

Simple Waves

Reference: Landau & Lifshitz, *Fluid Mechanics*

1 Nonlinear waves

1.1 Small amplitude waves

When we considered sound waves we considered small amplitude perturbations to the fluid equations. In 1D

$$\frac{\partial^2 \delta \rho}{\partial x^2} - \frac{1}{c_s^2} \frac{\partial^2 \delta \rho}{\partial t^2} = 0$$

The solution to this equation, of course, is:

$$\delta \rho = f_1(x - c_s t) + f_2(x + c_s t)$$

where f_1 and f_2 are arbitrary functions. The perturbation to the velocity is given by:

$$\begin{aligned} \frac{\partial \delta v_x}{\partial t} &= -\frac{c_s^2}{\rho} \frac{\partial \delta \rho}{\partial x} = -\frac{c_s^2}{\rho} [f_1'(x - c_s t) + f_2'(x + c_s t)] \\ \delta v_x &= \frac{c_s}{\rho} [f_1(x - c_s t) - f_2(x + c_s t)] \end{aligned}$$

Note that for the wave travelling in each direction

$$\delta v_x = \pm \frac{c_s}{\rho} \delta \rho$$

1.2 Simple waves

The above relationship suggests looking for a full nonlinear solution in which

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$$v = v(\rho)$$

Such a solution of the Euler equations is called a simple wave.

Take the 1D Euler equations

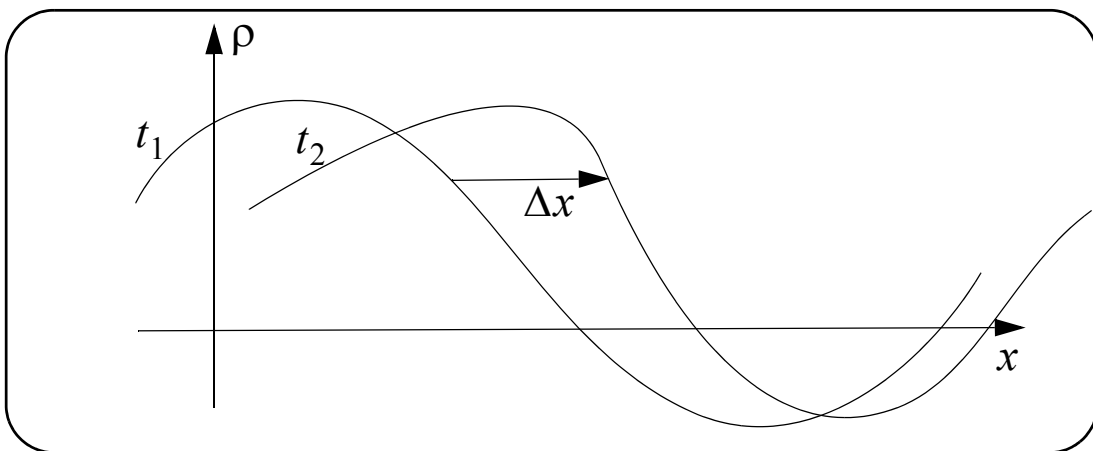
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

Since $v = v(\rho)$ and $p = p(\rho)$ then $p = p(v)$ and the above equations can be written:

$$\frac{\partial \rho}{\partial t} + \frac{d}{d\rho}(\rho v) \frac{\partial \rho}{\partial x} = 0$$
$$\frac{\partial v}{\partial t} + \left(v + \frac{1}{\rho} \frac{dp}{dv} \right) \frac{dv}{dx} = 0$$

1.2.1 Aside - Profile velocity

We would like to work out the velocity of a point on the density profile, at a fixed density, as shown in the figure. The aim of the following is to determine $\left. \frac{\partial x}{\partial t} \right|_{\rho}$.



