission processes (Bound-free recombination, free-free emission, collisional excitation followed by photon emission) involve binary processes between electrons, ions and atoms. Hence the emission is proportional to the product of the number densities (electron n_e and proton n_p) and is written:

$$j = n_e n_p \Lambda(T)$$

where $\Lambda(T)$ is the *cooling function*.

Units

From the above equation, we have for the dimensions:

$$\begin{aligned} \mathrm{W\,m^{-3}} &= \mathrm{m^3} \times \mathrm{m^3} \times [\Lambda] \\ \Rightarrow [\Lambda] &= \mathrm{Wm^3} \end{aligned} \tag{12}$$

Energy equation with optically thin radiative cooling

The energy equation becomes in this case

$$\frac{d\varepsilon}{dt} - \frac{(\varepsilon + p)}{\rho}\frac{d\rho}{dt} = -n_e n_p \Lambda(T)$$

This equation defines a cooling time:

$$t_{\rm cool} = \frac{\varepsilon}{n_e n_p \Lambda(T)} = \frac{3}{2}\frac{(n_e + n_p)}{n_e n_p}\frac{kT}{\Lambda(T)}$$
$$= 6.6\times 10^{11}\ {\rm y}\ \frac{(n_e + n_p)}{n_e n_p}\frac{T_7}{\Lambda_{-35}} \qquad (13)$$

where $T_7 = T/10^7 K$ and $\Lambda_{-35} = \Lambda/10^{-35}{\rm Wm}^3$.

Note:

The cooling function is often published in cgs units of ergs $\rm cm^3\ s^{-1}$.

