

Shocks

1 Steepening of sound waves

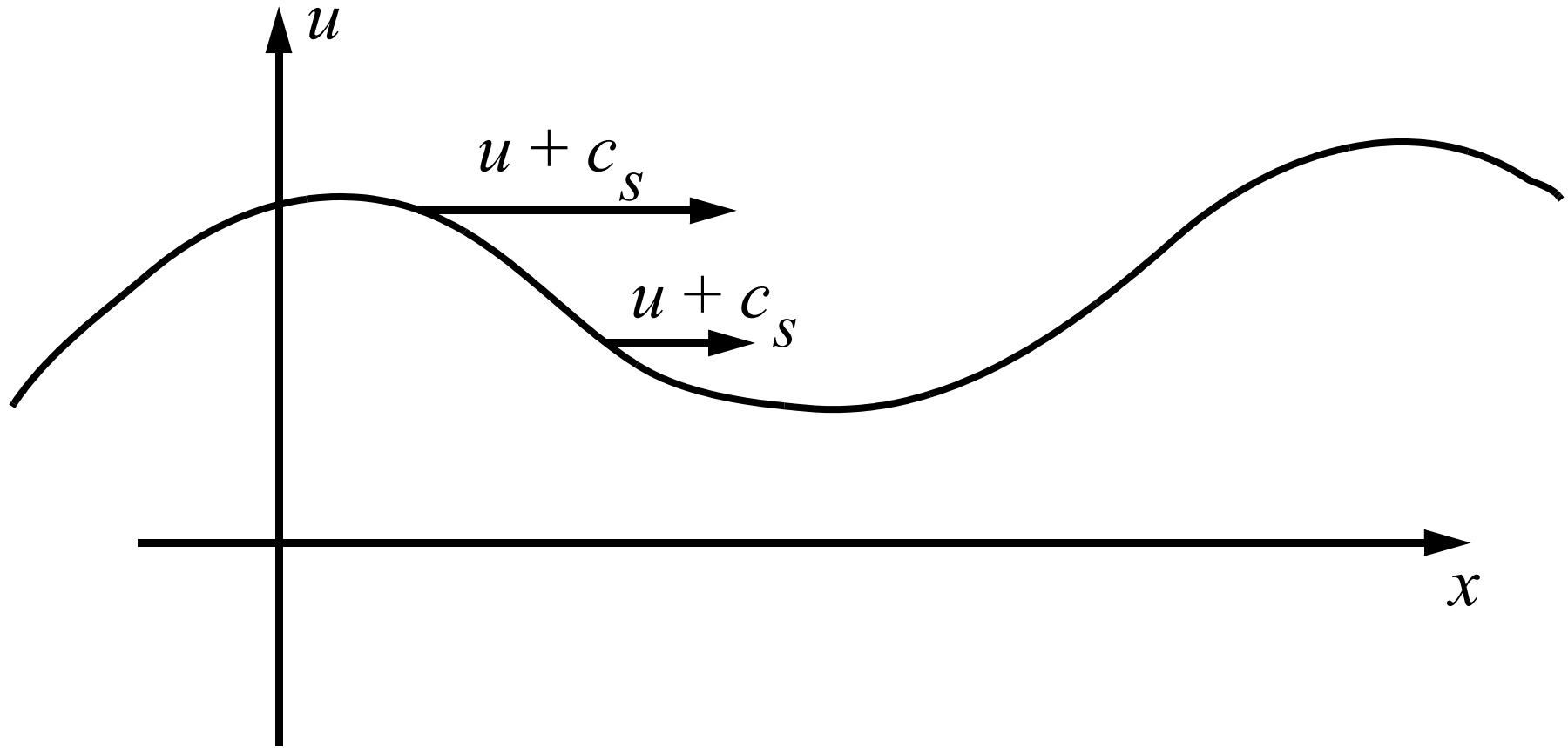
We have the result that the velocity of a sound wave in an arbitrary reference frame is given by:

$$\mathbf{v} = \mathbf{u} + c_s \mathbf{k} \quad (1)$$

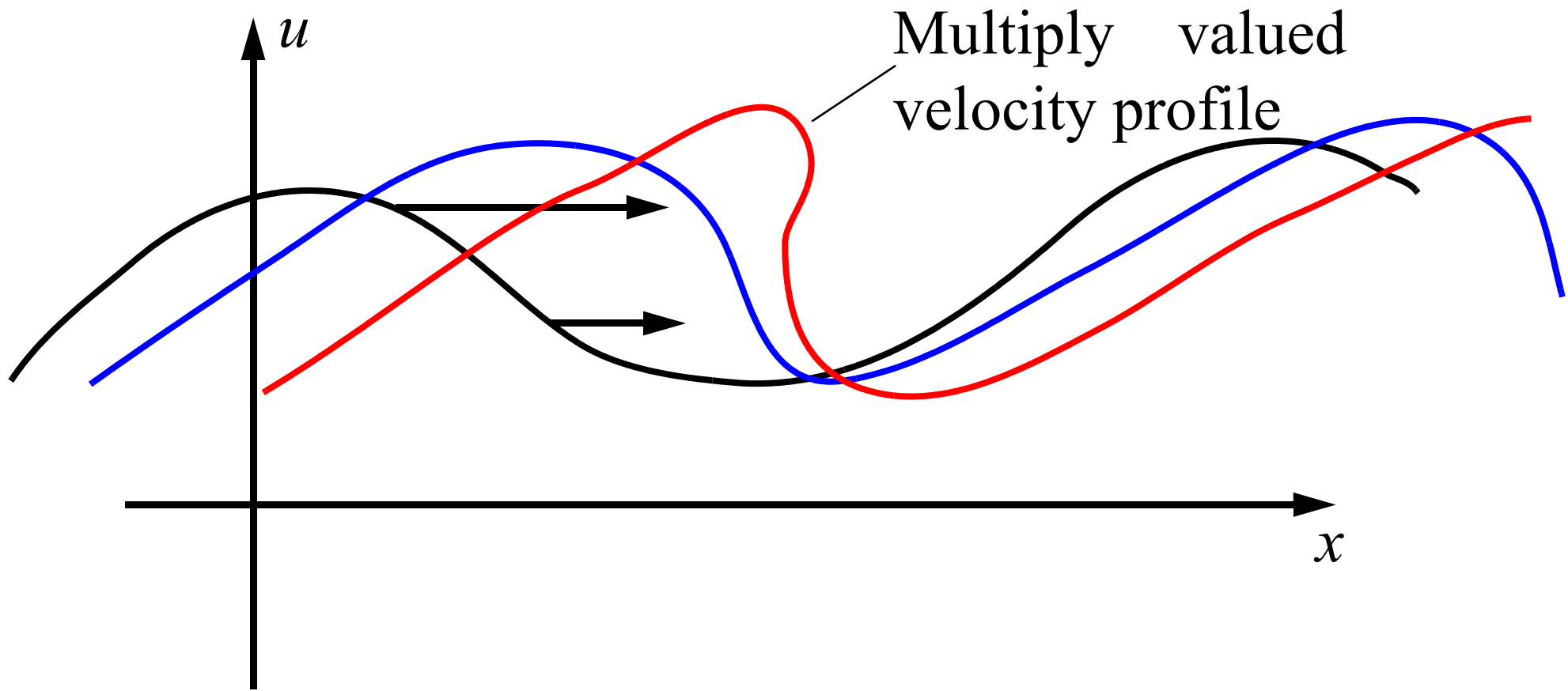
where \mathbf{u} is the velocity of the fluid and \mathbf{k} is the wave vector. In one dimension:

$$v = u + c_s \quad (2)$$

Consider the velocity profile in a sound wave that is non-linear:



The velocity of sound is larger at larger velocity amplitudes so that the wave profile evolves in the following way:



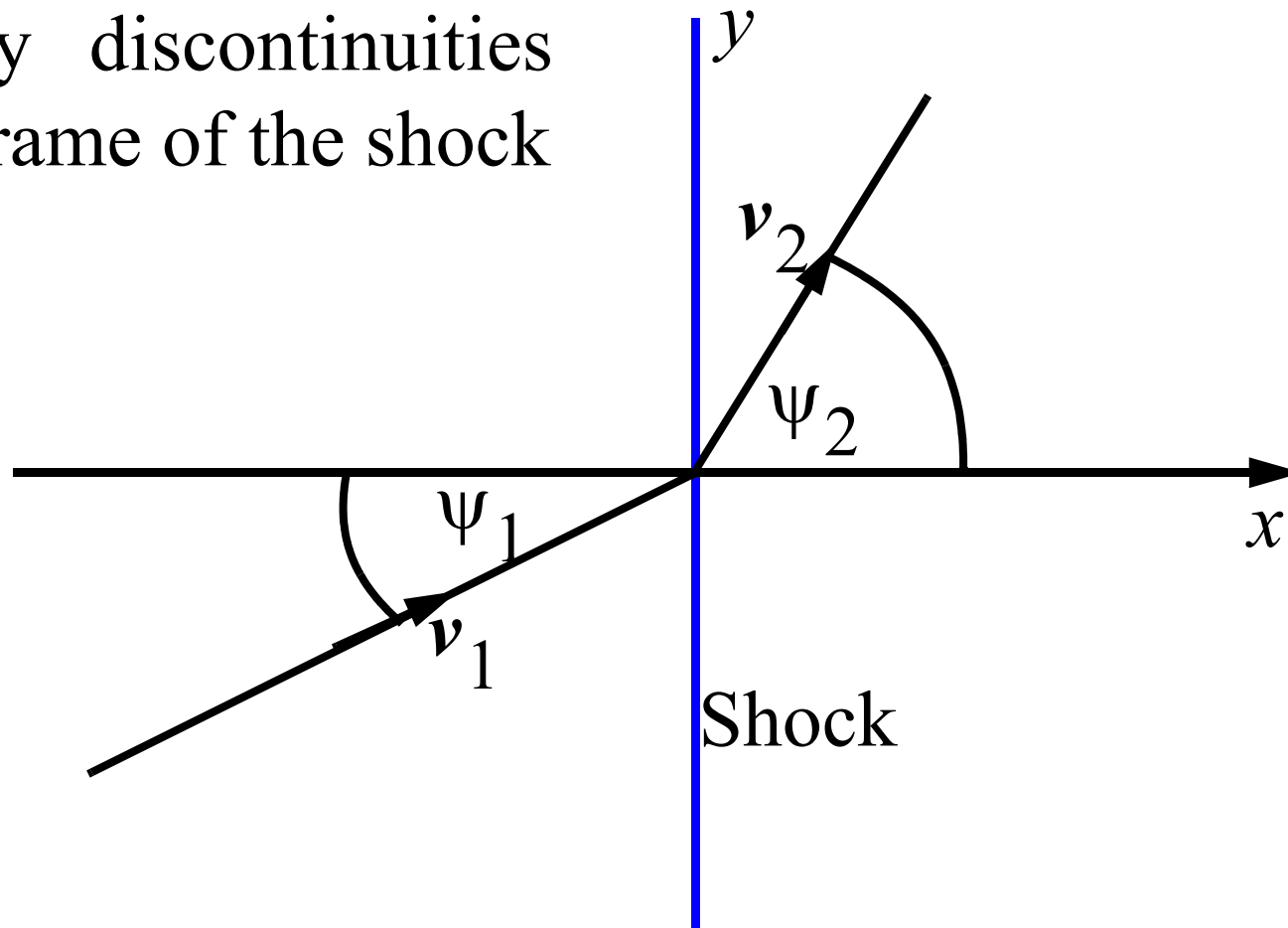
The velocity profile eventually becomes multiply valued and at this point the solution breaks down, requiring the insertion of a discontinuity into the solution.

