Unmagnetised Winds

1 Hydrostatic atmospheres

1.1 The pressure cooker model

In many cases we can approximate the atmosphere of a planet, star galaxy, cluster etc. as hydrostatic. Neglecting magnetic forces for the time being, and assuming spherical symmetry, the equation describing such an atmosphere is:

$$-\frac{dP}{dr} - \rho\frac{d\psi}{dr} = 0$$

where ψ is the gravitational potential. In the case of a star we can put

$$\psi = -\frac{GM}{r}$$

$$-\frac{d\psi}{dr} = -\frac{GM}{r^2}$$

Frequently, we assume that the atmosphere is isothermal and this is adequate for our purposes here. Thus,

$$P = \rho\frac{kT}{\mu m_{\mathrm{p}}}$$

and the hydrostatic equation is:

$$\frac{kT}{\mu m_{\mathrm{p}}}\rho^{-1}\frac{d\rho}{dr} = -\frac{GM}{r^2}$$

$$\ln\rho = \text{Constant} + \frac{\mu GMm_p}{kT}\frac{1}{r}$$

Taking the density to be ρ_0 at $r = r_0$, we obtain

$$\frac{\rho}{\rho_0} = \exp\left[\frac{GMm_p}{kT}\left(\frac{1}{r} - \frac{1}{r_0}\right)\right] = \exp\left[-\frac{GMm_p}{kTr_0}\left(1 - \frac{r_0}{r}\right)\right]$$

We can define a "virial" temperature by:

$$T_{\text{virial}} = \frac{GMm_{\text{p}}}{kr_0} = 2.4\times10^7\left(\frac{M}{M_O}\right)\left(\frac{r_0}{R_O}\right) \text{ K}$$

so that

$$\frac{\rho}{\rho_0} = \exp\left[-\frac{T_{\text{virial}}}{T}\left(1 - \frac{r_0}{r}\right)\right]$$

Now, the intriguing point about this expression is that the density does not drop to zero at $r = \infty$. Instead,

$$\frac{\rho_\infty}{\rho_0} = \exp\left[-\frac{T_{\text{virial}}}{T}\right]$$

and if $T \sim T_{\text{virial}}$, then the drop in density from ρ to ρ_∞ is minor. The physical interpretation of this is that such an atmosphere requires a background pressure,

$$P_\infty = \frac{\rho_\infty kT}{\mu m_{\text{p}}}$$

in order to keep the atmosphere hydrostatic. This is sometimes known as the "pressure cooker" model since we have to keep a lid on the atmosphere.

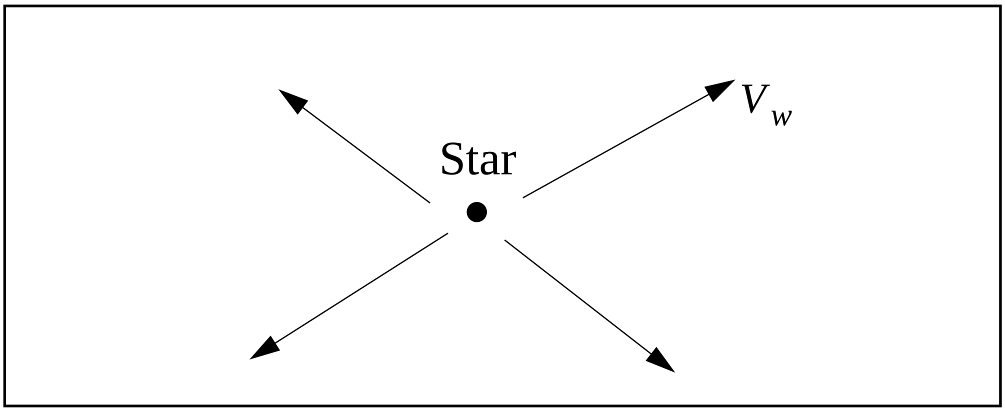
1.2 Application to the Sun

The sun has a hot corona with $T \approx 2\times10^6 K$ and therefore

$$\frac{\rho_\infty}{\rho_0} \approx \exp\left[-\frac{2.4\times10^7}{2\times10^6}\right] = \exp[-12] = 6.1\times10^{-6}$$

and although this ratio is fairly low it is not low enough that this density and the associated pressure $\sim 10^{-2}$ dynes cm^{-2} can be provided by the interstellar medium. Therefore, the sun's corona has to be driven out in ***a thermally driven wind,*** since the background medium does not have a high enough pressure to contain the hot gas. This is known as the ***solar wind.***