$$1\,{\rm ergs\ cm^3\ s^{-1}} = 10^{-13}\ {\rm W\ m^3} \qquad (14)$$

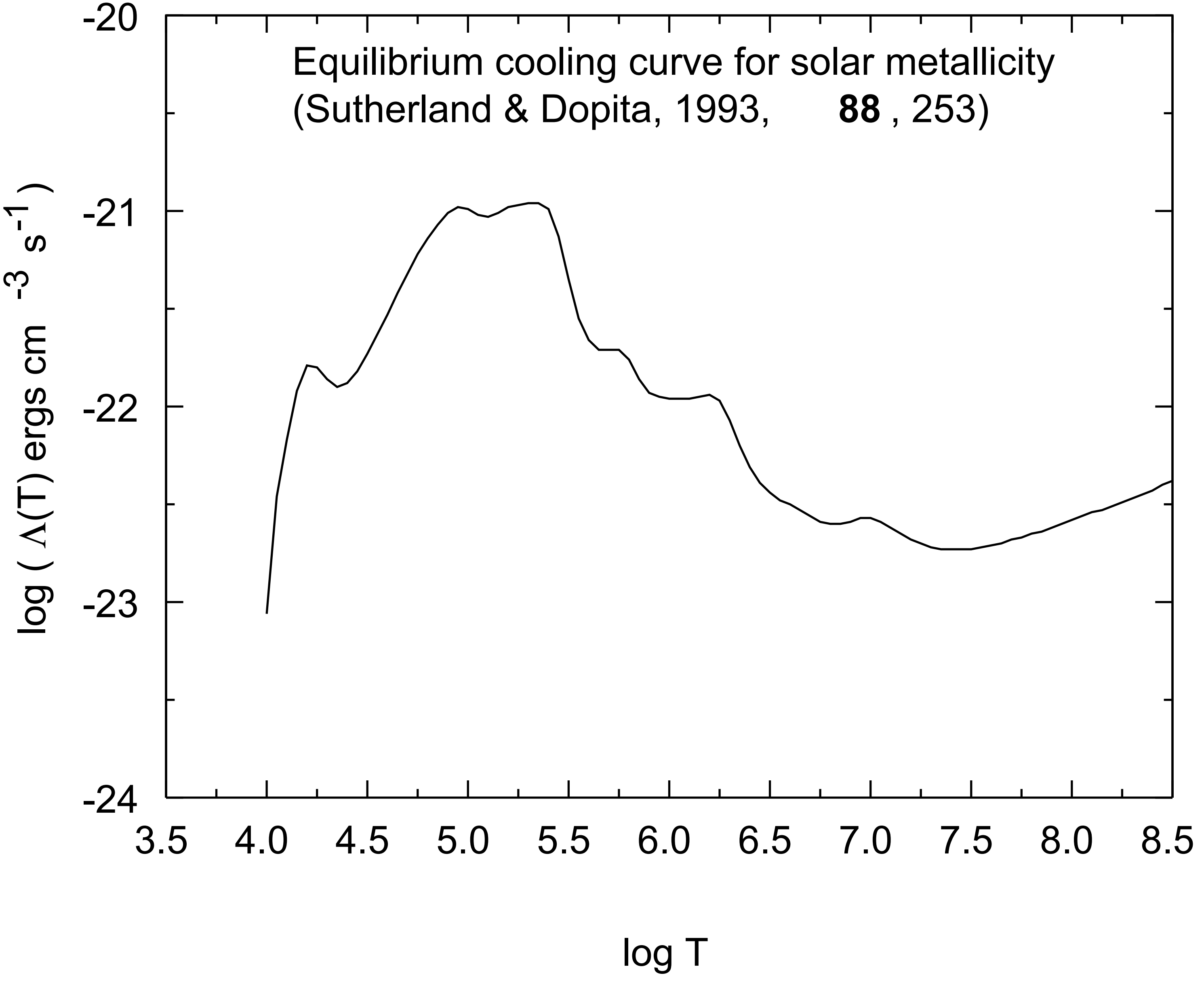
Equivalently

$$1\,\mathrm{W\,m^{-3}} = 10^{13}\ \mathrm{ergs\,s^{-1}\,cm^{3}} \qquad (15)$$

A typical value for $\Lambda(T)$ in cgs units is $10^{-22}\,\mathrm{ergs\,cm^{3}\,s^{-1}}$ so that a typical value in SI units is $10^{-35}\ \mathrm{W\,m^{3}\,s^{-1}}$.

Various workers have calculated cooling curves for astrophysical plasma, including Ralph Sutherland and Mike Dopita. One of theri cooling curves is shown in the following figure.

Equilibrium cooling curve



When the cooling time becomes comparable to the dynamical age then an adiabatic approximation is inadequate.

For typical ISM number densities, $n_e \approx n_p \sim 10^6 \text{ m}^{-3}$

$$t_{\text{cool}} \approx 7\times10^5 \text{ y}\frac{T_7}{n_6\Lambda_{-35}}$$

where $n_6 = n/10^6 \text{ m}^{-3}$.

There are two effects which are obvious from the above equation:

- The cooling time decreases as the temperature decreases and the cooling function Λ increases. The cooling function has a maximum of about 10^{-34} W m^3 around $T \approx 10^5 K$ so that when plasma is at this temperature we can be reasonably sure that cooling will be important.