2 Shocks as discontinuities

2.1 Basic approach

We have seen that as velocities approach the speed of sound the nonlinearity of the Euler equations forces waves to become steeper and multiple valued. At the point where the velocity profile becomes infinitely steep we intervene and insert a surface of discontinuity into the fluid. Physically this discontinuity represents a region where the fluid variables are rapidly varying and we shall assess later some of the details of this region. Here we look at the consequences of energy and momentum conservation across the surface of discontinuity which we refer to as a *shock wave*. We develop equations for magnetised shocks here and extend this to magnetised shocks later in the course.

Velocity discontinuities
in the frame of the shock



The shock relations are best analysed in the frame of the shock (in which its velocity is zero). The coordinate x is normal to the shock; y is in the plane of the pre-shock velocity \mathbf{v}_1 and is along the shock plane; z is normal to x and y . The general situation depicted above is an *oblique shock*. When $v_{y,1} = v_{y,2} = 0$ the shock is normal.

In general many fluid variables, specifically density, pressure and the normal component of the velocity are discontinuous at a shock.

2.2 Conservation laws satisfied at the shock

Mass flux

There is no creation of mass at the shock so that the mass flux into the shock must equal the mass flux emerging from the shock.

$$\rho_1 v_{x,1} = \rho_2 v_{x,2}$$

i.e.

$$[\rho v_x] = 0$$

where the square brackets refer the jump in a variable across the shock.

Momentum Flux

Likewise there is no creation of momentum at the shock front and the x and y components of the momentum flux are the same on both sides of the shock. The momentum flux

$$\Pi_{ij} = \rho v_i v_j + p \delta_{ij} \quad (3)$$

so that the flux of x -momentum normal to the shock is:

$$\Pi_{xx} = \rho_1 v_{x,1}^2 + p_1 = \rho_2 v_{x,2}^2 + p_2 \quad (4)$$

and the flux of y -momentum normal to the shock

$$\Pi_{xy} = \rho_1 v_{y,1} v_{x,1} = \rho_2 v_{y,2} v_{x,2} \quad (5)$$

Energy Flux

The energy flux into and out of either side of the shock is

$$F_{E, x} = \rho v_x \left(\frac{1}{2} v^2 + h \right) \quad (6)$$

so that

$$\rho_1 v_{x, 1} \left(\frac{1}{2} v_1^2 + h_1 \right) = \rho_2 v_{x, 2} \left(\frac{1}{2} v_2^2 + h_2 \right)$$

Summary

The above equations all collected into one place are:

$$\begin{aligned}\rho_1 v_{x,1} &= \rho_2 v_{x,2} \\ \rho_1 v_{x,1}^2 + p_1 &= \rho_2 v_{x,2}^2 + p_2 \\ \rho_1 v_{y,1} v_{x,1} &= \rho_2 v_{y,2} v_{x,2} \\ \rho_1 v_{x,1} \left(\frac{1}{2} v_1^2 + h_1 \right) &= \rho_2 v_{x,2} \left(\frac{1}{2} v_2^2 + h_2 \right)\end{aligned}\tag{7}$$