To determine the velocity of a point with constant density, we consider a transformation of independent variables from (x, t) to (ρ, t) defined by:

$$\begin{aligned}\rho &= \rho(x, t) \\ t' &= t\end{aligned}$$

the reverse transformation being:

$$\begin{aligned}x &= x(t', \rho) \\ t &= t'\end{aligned}$$

In the new variables (ρ, t)

$$\begin{aligned}\left. \frac{\partial \rho}{\partial t'} \right|_{\rho} &= \left. \frac{\partial \rho}{\partial x} \right|_t \left. \frac{\partial x}{\partial t'} \right|_{\rho} + \left. \frac{\partial \rho}{\partial t} \right|_x \left. \frac{\partial t}{\partial t'} \right|_{\rho} \\ &= \left. \frac{\partial \rho}{\partial x} \right|_t \left. \frac{\partial x}{\partial t'} \right|_{\rho} + \left. \frac{\partial \rho}{\partial t} \right|_x = 0 \\ \Rightarrow \left. \frac{\partial x}{\partial t} \right|_{\rho} &= - \frac{\left. \frac{\partial \rho}{\partial t} \right|_x}{\left. \frac{\partial \rho}{\partial x} \right|_t}\end{aligned}$$

Similarly, in order to consider the profile velocity at a given velocity value, we consider a transformation

$$\begin{aligned}v &= v(x, t) \\ t' &= t\end{aligned}$$

from (x, t) to (v, t) and the rate of change of x at a fixed t is given by

$$\left. \frac{\partial x}{\partial t} \right|_v = - \frac{\frac{\partial v}{\partial t}}{\frac{\partial v}{\partial x}}$$

1.3 Continued solution for simple wave:

Given the above expression for a simple wave and the continuity equation we have

$$\left. \frac{\partial x}{\partial t} \right|_\rho = \frac{d}{d\rho}(\rho v) = v + \rho \frac{dv}{d\rho}$$

The momentum equation implies that

$$\left. \frac{\partial x}{\partial t} \right|_v = v + \frac{1}{\rho} \frac{dp}{dv}$$

Now the crucial point in the development of this solution is that since $v = v(\rho)$ then $v = \text{constant}$ corresponds to $\rho = \text{constant}$ and

$$\left. \frac{\partial x}{\partial t} \right|_v = \left. \frac{\partial x}{\partial t} \right|_\rho$$

Hence,

$$\begin{aligned}
v + \frac{1}{\rho} \frac{dp}{dv} &= v + \rho \frac{dv}{d\rho} \\
\Rightarrow \frac{c_s^2 d\rho}{\rho dv} &= \rho \frac{dv}{d\rho} \\
&\Rightarrow \left(\frac{dv}{d\rho} \right)^2 = \frac{c_s^2}{\rho^2} \\
\frac{dv}{d\rho} &= \pm \frac{c_s}{\rho}
\end{aligned}$$

Thus V as a function of ρ is given by:

$$v = \pm \int \frac{c_s}{\rho} d\rho$$

We take a polytropic equation of state

$$p = K(s)\rho^\gamma$$

and take ρ_0 to be a reference density corresponding to $v = 0$. Thus

$$\begin{aligned}
K &= p_0 \rho_0^{-\gamma} & c_0^2 &= \gamma K \rho_0^{\gamma-1} \\
\frac{p}{p_0} &= \left(\frac{\rho}{\rho_0} \right)^\gamma & c_s^2 &= c_0^2 \left(\frac{\rho}{\rho_0} \right)^{\gamma-1} & c_s &= c_0 \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}}
\end{aligned}$$

Hence the relationship between V and ρ for a polytropic gas is:

$$\begin{aligned}
v &= \pm \int \frac{c_s}{\rho} d\rho = \pm c_0 \int \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-3}{2}} d\left(\frac{\rho}{\rho_0} \right) \\
&= \pm \frac{2c_0}{\gamma-1} \left[\left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{2}} - 1 \right] = \pm \frac{2c_0}{\gamma-1} \left[\frac{c_s}{c_0} - 1 \right]
\end{aligned}$$

Solving for the sound speed:

$$c_s = c_0 \pm \frac{\gamma - 1}{2} v$$

1.3.1 Wave velocity

$$\left(\frac{\partial x}{\partial t}\right)_v = v + \frac{1}{\rho} \frac{dp}{dv} = v + \frac{c_s^2}{\rho} \frac{d\rho}{dv} = v \pm c_s$$

Since $c_s = c_s(\rho) = c_s(v)$ then

$$x = t(v \pm c_s) + f(v)$$

where $f(v)$ is an arbitrary function of v . Using the solution for c_s then

$$x = t \left[\frac{\gamma + 1}{2} v \pm c_0 \right] + f(v)$$

This is one of the most useful forms of the solution for a simple wave.