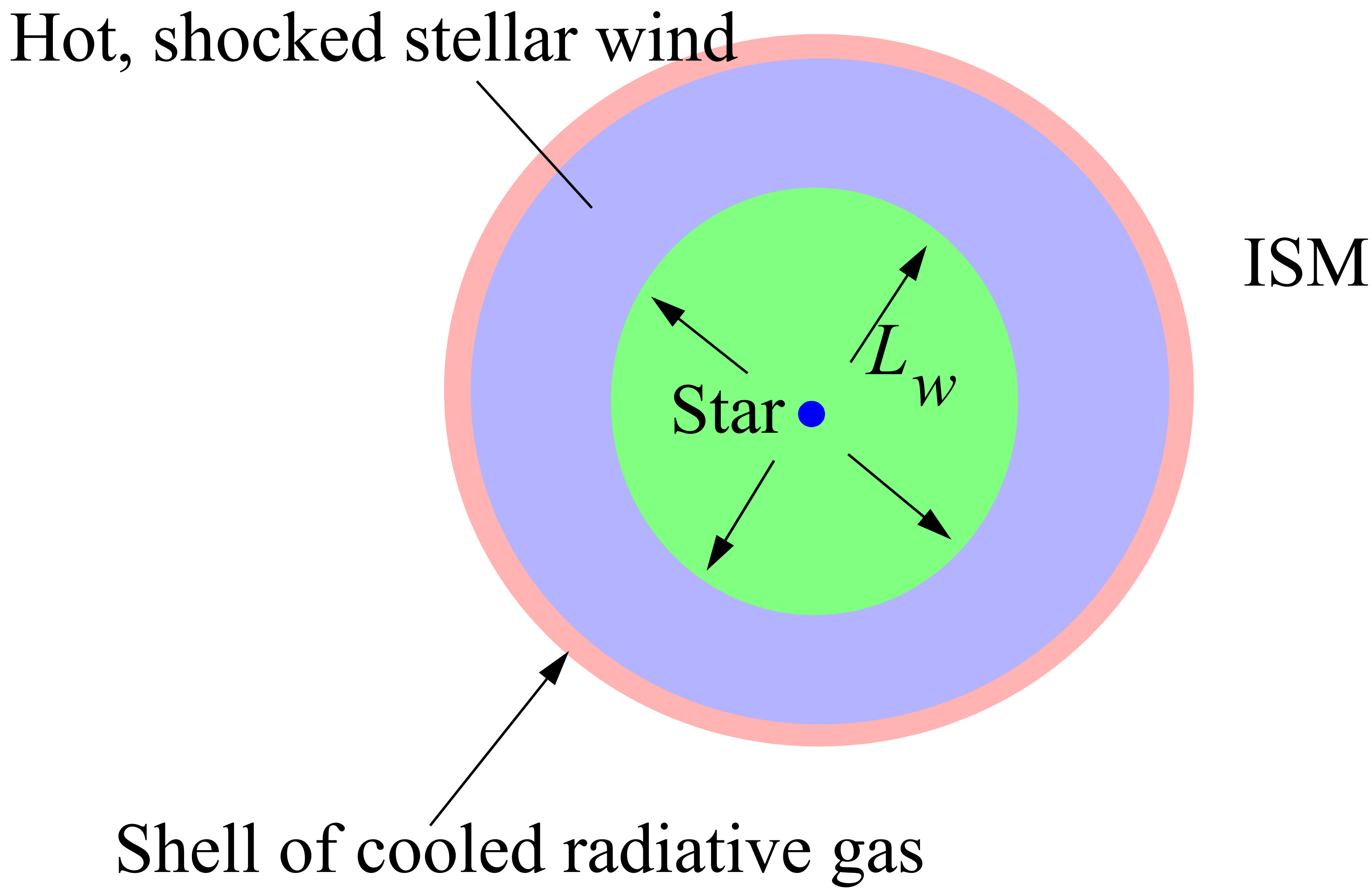
- The density dependence

$$t_{\mathrm{cool}} \propto \frac{1}{n}$$

so that as the density increases, the cooling time becomes shorter. We expect cooling to be important in very dense gases.

4 The effect of cooling on bubble dynamics

4.1 Analysis of a radiative bubble

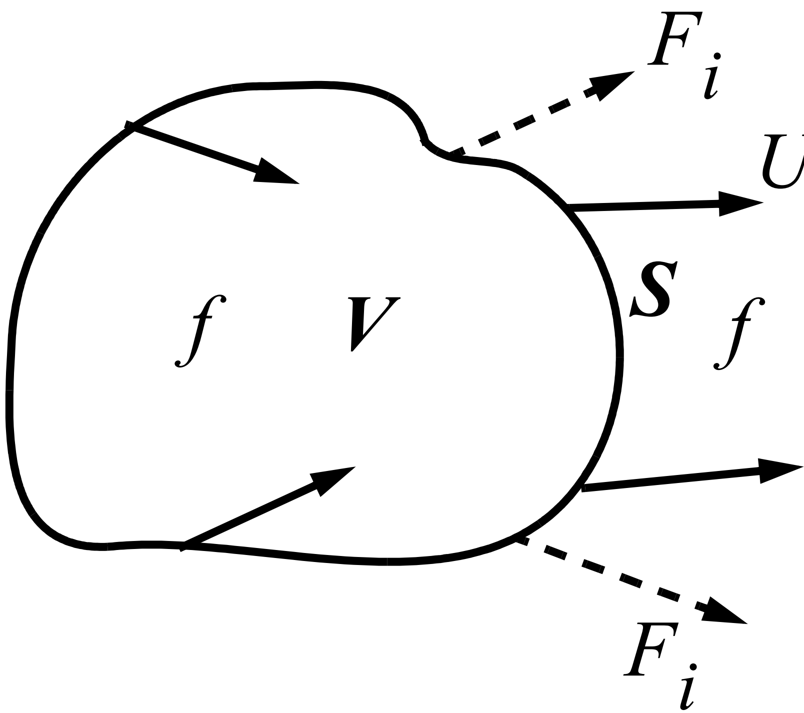


As a wind blown bubble expands into the ISM it compresses the surrounding medium and, in fact, forms a shocked layer of gas outside the bubble. Since the shocked ISM is fairly dense, it cools rapidly and loses pressure support. The compressed region of hot gas external to the bubble, collapses and forms a dense shell at the exterior of the bubble as shown in the diagram.

We can analyse the dynamics of this situation using conservation laws for mass, momentum and energy.

4.2 Generic conservation law for a moving surface

Recall the generic conservation law for a conserved quantity, f, with associated flux F_i



$$\frac{\partial f}{\partial t} + \frac{\partial F_i}{\partial x_i} = 0 \qquad (16)$$