2.3 Two types of discontinuity

Tangential or contact discontinuity

The first type of discontinuity is a *tangential discontinuity* or *contact discontinuity*. This occurs when there is no mass flux across the surface, i.e.

$$\rho_1 v_{x,1} = \rho_2 v_{x,2} = 0 \quad (8)$$

For non-zero densities

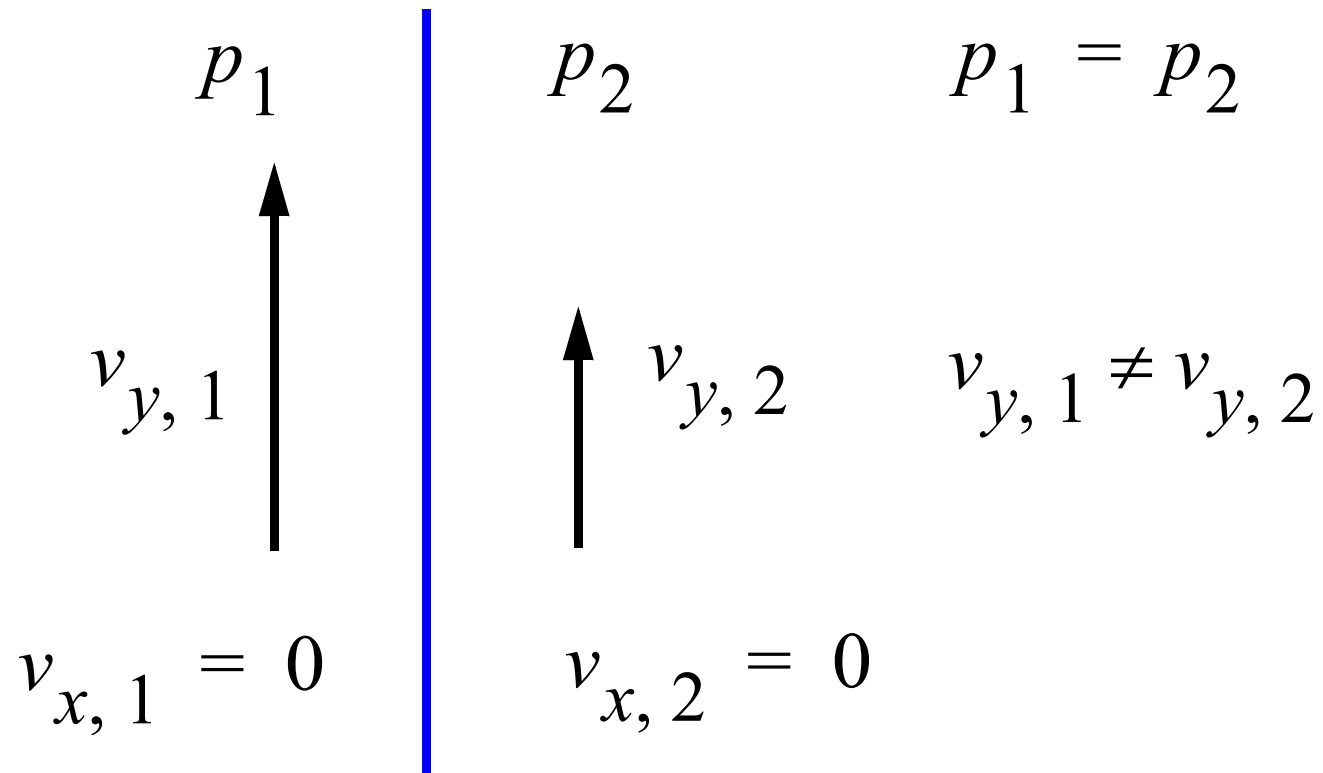
$$v_{x,1} = v_{x,2} = 0 \quad (9)$$

Since this is the case then the continuity of the y -component of momentum can be satisfied with

$$v_{y,1} \neq v_{y,2} \quad (10)$$

The xx -component of the momentum flux is continuous if $p_1 = p_2$

This situation is depicted in the following figure:



Such a situation is unstable to the *Kelvin-Helmholtz instability*.

2.4 Shock discontinuity

In this case there is a non-zero mass flux across the discontinuity.

The continuity of the xy -component of momentum flux implies that

$$v_{y, 1} = v_{y, 2} \quad (11)$$

i.e. the component of velocity tangential to the shock is conserved.