NB. Waves travelling to the right represented by the + sign in the above expressions; waves travelling to the left by the - sign.

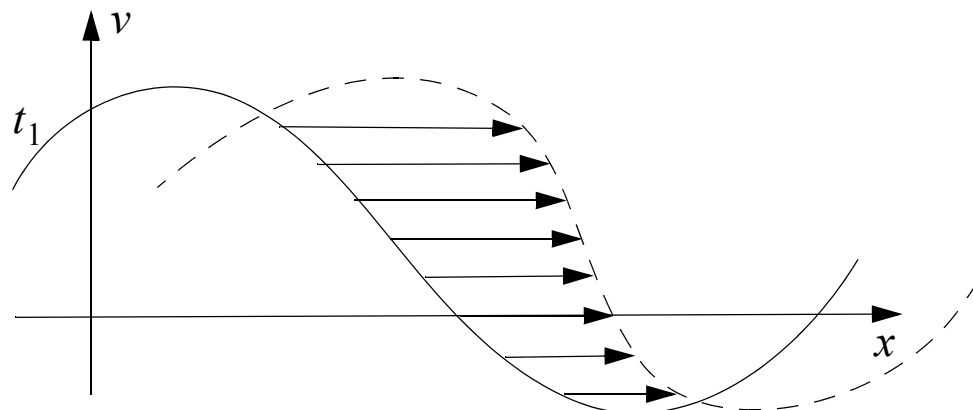
1.3.2 Velocity and the formation of shocks

From the above

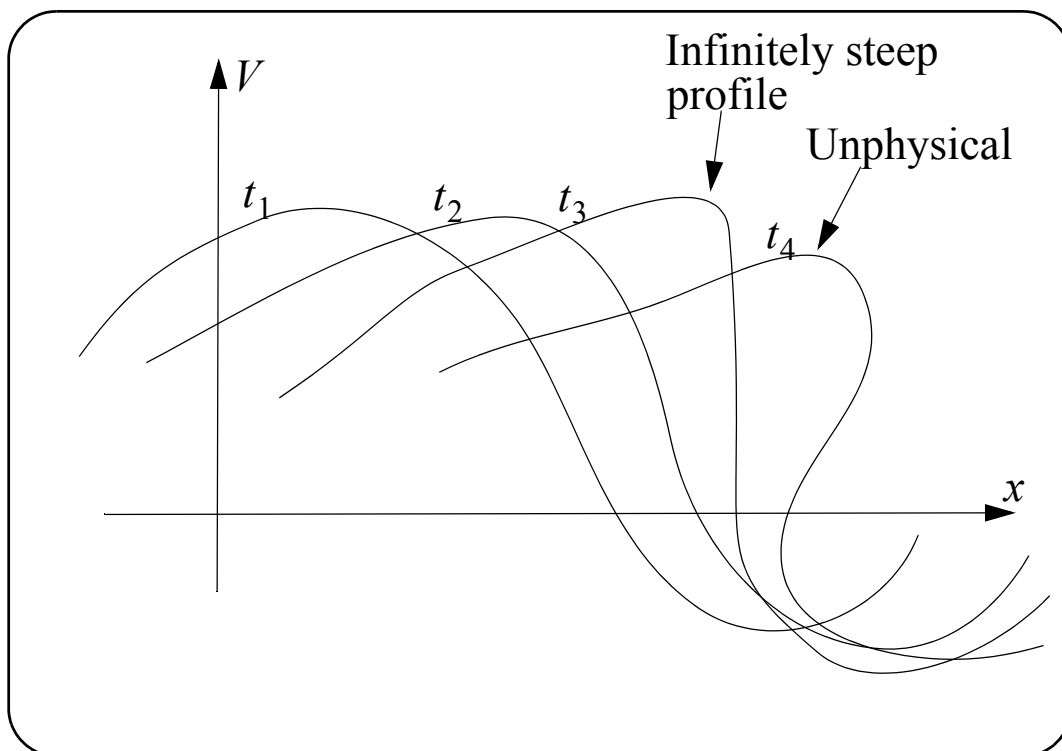
$$\left(\frac{\partial x}{\partial t}\right)_v = \frac{\gamma + 1}{2} v \pm c_0$$