When the surface enclosing a given volume is fixed, then the integral form of this conservation law is:

$$\frac{d}{dt}\int_V f d^3x = -\int_S F_i n_i dS \qquad (17)$$

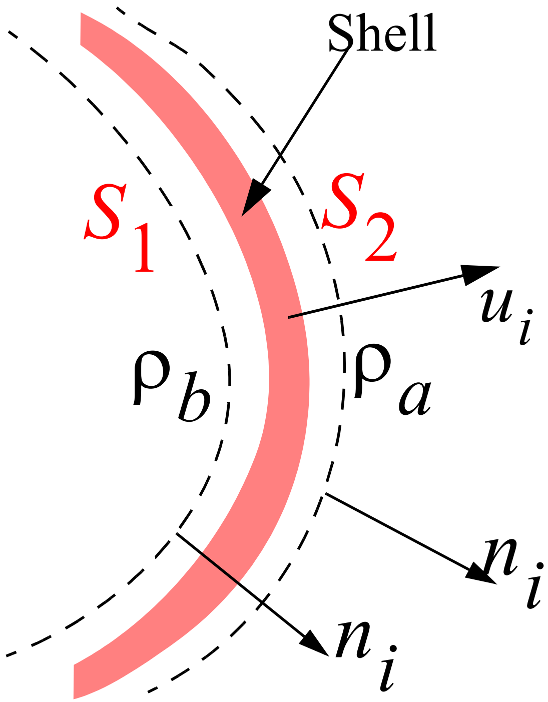
When the surface enclosing a volume V is moving at velocity u_i the relevant conservation law is:

$$\frac{d}{dt}\int_V f d^3x = -\int_S F_i n_i dS + \int_S f u_i n_i dS$$

$$\frac{d}{dt}\int_V f d^3x + \int_S (F_i - f u_i) n_i dS = 0 \tag{18}$$

The extra term describes the flux of the quantity f in to the volume as a result of it being swept up by the moving surface.

4.3 Conservation of mass



Consider two surface S_1 and S_2 where S_1 is just inside the moving shell and S_2 is just outside the shell. We integrate the mass conservation law over the volume in between S_1 and S_2. Then the surface integral involves integration over the 2 spherical surfaces. In the integral, the normal is always oriented outwards, so that we reverse the sign of the integral for S_1 when we keep the orientation of the normal in the outward radial direction in both cases. Hence the mass conservation law reads:

$$\frac{d}{dt}\int_{\text{Shell}} \rho d^3x = -\int_{S_2} (\rho v_i - \rho u_i) n_i dS + \int_{S_1} (\rho v_i - \rho u_i) n_i dS \qquad (19)$$

On S_2, the gas velocity just outside S_2 is zero and the radial velocity of the surface is the radial velocity of the shell:

$$v_i = 0 \qquad u_i = \frac{dR_s}{dt} n_i \qquad \rho = \rho_a \qquad (20)$$

On S_1, the gas is moving with the velecity of the surface:

$$v_i = u_i \qquad \rho = \rho_b \tag{21}$$

Substitute these into

$$\frac{d}{dt} \int_{\text{Shell}} \rho d^3x = -\int_{S_2} (\rho v_i - \rho u_i) n_i dS + \int_{S_1} (\rho v_i - \rho u_i) n_i dS \tag{22}$$

gives

$$\frac{d}{dt}\int_{\text{Shell}} \rho d^3x = \int_{S_2} \rho_a \frac{dR_s}{dt} dS = 4\pi R_s^2 \rho_a \frac{dR_s}{dt} \qquad (23)$$

i.e. in terms of the mass M_s of the shell:

$$\frac{dM_s}{dt} = 4\pi R_s^2 \rho_a \frac{dR_s}{dt} \qquad (24)$$

Physical description

In simple physical terms, the time rate of change of mass of the shell is determined by the rate of sweeping up of the ambient ISM.

4.4 Momentum

Applying the generic result for a conservation law to the conservation of momentum, we use:

$$f \to \rho v_i \qquad F_i \to \rho v_i v_j + p\delta_{ij} \tag{25}$$

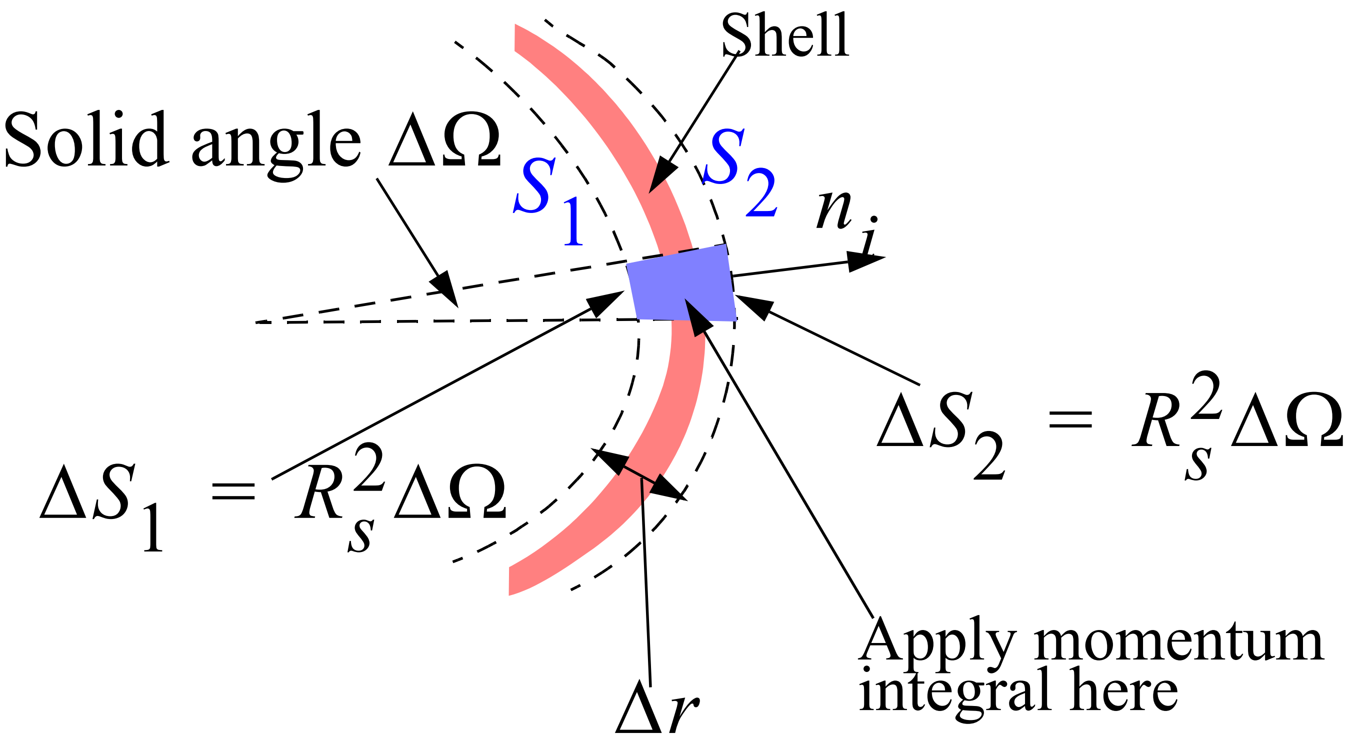
Therefore, the conservation of momentum within a moving surface is:

$$\frac{d}{dt}\int_V \rho v_i d^3x + \int_S (\rho v_i v_j + p\delta_{ij} - \rho v_i u_j) n_j d^3x = 0 \tag{26}$$

We have to be a bit more judicious in applying the momentum equation. If it is applied to the entire spherical surface the momentum integral is identically zero and we learn nothing. What we can do is apply the conservation of momentum to a section of small solid angle $\Delta\Omega$.

We apply momentum conservation to the small volume bounded by the dashed curves in the figure. Noting that the normal is directed to the interior of the volume on ΔS_1, we have:

$$\frac{a}{dt}\int_{\Delta V_1} \rho v_i d^3x = -\int_{\Delta S_2} (\rho v_i v_j + p\delta_{ij} - \rho v_i u_j) n_j dS + \int_{\Delta S_1} (\rho v_i v_j + p\delta_{ij} - \rho v_i u_j) n_j dS \qquad (27)$$



There is no contribution from the sides of the dashed region since $v_j n_j = 0$ there because the flow is spherically symmetric.

Now on S_1:

$$v_i = u_i = \frac{dR_s}{dt} n_i \qquad (28)$$

$$p = p_b$$

where p_b is the pressure interior to the bubble.

Therefore, the integrand of the surface integral is, on S_1:

$$(\rho v_i v_j + p\delta_{ij} - \rho v_i u_j) n_j = (\rho v_i u_j + p_b \delta_{ij} - \rho v_i u_j) n_i$$
$$= p_b n_i \qquad (29)$$

On S_2:

$$v_i = 0$$
$$p = p_a \qquad (30)$$

so that

$$(\rho v_i v_j + p\delta_{ij} - \rho v_i u_j)n_j = (0 + p_a\delta_{ij} - 0)n_j = p_a n_i \qquad (31)$$

Also the integral over the small volume can be expressed as:

$$\int_{\Delta V_1} \rho u_i d^3x = \left[\Delta m \frac{dR_s}{dt}\right] n_i = \left[M_s\left(\frac{\Delta\Omega}{4\pi}\right)\frac{dR_s}{dt}\right] n_i \qquad (32)$$

The mass, Δm of the elementary volume indicated in the diagram is:

$$\Delta m = \rho_s R_s^2 \Delta\Omega \Delta r \qquad (33)$$

where Δr is the thickness of the shell. Comparing this to the mass of the shell:

$$M_s = 4\pi\rho_s R_s^2 \Delta r \qquad (34)$$

we see that

$$\Delta m = M_s \frac{\Delta\Omega}{4\pi} \qquad (35)$$

Hence, the momentum equation, (28), becomes:

$$\frac{d}{dt}\left[\left(M_s\frac{\Delta\Omega}{4\pi}\frac{dR_s}{dt}\right)n_i\right] = -(p_a\Delta\Omega R_s^2)n_i + (p_b\Delta\Omega R_s^2)n_i = (p_b - p_a)\Delta\Omega R_s^2 n_i \qquad (36)$$