2 Characteristics of stellar winds



Solar wind:

Velocity at earth's orbit:

$$v \approx 400 \text{ km/s}$$

Density:

$$n \approx 10^7 \text{ m}^{-3}$$

Temperature:

$$T \approx 10^5 \text{ K}$$

Speed of sound:

$$c_s = 50 \text{ km/s}$$

Mass flux (spherically symmetric wind):

$$\dot{M} = 4\pi\mu n m_p v_r r^2 = 3\times 10^{-14} M_O/\mathrm{yr}$$

Other stars:

Red giants: $\dot{M} \approx 10^{-11} M_O/\mathrm{yr}$

O&B type stars: $\dot{M} \approx (10^{-7} - 10^{-6}) M_O/\mathrm{yr}$

Protostars: $\dot{M} \approx 10^{-4} M_O/\mathrm{yr}$

In the above cases, not all of the winds are thermally driven. In O&B stars, radiation pressure is the driving force. We do not know the mechanism in protostars, although magnetic fields seem to be implicated. However, an analysis of thermally driven winds introduces many of the physical and mathematical concepts that are relevant to all winds. The following treatment also introduces many of the concepts that are relevant to the more complex analysis of magnetised winds.

3 Analysis of spherically symmetric unmagnetised winds

3.1 Fundamental equations

Euler equations:

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial \phi}{\partial x_i}$$

$$\text{Spherical symmetry} \Rightarrow V_r \frac{\partial V_r}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial \phi}{\partial r}$$

For a spherical star:

$$\phi = -\frac{GM}{r}$$

$$-\frac{\partial\phi}{\partial r} = -\frac{GM}{r^2}$$

Equation of state:

$$\text{Adiabatic:} \quad p = K(s)\rho^{\gamma}$$

$$\text{Polytropic:} \quad p = C\rho^{\gamma}$$

In the latter, γ not necessarily c_p/c_v. $\gamma < 5/3 \Rightarrow$ Heating .

Isothermal:

$$P = \frac{kT}{\mu m_p}\rho \qquad T = \text{constant}$$

Source of heat for the solar wind: Dissipation by waves in solar wind generated in photosphere/chromosphere.

Speed of sound

$$a_s^2 = \frac{dp}{d\rho} = \gamma C\rho^{\gamma-1}$$

where the derivative is no longer at constant entropy.

$$\text{Isothermal sound speed} = \sqrt{\frac{kT}{\mu m_p}}$$

Radial momentum equation

$$V_r\frac{dV_r}{dr} = -\frac{a_s^2}{\rho}\frac{d\rho}{dr} - \frac{GM}{r^2}$$

Mass flux

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = \frac{1}{r^2}\frac{d}{dr}(r^2\rho V_r) = 0$$

$$\Rightarrow \rho r^2 V_r = \text{Constant} = \frac{\dot{M}}{4\pi}$$

$$\dot{M} = \int_{\text{Spherical surface}} \rho V_i n_i dS = 4\pi r^2 \rho V_r$$

From now on $V_r = V$.

3.2 Sonic point

$$\ln 4\pi + \ln\rho + \ln V + 2\ln R = \ln\dot{M}$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dr} + \frac{1}{V}\frac{dV}{dr} + \frac{2}{r} = 0$$

Substitute $\rho^{-1}\frac{d\rho}{dr}$ into momentum equation:

$$V\frac{dV}{dr} = \frac{a_s^2}{V}\frac{dV}{dr} + 2\frac{a_s^2}{r} - \frac{GM}{r^2}$$

$$(V^2 - a_s^2)\frac{dV}{dr} = V\left(2\frac{a_s^2}{r} - \frac{GM}{r^2}\right)$$

$$\frac{dV}{dr} = \frac{V\left(2\frac{a_s^2}{r} - \frac{GM}{r^2}\right)}{(V^2 - a_s^2)}$$