and this shows the nonlinear effect on the profile velocity. When

$v \ll c_0$ then $\left(\frac{\partial x}{\partial t}\right)_v = \pm c_0$, i.e. the standard relation for a sound wave. However, when $v \sim c_0$ nonlinear effects have a marked effect on the profile.



The increase of the profile velocity with increasing positive v and the decrease with decreasing negative v means that the profile steepens as shown in the figure. The end result is a series of ever steepening profiles which eventually become unphysical.



When the profile becomes infinitely steep a shock forms. In this region the assumptions embodied in the Euler equations break down i.e. it is no longer possible to neglect viscosity. (More about this later.)

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At the time of formation of the shock:

$$\left. \frac{\partial x}{\partial v} \right|_t = 0$$

Moreover, the shock represents a point of inflection in the profile so that

$$\left. \frac{\partial^2 x}{\partial v^2} \right|_t = 0$$

If we use the solution:

$$x = t \left(\pm c_0 + \frac{\gamma + 1}{2} v \right) + f(v)$$

then the conditions for the formation of a shock are:

$$\frac{\partial x}{\partial v} = \frac{\gamma + 1}{2} t + f'(v) = 0 \Rightarrow t = -\frac{2f'(v)}{\gamma + 1}$$

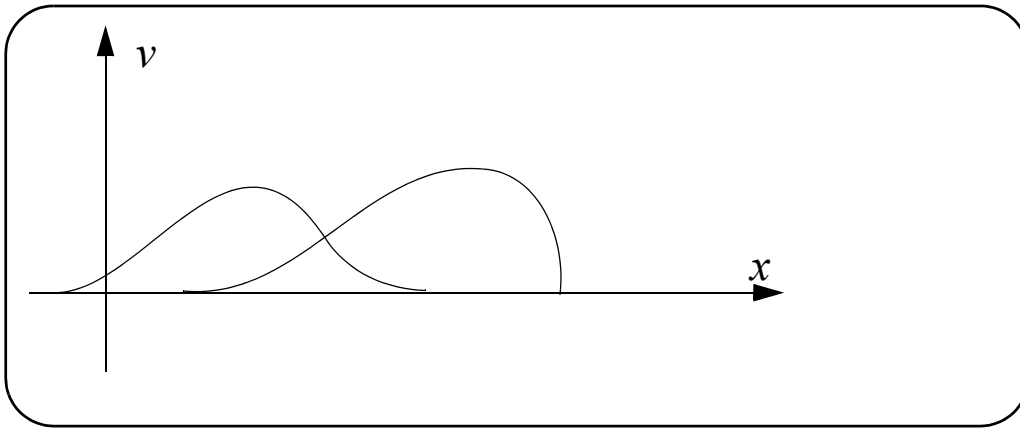
and

$$f''(v) = 0$$

Both of these conditions have to be satisfied simultaneously.

1.3.3 Fluid outside the region at rest

This is a special case.