Now consider the continuity of energy flux which can be expressed in the form:

$$\begin{aligned} \rho_1 v_{x,1} \left(\frac{1}{2} (v_{x,1}^2 + v_{y,1}^2) + h_1 \right) \\ = \rho_2 v_{x,2} \left(\frac{1}{2} (v_{x,2}^2 + v_{y,2}^2) + h_2 \right) \end{aligned} \quad (12)$$

Since $[\rho v_x] = 0$ and $[v_y] = 0$ then conservation of energy can be expressed in the form:

$$\frac{1}{2} v_{x,1}^2 + h_1 = \frac{1}{2} v_{x,2}^2 + h_2 \quad (13)$$

Hence, the complete set of equations for shocks can be expressed in the form

$$\begin{aligned}
 \rho_1 v_{x,1} &= \rho_2 v_{x,2} = j = \text{Mass Flux} \\
 p_1 + \rho_1 v_{x,1}^2 &= p_2 + \rho_2 v_{x,2}^2 && \text{x-Momentum flux} \\
 v_{y,1} &= v_{y,2} && \text{y-Momentum flux} \\
 h_1 + \frac{1}{2} v_{x,1}^2 &= h_2 + \frac{1}{2} v_{x,2}^2 && \text{Energy Flux}
 \end{aligned} \tag{14}$$

2.5 The shock adiabat (the Rankine-Hugoniot equations)

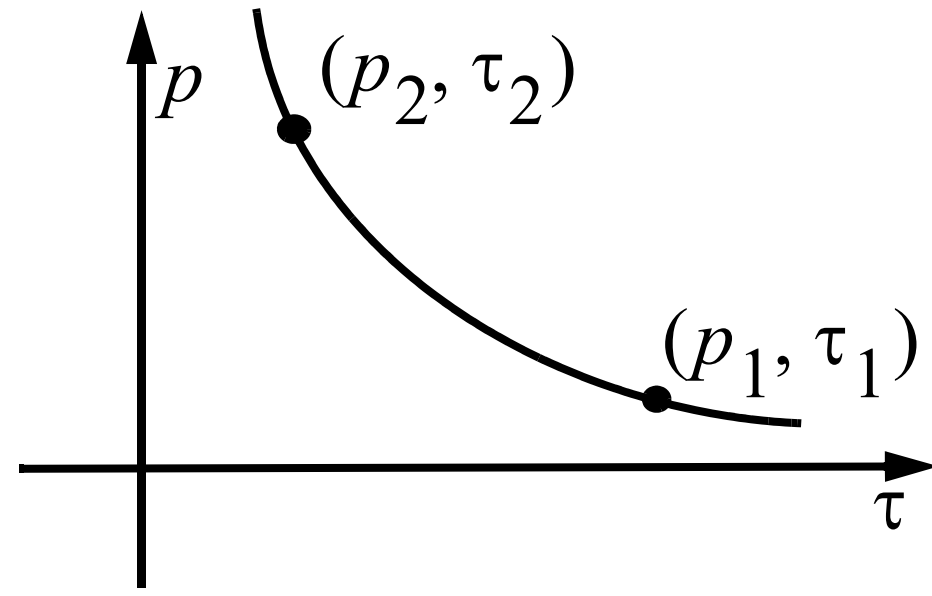
Introduce

$$\text{Specific Volume} = \tau = \frac{1}{\rho} \tag{15}$$

For a perfect gas there is an adiabatic relationship between pressure and specific volume, viz.

$$P\tau^\gamma = \text{constant} \quad (16)$$

Similarly, there is a relationship between P and τ for shocks - the shock adiabat.



The purpose of the following is to derive the shock adiabat.

Continuity of mass flux implies

$$v_{x, 1} = \frac{j}{\rho_1} = j\tau_1 \quad v_{x, 2} = j\tau_2 \quad (17)$$

Using these expressions we can write the xx component of the momentum flux as:

$$p + \rho v_x^2 = p + \frac{1}{\tau} j^2 \tau^2 = p + j^2 \tau \quad (18)$$

Substitute these expressions into the momentum flux equation allows us to solve for j :

$$\begin{aligned} p_1 + j^2 \tau_1 &= p_2 + j^2 \tau_2 \\ \Rightarrow j^2 &= \frac{p_2 - p_1}{\tau_1 - \tau_2} = -\frac{\Delta p}{\Delta \tau} \end{aligned} \tag{19}$$

i.e. the mass flux through the shock is determined by the difference in pressures and specific volumes.

Velocity difference

We have for the velocities on either side of the shock:

$$v_{x, 1} = j \tau_1 \quad v_{x, 2} = j \tau_2 \tag{20}$$

Hence the velocity difference is given by:

$$v_{x, 1} - v_{x, 2} = j(\tau_1 - \tau_2) \quad (21)$$

and we can write the mass flux as

$$j^2 = \frac{(v_{x, 1} - v_{x, 2})^2}{(\tau_1 - \tau_2)^2} = \frac{p_2 - p_1}{\tau_1 - \tau_2} \quad (22)$$

Hence the velocity difference is given by:

$$\begin{aligned} v_{x, 1} - v_{x, 2} &= \sqrt{(p_2 - p_1)(\tau_1 - \tau_2)} \\ &= \sqrt{-\Delta p \Delta \tau} \\ &= \sqrt{(p_2 - p_1)(\rho_1^{-1} - \rho_2^{-1})} \end{aligned} \quad (23)$$

Hence the velocity difference is also determined by the pressure difference and the specific volume difference.