The equation for V has a ***critical point*** at:

$$V = \pm a_s = \sqrt{\frac{\gamma k T_c}{\mu m_p}}$$

$$2\frac{a_s^2}{r} = \frac{GM}{r^2} \Rightarrow r_c = \frac{GM}{2a_s^2} = \frac{\mu G M m_p}{2\gamma k T_c}$$

e.g. the Sun ($\gamma \approx 1$):

$$V_c \approx 170 \text{ km/s}\left(\frac{T}{10^6}\right)^{1/2}$$

$$r_c \approx 4.8\times10^9 \text{ m}\left(\frac{T}{10^6}\right)^{-1} = 6.9R_O\left(\frac{T}{10^6}\right)^{-1}$$

Solar radius: $R_O = 6.96\times10^8$ m .

Actual sonic radius: $r_c \approx 3.5R_O \Rightarrow T \approx 2\times10^6$ and $V_c \approx 340$ km/s .

4 Critical point analysis

If we want a wind solution which accelerates to supersonic then we need to negotiate the critical point (cf. deLaval nozzle).

$$\frac{dV}{dr} = \frac{V\left(\frac{2a_s^2}{r} - \frac{GM}{r^2}\right)}{(V^2 - a_s^2)}$$

Split this equation into 2 by introducing a parameter u and writing the equations as:

$$\frac{dr}{du} = r^2(V^2 - a_s^2)$$

$$\frac{dV}{du} = V(2a_s^2 r - GM)$$

These equations have no potential infinities anywhere and are much better behaved numerically and easier to analyse mathematically.

Critical point:

$$\frac{dr}{du} = \frac{dV}{du} = 0$$

Expansion in neighbourhood of critical point:

$$r = r_c + r' \qquad V = V_c + V'$$

$$\frac{dr}{du} = \frac{d}{du}r' \qquad \frac{dV}{du} = \frac{d}{du}V'$$

$$\frac{d}{u}\begin{bmatrix} r' \\ V' \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial V} \end{bmatrix} \begin{bmatrix} r' \\ V' \end{bmatrix}$$

where

$$f_1 = r^2(V^2 - a_s^2) \qquad f_2 = V(2a_s^2 r - GM)$$

In order to evaluate the partial derivatives of f_α we need to evaluate the partial derivatives of a_s^2.

Bernoulli's equation:

$$
\begin{aligned}
V\frac{dV}{dr} &= -\gamma C\rho^{\gamma-2}\frac{d\rho}{dr} - \frac{GM}{r^2} \\
\Rightarrow \frac{d}{dr}\left(\frac{1}{2}V^2\right) &= -\gamma C\rho^{\gamma-2}\frac{d\rho}{dr} - \frac{GM}{r^2} \\
\frac{1}{2}V^2 &= -\frac{\gamma C}{\gamma-1}\rho^{\gamma-1} + \frac{GM}{r} + \text{constant} \\
\frac{1}{2}V^2 + \frac{C\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} &= \text{constant} \\
a_s^2 &= (\gamma-1)\left(\text{constant} + \frac{GM}{r} - \frac{1}{2}V^2\right)
\end{aligned}
$$

The last equation implies:

$$
\frac{\partial}{\partial r}a_s^2 = -(\gamma-1)\frac{GM}{r^2} \qquad \frac{\partial}{\partial V}a_s^2 = -(\gamma-1)V
$$

Hence, at the critical point:

$$
\begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial V} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial V} \end{bmatrix} = \begin{bmatrix} (\gamma-1)GM & (\gamma+1)V_c r_c^2 \\ \frac{GM}{r_c}V_c(3-2\gamma) & -(\gamma-1)GM \end{bmatrix}
$$

and

$$
\frac{d}{du}\begin{bmatrix} r' \\ V' \end{bmatrix} = \begin{bmatrix} (\gamma-1)GM & (\gamma+1)V_c r^2 \\ \frac{GM}{r_c}V_c(3-2\gamma) & -(\gamma-1)GM \end{bmatrix}\begin{bmatrix} r' \\ V' \end{bmatrix}
$$

The solution of these equations requires the eigenvalues and eigenvectors of the matrix. Denoting the eigenvalues by λ:

$$\begin{vmatrix} \lambda-(\gamma-1)GM & -(\gamma+1)V_c r_c^2 \\ -\dfrac{GM}{r_c}V_c(3-2\gamma) & \lambda+(\gamma-1)GM \end{vmatrix} = 0$$