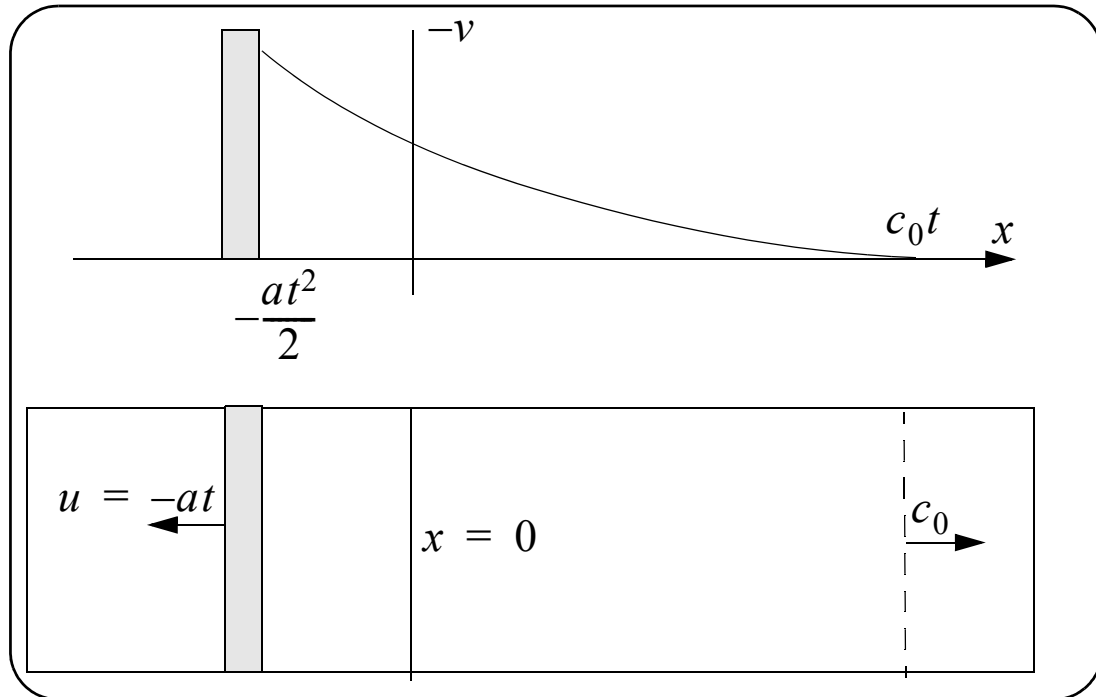
In this case the condition for a shock to form is simply

$$\left. \frac{\partial x}{\partial v} \right|_t = 0 \quad \text{at } v = 0$$

since the velocity profile beyond this point is simply $v = 0$. This gives the condition $t = -\frac{2f'(0)}{\gamma + 1}$

2 Piston moving out of a pipe as an example of a simple wave

2.1 Formation of solution



The piston is initially at $x = 0$ and moves towards the left with acceleration a . A sound wave travels towards the right with velocity c_0 since the gas is at rest to the right of the piston.

We can solve for this flow as a simple wave. Take the general solution:

$$x = t \left[\pm c_0 + \frac{1}{2}(\gamma + 1)v \right] + f(v)$$

The $+$ sign applies to a wave travelling to the right so that

$$x = t \left[c_0 + \frac{1}{2}(\gamma + 1)v \right] + f(v)$$

(Note: the *wave* is travelling to the right although the piston is travelling to the left.)

The arbitrary function may be calculated using the boundary condition that the gas remains in contact with the piston, i.e.

$$v = -at \quad \text{at} \quad x = -\frac{at^2}{2}$$

Write the solution in the form:

$$\begin{aligned} f(v) &= x - t \left[c_0 + \frac{\gamma + 1}{2}v \right] \\ \Rightarrow f(-at) &= -\frac{at^2}{2} - t \left[c_0 - \frac{(\gamma + 1)}{2}at \right] \\ &= \frac{c_0}{a}(-at) + \frac{\gamma}{2a}(-at)^2 \end{aligned}$$

Therefore

$$\begin{aligned} f(v) &= \frac{c_0}{a}v + \frac{\gamma}{2a}v^2 \\ \Rightarrow x &= t \left[c_0 + \frac{(\gamma + 1)}{2}v \right] + \frac{c_0}{a}v + \frac{\gamma}{2a}v^2 \end{aligned}$$

and the solution for x as a function of x and t is given by the solution of the quadratic:

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$$\frac{\gamma}{2a}v^2 + \left(\frac{c_0}{a} + \frac{\gamma+1}{2}t\right)v + c_0t - x = 0$$
$$\Rightarrow -v = \frac{c_0}{\gamma} + \frac{\gamma+1}{2\gamma}(at) \pm \sqrt{\left(\frac{c_0}{\gamma} + \frac{\gamma+1}{2\gamma}at\right)^2 + \frac{2a}{\gamma}(x - c_0t)}$$

The other boundary condition to incorporate is the condition that a sound wave moves away from the piston. i.e. $v = 0$ at $x = c_0t$. We can readily see that the - sign in the above expression gives the correct boundary condition so that

$$-v = \frac{c_0}{\gamma} + \frac{\gamma+1}{2\gamma}(at) - \sqrt{\left(\frac{c_0}{\gamma} + \frac{\gamma+1}{2\gamma}at\right)^2 + \frac{2a}{\gamma}(x - c_0t)}$$

Does a shock form in this solution?

$$f(v) = \frac{c_0}{a}v + \frac{\gamma}{2a}v^2$$

$$f'(v) = \frac{c_0}{a} + \frac{\gamma}{a}v$$

$$f''(v) = \frac{\gamma}{a}$$

The second derivative is always non zero so that if a shock forms it does so when $v = 0$ i.e. when

$$t = -\frac{2f'(0)}{\gamma+1} = -\frac{2}{\gamma+1}\frac{c_0}{a} < 0$$

Negative values of t are outside the domain of validity of our solution so that a shock does **not** form.

2.2 Maximum Velocity

An intriguing aspect of the above solution is that a velocity is reached at which a cavity forms behind the piston. Consider our solution for the sound speed in a simple wave:

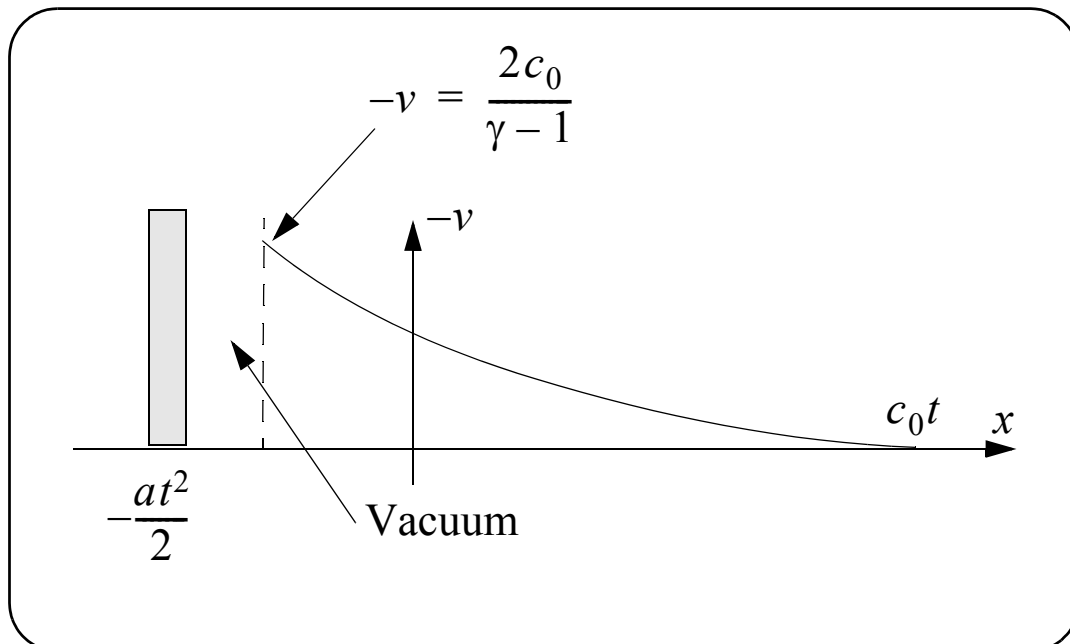
$$c_s = c_0 \pm \frac{\gamma-1}{2} v = c_0 + \frac{\gamma-1}{2} v$$