Energy equation: Enthalpy difference

The difference in enthalpy between the pre-shock and post-shock fluids can be determined from the energy equation. First note that the combination of enthalpy and velocity resulting from the continuity equations can be written in terms of the mass flux and specific volume as:

$$h + \frac{1}{2}v_x^2 = h + \frac{1}{2}j^2\tau^2 \quad (24)$$

Therefore, the energy continuity equation becomes:

$$\begin{aligned}h_1 + \frac{1}{2}j^2\tau_1^2 &= h_2 + \frac{1}{2}j^2\tau_2^2 \\ \Rightarrow h_2 - h_1 &= \frac{1}{2}j^2(\tau_1^2 - \tau_2^2)\end{aligned}\tag{25}$$

Using the expression for j^2 developed above, viz.

$$j^2 = \frac{p_2 - p_1}{\tau_1 - \tau_2}\tag{26}$$

we have

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)(\tau_1 + \tau_2)\tag{27}$$

Summary of Rankine-Hugoniot relations

$$j^2 = \frac{p_2 - p_1}{\tau_1 - \tau_2}$$

$$v_{x,1} - v_{x,2} = \sqrt{(p_2 - p_1)(\tau_1 - \tau_2)} \quad (28)$$

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)(\tau_1 + \tau_2)$$

Note one feature of shocks that is apparent from the above. We require the sign of $\tau_1 - \tau_2$ to be the same as that of $p_2 - p_1$ for the Rankine-Hugoniot relations to be physically valid. Hence if

the pressure increases ($p_2 > p_1$) then so also should the density ($\tau_1 > \tau_2 \Rightarrow \rho_2 > \rho_1$). Also the specific enthalpy increases, implying an increase in temperature across a shock.

2.6 Velocity difference

The result

$$v_{x,1} - v_{x,2} = \sqrt{(p_2 - p_1)(\tau_1 - \tau_2)} \quad (29)$$

whilst derived in the shock frame is actually independent of the frame of the shock. This relationship is useful in a number of contexts.

3 The shock adiabat for a polytropic gas

In order to complete the relationship between P and τ we require an equation of state. This provides a relationship between enthalpy and pressure. Since

$$h = \frac{\varepsilon + p}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{\gamma}{\gamma - 1} p \tau \quad (30)$$

for a polytropic gas, then the last of the RH equations implies

$$\begin{aligned} \frac{\gamma}{\gamma - 1} (p_2 \tau_2 - p_1 \tau_1) &= \frac{1}{2} (p_2 - p_1) (\tau_1 + \tau_2) \\ \Rightarrow \frac{\tau_2}{\tau_1} &= \frac{\rho_1}{\rho_2} = \frac{v_{x,2}}{v_{x,1}} = \frac{(\gamma + 1)p_1 + (\gamma - 1)p_2}{(\gamma - 1)p_1 + (\gamma + 1)p_2} \end{aligned} \quad (31)$$

This equation is the shock adiabat we have been aiming for. As one can easily see, once p_1 , p_2 and τ_1 are known then τ_2 is determined.

The inverse of the above relationship is:

$$\frac{p_2}{p_1} = \frac{(\gamma + 1)\tau_1 - (\gamma - 1)\tau_2}{(\gamma + 1)\tau_2 - (\gamma - 1)\tau_1} \quad (32)$$

and this equation tells us that if τ_1 , τ_2 and p_1 are known then p_2 is also determined.

3.1 Temperature

For an ideal gas:

$$p = \frac{\rho k T}{\mu m_p} \Rightarrow T = \frac{\mu m_p}{k} \left(\frac{p}{\rho} \right) \quad (33)$$

Hence,

$$\begin{aligned} \frac{T_2}{T_1} &= \frac{p_2 / \rho_2}{p_1 / \rho_1} = \frac{p_2 \tau_2}{p_1 \tau_1} \\ &= \frac{p_2 (\gamma + 1) p_1 + (\gamma - 1) p_2}{p_1 (\gamma - 1) p_1 + (\gamma + 1) p_2} \end{aligned} \quad (34)$$

3.2 Pre- and post-shock velocities

Use

$$\begin{aligned} j^2 &= \frac{p_2 - p_1}{\tau_1 - \tau_2} \\ &= \frac{1}{\tau_1} \frac{(p_2 - p_1)}{1 - \tau_2/\tau_1} \\ &= \frac{(\gamma - 1)p_1 + (\gamma + 1)p_2}{2\tau_1} \end{aligned} \tag{35}$$

To determine the velocity $v_{x, 1}$ we use

$$\begin{aligned} v_{x, 1}^2 &= j^2 \tau_1^2 = \tau_1 \times j^2 \tau_1 \\ &= \frac{(\gamma - 1)p_1 + (\gamma + 1)p_2}{2\rho_1} \end{aligned} \tag{36}$$

$$= \frac{c_{s, 1}^2}{2\gamma} \left[(\gamma - 1) + (\gamma + 1) \frac{p_2}{p_1} \right]$$