The solid angle, $\Delta\Omega$ and the unit normal n_i are constant. Therefore,

$$\frac{d}{dt}\left(M_s\frac{dR_s}{dt}\right) = 4\pi R_s^2(p_b - p_a) \qquad (37)$$

Physical interpretation

$$\begin{array}{c}\text{Rate of change of}\\ \text{momentum of shell}\end{array} = \begin{array}{c}\text{Force due to excess of}\\ \text{pressure on shell}\end{array} \qquad (38)$$

4.5 Energy

For energy

$$f \to \varepsilon + \frac{1}{2}\rho v^2 \qquad F_i \to \rho\left(\frac{1}{2}v^2 + h\right)v_i \qquad (39)$$

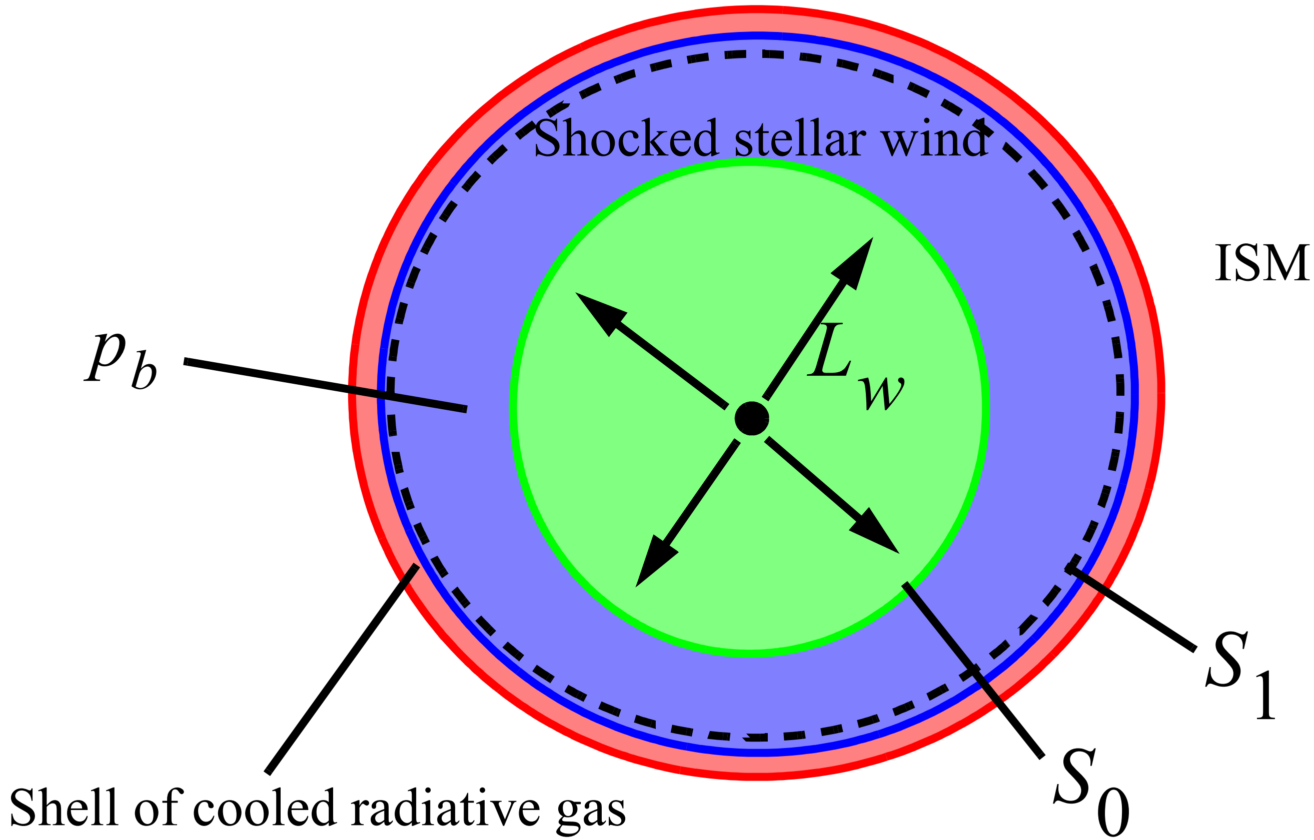
The conservation of energy for a moving surface is expressed as:

$$\frac{d}{dt}\int_V \left(\varepsilon + \frac{1}{2}\rho v^2\right) d^3x$$

$$= -\int_S \left[\rho\left(\frac{1}{2}v^2 + h\right)v_i - \left(\varepsilon + \frac{1}{2}\rho v^2\right)u_i\right] n_i dS \qquad (40)$$

$$= -\int_S \left[\frac{1}{2}\rho v^2 (v_i - u_i) + \varepsilon(v_i - u_i) + p v_i\right] n_i dS$$

We apply this integral to the region between the surface just exterior to the stellar wind shock and the surface just interior to the shell. The motivation for this is to obtain an equation for the interior pressure of the bubble.