Now in order that the density remain positive, the sound speed must also remain positive and

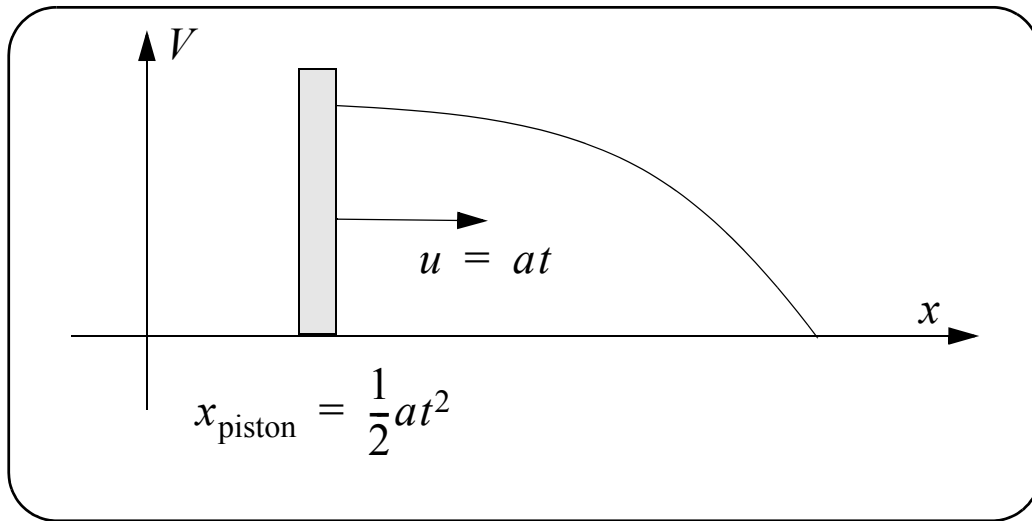
$$c_0 + \frac{\gamma-1}{2} v > 0$$

$$-v < \frac{2c_0}{\gamma-1}$$

Once the gas (and hence the piston) reaches this velocity a vacuum forms.



3 Piston moving into gas



3.1 Solution

Piston accelerates from zero velocity to a velocity $u = at$ and a position $x = \frac{1}{2}at^2$ at time t .

Again we use the general simple wave for a wave moving to the right

$$x = t \left[c_0 + \frac{\gamma + 1}{2} v \right] + f(v)$$

and we use the boundary condition $v = at$ for gas at the surface of the piston:

$$\Rightarrow \frac{1}{2}at^2 = t \left[c_0 + \frac{\gamma + 1}{2} at \right] + f(at)$$

$$\Rightarrow f(at) = -\frac{c_0}{a}(at) - \frac{\gamma}{2a}(at)^2$$

$$f(v) = -\frac{c_0}{a}v - \frac{\gamma}{2a}v^2$$

3.2 Formation of shock:

We have

$$f(v) = -\frac{c_0}{a} - \frac{\gamma}{a}v \quad \text{and} \quad f''(v) = -\frac{\gamma}{a} < 0$$

so that the only place where a shock can form is at $v = 0$. The time of formation of the shock in this case is:

$$t = -\frac{2f(0)}{\gamma + 1} = \frac{2}{\gamma + 1} \frac{c_0}{a}$$

Note that the velocity of the piston is not supersonic with respect to the undisturbed gas when the shock forms:

$$u = at = \frac{2}{\gamma + 1}c_0 < c_0 \quad \text{for} \quad \gamma > 1$$

3.3 Velocity of gas

The expression for the velocity profile is, in this case (exercise):

$$v = -\left(\frac{c_0}{\gamma} - \frac{\gamma + 1}{2\gamma}at\right) + \sqrt{\left(\frac{c_0}{\gamma} - \frac{\gamma + 1}{2\gamma}at\right)^2 - \frac{2a}{\gamma}(x - c_0t)}$$

One can show directly from this expression that the velocity profile

becomes infinitely steep at $x = c_0t$ when $t = \frac{2}{\gamma + 1} \frac{c_0}{a}$.