Similarly (just swap 1 and 2)

$$v_{x, 2}^2 = \frac{c_{s, 2}^2}{2\gamma} \left[(\gamma - 1) + (\gamma + 1) \frac{p_1}{p_2} \right] \tag{37}$$

The pre- and post-shock Mach numbers are given by:

$$M_{x,1}^2 = \frac{(\gamma - 1) + (\gamma + 1)\frac{p_2}{p_1}}{2\gamma} \quad (38)$$

$$M_{x,2}^2 = \frac{(\gamma - 1) + (\gamma + 1)\frac{p_1}{p_2}}{2\gamma}$$

It is easy to show that for $p_2 > p_1$

$$M_{x,1}^2 > 1 \quad \text{and} \quad M_{x,2}^2 < 1 \quad (39)$$

i.e. the normal component of pre-shock velocity is supersonic and the normal component of post-shock velocity is subsonic. This reflects the dissipation which occurs at a shock discontinuity which increases the temperature at the expense of the velocity.

It is important to recognize that these constraints do not apply to the transverse component of shock velocity since this is arbitrary and conserved.

3.3 Velocity difference

We can express the velocity change across a shock in terms of the pressure difference.

Previously we had

$$\begin{aligned} v_{x, 1} - v_{x, 2} &= \sqrt{(p_2 - p_1)(\tau_1 - \tau_2)} \\ &= (p_2 - p_1)^{1/2} \tau_1^{1/2} \left(1 - \frac{\tau_2}{\tau_1}\right)^{1/2} \end{aligned} \quad (40)$$

and as we have seen, we have for a polytropic gas:

$$\frac{\tau_2}{\tau_1} = \frac{(\gamma + 1)p_1 + (\gamma - 1)p_2}{(\gamma - 1)p_1 + (\gamma + 1)p_2} \quad (41)$$

Therefore,

$$1 - \frac{\tau_2}{\tau_1} = \frac{2(p_2 - p_1)}{(\gamma - 1)p_1 + (\gamma + 1)p_2} \quad (42)$$