Shocked stellar wind
ISM
p_b
L_w
S_1
S_0
Shell of cooled radiative gas

At the surface S_1:

$$\begin{aligned} v_i &= u_i \\ p &= p_b \end{aligned} \tag{41}$$

and the surface integral

$$\begin{aligned} &\int_{S_1} \left[\frac{1}{2}\rho v^2 (v_i - u_i) + \varepsilon(v_i - u_i) + p v_i\right] n_i dS \\ &= 4\pi R_s^2 p_b \frac{dR_s}{dt} \end{aligned} \tag{42}$$

At S_0, the wind passes through the surface at approximately 1/4 of the asymptotic wind velocity from the star. This is generally much larger than the expansion speed of the bubble.

$$v \gg u \qquad p = p_b \tag{43}$$

and the surface integral

$$\int_{S_0} \left[\frac{1}{2}\rho v^2 (v_i - u_i) + \varepsilon(v_i - u_i) + p v_i\right] n_i dS$$

$$= \int_{S_0} \left[\frac{1}{2}\rho v^2 v_i + (\varepsilon + p) v_i\right] n_i dS \qquad (44)$$

$$= \text{Flux of energy across } S_0$$

$$= L_w = \text{Mechanical luminosity of the wind}$$

In the shocked wind region

$$\varepsilon \gg \frac{1}{2}\rho v^2 \qquad (45)$$

and we take the pressure and energy density to be constant within this region. Hence, the integrated from of the energy equation for the region between the stellar wind shock and the radiative shell is:

$$\frac{dE_b}{dt} = \frac{d}{dt}\int_b \varepsilon d^3x = L_w - 4\pi R_s^2 p_b \frac{dR_s}{dt} \qquad (46)$$

The physical interpretation of this equation is that the energy of region (b) increases due to the input from the stellar wind and decreases as a result of the work done in expanding the bubble.

NB. This equation also holds for bubbles which are adiabatic in the external region (c).

In using this equation we can make use of the fact that

$$\varepsilon = \frac{3}{2}p \Rightarrow E_b = 2\pi p_b(R_s^3 - R_0^3) \qquad (47)$$

where, R_0 is the radius of the wind shock. We usually assume that $R_0^3 \ll R_s^3$ so that

$$E_b \approx 2\pi p_b R_s^3 \qquad (48)$$

4.6 Complete set of equations

$$\frac{dM_s}{dt} = 4\pi\rho_a R_s^2 \frac{dR_s}{dt}$$

$$\frac{d}{dt}\left(M_s \frac{dR_s}{dt}\right) = 4\pi(p_b - p_a)R_s^2$$

$$= 4\pi p_b R_s^2 \quad \text{for} \quad p_b \gg p_a \tag{49}$$

$$\frac{d}{dt}(p_b R_s^3) = \frac{L_w}{2\pi} - 2p_b R_s^2 \frac{dR_s}{dt}$$

4.7 Power-law solution

We assume that the ambient density ρ_a is a constant and look for a solution of these equations of the form:

$$M_s = a_1 t^{\alpha_1} \qquad R_s = a_2 t^{\alpha_2} \qquad p_b = a_3 t^{\alpha_3} \tag{50}$$

Substituting these expressions into the dynamical equations and equating powers of t leads to the equations:

$$\begin{aligned} \alpha_1 - 3\alpha_2 &= 0 & \alpha_1(a_1 a_2^{-3}) &= \alpha_2(4\pi\rho_a) \\ \alpha_1 - \alpha_2 - \alpha_3 &= 2 & \alpha_2(\alpha_1 + \alpha_2 - 1)a_1 a_2^{-1} a_3^{-1} &= 4\pi \\ 3\alpha_2 + \alpha_3 &= 1 & (\alpha_3 + 5\alpha_2)a_2^3 a_3 &= \frac{L_w}{2\pi} \end{aligned} \tag{51}$$

The solution of the α equations is

$$\alpha_1 = \frac{9}{5} \qquad \alpha_2 = \frac{3}{5} \qquad \alpha_3 = -\frac{4}{5} \tag{52}$$

and these values substituted into the equations for the a_i give

$$\begin{aligned} a_1 a_2^{-3} &= \frac{4\pi}{3}\rho_a \\ a_1 a_2^{-1} a_3^{-1} &= \frac{100\pi}{21} \\ a_2^3 a_3 &= \frac{5}{22\pi} L_w \end{aligned} \tag{53}$$