This simplifies to:

$$\lambda^2 = (GM)^2\frac{(5-3\gamma)}{2}$$

$$\Rightarrow \lambda = \pm GM\left(\frac{5-3\gamma}{2}\right)^{1/2}$$

Eigenvectors:

$$\begin{bmatrix} \lambda-(\gamma-1)GM & -(\gamma+1)Vr_c^2 \\ X & Y \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow [\lambda-(\gamma-1)GM]u_1-(\gamma+1)V_c r_c^2 u_2 = 0$$

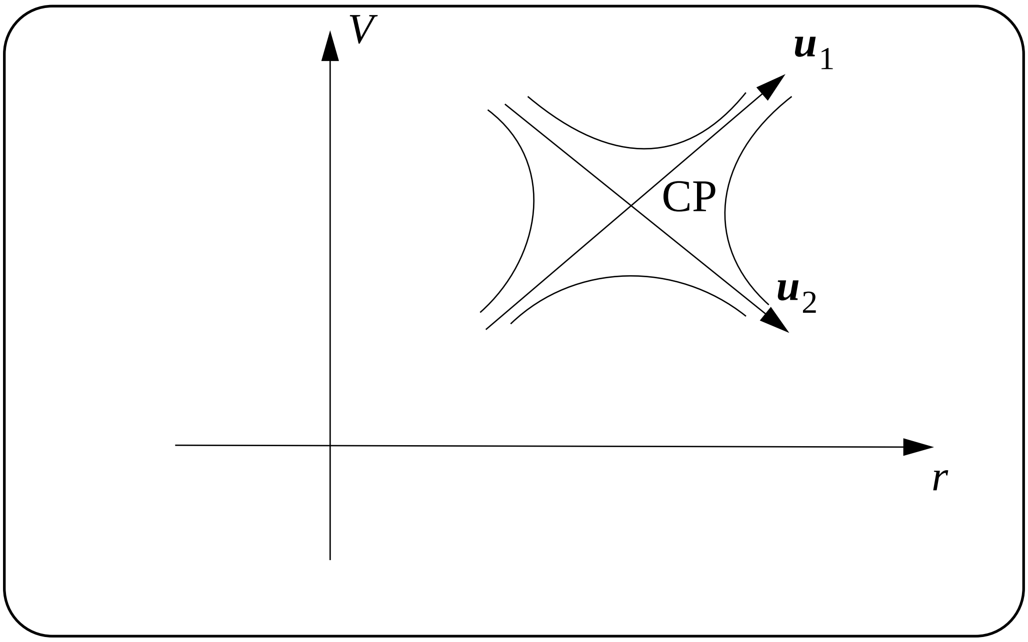
Take

$$u_2 = 1 \Rightarrow u_1 = \frac{(\gamma+1)\dfrac{V_c r_c^2}{GM}}{\pm\left(\dfrac{5-3\gamma}{2}\right)^{1/2}-(\gamma-1)}$$

The general solution in the neighbourhood of the critical point is:

$$\begin{bmatrix} r' \\ V' \end{bmatrix} = A_1\boldsymbol{u}_1\exp\lambda_1 u + A_2\boldsymbol{u}_2\exp\lambda_2 u$$

where $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are the two independent eigenvectors.



Slopes of lines through critical point:

$$\frac{dV}{dr} = \frac{u_2}{u_1} = \frac{2V_c}{r_c}\left[\frac{\pm\left(\frac{5-3\gamma}{2}\right)^{1/2} - (\gamma-1)}{\gamma+1}\right]$$

5Asymptotic wind solutions

As $r \to \infty$ the density in the wind goes to zero and therefore since

$$\frac{1}{2}V^2 + \frac{C\gamma}{\gamma-1}\rho^{\gamma-1} - \frac{GM}{r} = \text{constant}$$

the velocity tends to a constant.

Since

$$\dot{M} = 4\pi\rho V r^2$$

then asymptotically

$$\rho = \frac{\dot{M}}{4\pi V_\infty r^2}$$

This expression for the density is frequently used for winds well outside the sonic point where the wind can be considered to have achieved its terminal velocity, V_∞.