One solves these equations by taking $b_i = \ln a_i$; this makes the equations linear in b_i

The linear set of equations is:

$$b_1 - 3b_2 = c_1 = \ln\left(\frac{4\pi}{3}\rho_a\right) \quad (54)$$

$$b_1 - b_2 - b_3 = c_2 = \ln\left(\frac{100\pi}{21}\right) \quad (55)$$

$$3b_2 + b_3 = c_3 = \ln\left(\frac{5}{22\pi}L_w\right) \quad (56)$$

Take -(54) + (55) + (56) gives:

$$b_2 = \frac{1}{5}(-c_1 + c_2 + c_3) \qquad (57)$$

and

$$a_2 = \left[\frac{5^3}{7 \cdot 22\pi}\frac{L_w}{\rho_a}\right]^{1/5} \qquad (58)$$

Using the other equations

$$a_1 = \frac{4\pi}{3}\rho_a\left[\frac{5^3}{7\cdot 22\pi}\frac{L_w}{\rho_a}\right]^{3/5}$$

$$a_3 = \frac{5}{22\pi}L_w\left[\frac{5^3}{7\cdot 22\pi}\frac{L_w}{\rho_a}\right]^{-3/5} \qquad (59)$$

Numerical value

The radius of the bubble is given numerically by:

$$R_s = 25\ \mathrm{pc}\left(\frac{\dot{M}_w}{10^{-6} M_{sun}\ \mathrm{y}^{-1}}\right)^{0.2}\left(\frac{v_w}{1000\ \mathrm{km/s}}\right)^{0.4} \times \left(\frac{n_a}{10^6\ \mathrm{m}^{-3}}\right)^{-0.2}\left(\frac{t}{10^6\ \mathrm{y}}\right)^{0.6} \tag{60}$$

This is quite a good result since this is a typical bubble radius for these parameters.

4.8 Momentum balance

The total momentum of the shell is

$$\Pi_s = M_s \frac{dR_s}{dt} = \alpha_2 a_1 a_2 t^{\alpha_1 + \alpha_2 - 1}$$
$$= \frac{4\pi}{5}\rho_a \left[\frac{125}{7 \cdot 22\pi} \frac{L_w}{\rho_a}\right]^{4/5} t^{7/5} \qquad (61)$$

Total radial momentum flux:

$$\text{Momentum flux of wind} = 4\pi\rho_w v_w^2 R_1^2 = \frac{2L_w}{v_w} \qquad (62)$$

Therefore, the total momentum of the shell divided by the total momentum input from the star is:

$$\frac{2\pi}{5}\frac{\rho_a v_w}{L_w}\left[\frac{125}{7\cdot 22\pi}\frac{L_w}{\rho_a}\right]^{4/5} t^{2/5} = 4.8 \qquad (63)$$

for the fiducial parameters we have been using.

This is in apparent violation of the conservation of momentum. What's wrong!?

5 Energy driven and momentum driven bubbles

To resolve the above question it is useful to look at the momentum equation for the bubble.

$$\frac{d}{dt}\left(M_s \frac{dR_s}{dt}\right) = 4\pi p_b R_s^2 \qquad (64)$$

Now the pressure of the shocked stellar wind is given by:

$$p_b = \frac{3}{4}\rho_w v_w^2 \qquad (65)$$

The momentum flux of the stellar wind is

$$\Pi = 4\pi\rho_w v_w^2 R_1^2 = 2\frac{L_w}{v_w} \qquad (66)$$

so that

$$4\pi p_b R_s^2 = \frac{3}{4}\Pi \frac{R_s^2}{R_1^2} \tag{67}$$

and

$$\frac{d}{dt}\left(M_s \frac{dR_s}{dt}\right) = \frac{3}{4}\Pi \frac{R_s^2}{R_1^2} \tag{68}$$

clearly showing that the momentum flux of the wind is amplified by the factor R_s^2/R_1^2. The reason that there is a large amplification is that the dissipation of kinetic energy into pressure of the post-shock gas causes a large expansion of the post-shock region and the hot gas exerts its pressure on the outer surface of the bubble. Thus, it is the energy deposition from the stellar wind that is driving the bubble. There is really no problem with momentum conservation since the total momentum of the bubble system is zero and you cannot get any more conserved than that! Such bubbles are referred to as “energy driven”.

If, in addition to the exterior of the bubble being radiative, the interior is radiative as well, the post-shock stellar wind can radiate to the extent that region (b) collapses and the “mechanical advantage” resulting from the factor (R_s^2/R_1^2) is lost. In this case, one says that the bubble is “momentum driven”.

The distinction between energy driven and momentum driven bubbles was first made by John Dyson in an attempt to explain the momentum deficit in bipolar flows. The attempt was unsuccessful since the interior regions of bipolar flows are probably radiative. However, the physical distinction is important and useful.