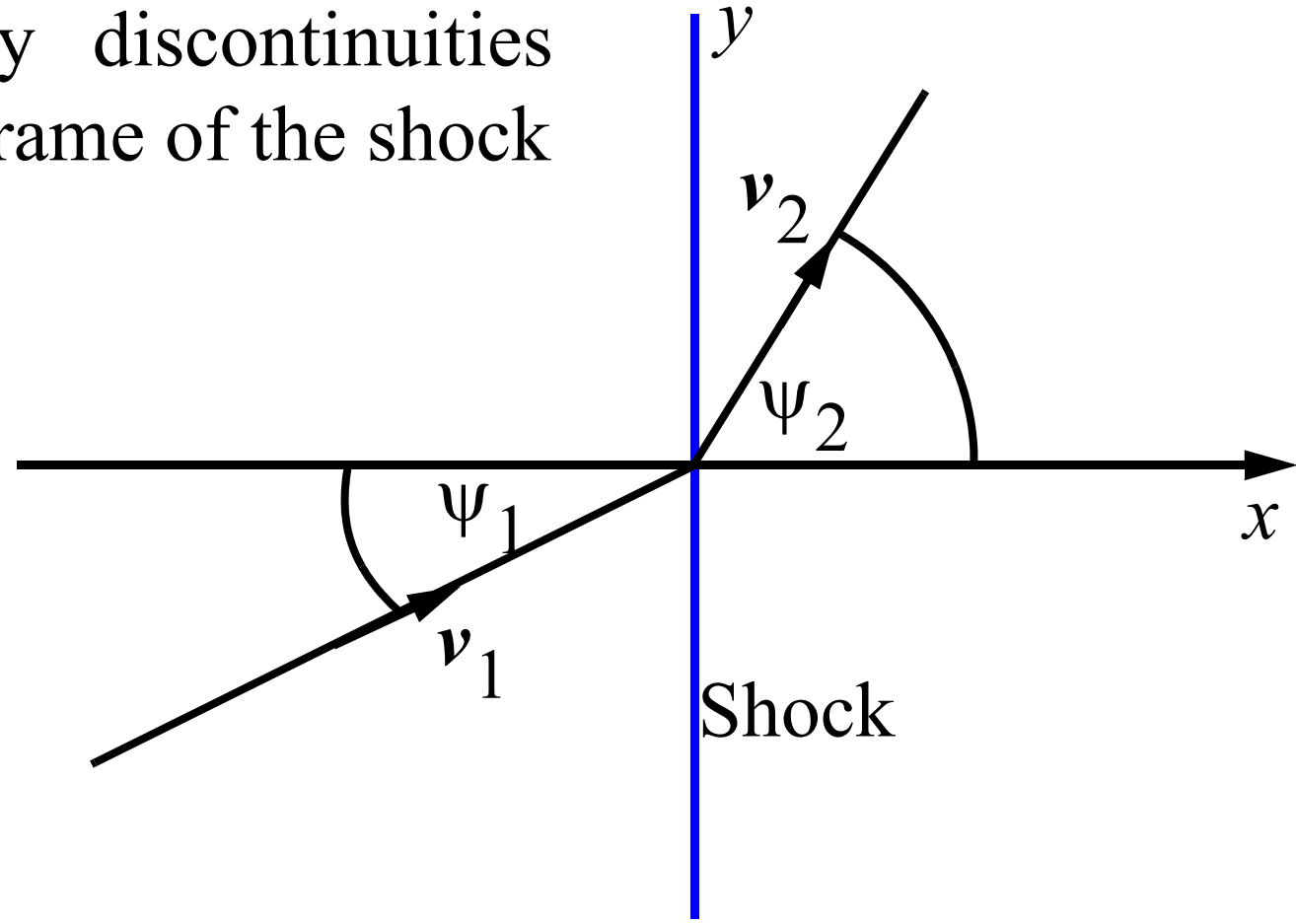
and

$$v_{x,1} - v_{x,2} = (2\tau_1)^{1/2} \frac{(p_2 - p_1)}{[(\gamma - 1)p_1 + (\gamma + 1)p_2]^{1/2}} \quad (43)$$

As before this result is independent of the shock frame.

3.4 Shock angles

Velocity discontinuities
in the frame of the shock



From the above diagram

$$\begin{aligned}\tan \psi_1 &= \frac{v_{y, 1}}{v_{x, 1}} = \frac{v_y}{v_{x, 1}} & \tan \psi_2 &= \frac{v_{y, 2}}{v_{x, 2}} = \frac{v_y}{v_{x, 2}} \\ & & \Rightarrow \frac{\tan \psi_2}{\tan \psi_1} &= \frac{v_{x, 1}}{v_{x, 2}}\end{aligned}\tag{44}$$

Now it is readily shown from the above that

$$\frac{v_{x, 1}}{v_{x, 2}} > 1 \Rightarrow \psi_2 > \psi_1\tag{45}$$

i.e. the fluid velocity bends away from the normal as shown in the diagram.

4 Strong shocks

The “strength” of a shock is characterised by the pressure ratio p_2/p_1 .

As this ratio becomes infinite one can see from the above expression for ρ_1/ρ_2 that

$$\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma + 1}{\gamma - 1} \quad (46)$$

and that this is finite. For a monatomic gas this ratio is 4. This limiting case is often used in astrophysics since shocks are often quite strong.

Note also the corresponding limit for the velocities:

$$\frac{v_{x,1}}{v_{x,2}} \rightarrow \frac{\gamma + 1}{\gamma - 1} = 4 \quad \text{for} \quad \gamma = \frac{5}{3} \quad (47)$$

5 Weak shock waves

The properties of weak shocks, as well as being interesting in themselves, can be used to derive interesting properties that are valid for shock waves in general.

The following is valid for any equation of state.

We take the thermodynamic variable specific enthalpy to be a function of state of the specific entropy and the pressure, i.e.

$$h = h(P, s) \quad (48)$$

Consider first the enthalpy jump in a shock. We expand to first order in the entropy and up to third order in the pressure.

$$\begin{aligned} h_2 - h_1 = & \left. \frac{\partial h}{\partial s} \right|_p (s_2 - s_1) + \left. \frac{\partial h}{\partial p} \right|_s (p_2 - p_1) \\ & + \frac{1}{2} \left. \frac{\partial^2 h}{\partial p^2} \right|_s (p_2 - p_1)^2 + \frac{1}{6} \left. \frac{\partial^3 h}{\partial p^3} \right|_s (p_2 - p_1)^3 \end{aligned} \quad (49)$$

Now use the thermodynamic relation between enthalpy, entropy and pressure

$$dh = kTds + \tau dp \quad (50)$$

This implies

$$\begin{aligned}\left. \frac{\partial h}{\partial s} \right|_p &= kT & \left. \frac{\partial h}{\partial p} \right|_s &= \tau \\ \left. \frac{\partial^2 h}{\partial p^2} \right|_s &= \left. \frac{\partial \tau}{\partial p} \right|_s & \left. \frac{\partial^3 h}{\partial p^3} \right|_s &= \left. \frac{\partial^2 \tau}{\partial p^2} \right|_s\end{aligned}\tag{51}$$

and therefore the enthalpy jump is given by:

$$\begin{aligned}h_2 - h_1 &= kT_1(s_2 - s_1) + \tau_1(p_2 - p_1) \\ &+ \frac{1}{2} \left. \frac{\partial \tau}{\partial p} \right|_s (p_2 - p_1)^2 + \frac{1}{6} \left. \frac{\partial^2 \tau}{\partial p^2} \right|_s (p_2 - p_1)^3\end{aligned}\tag{52}$$

We similarly take the specific volume to be a function of (p, s) and expand:

$$\begin{aligned}\tau &= \tau(p, s) \\ \tau_2 - \tau_1 &= \left. \frac{\partial \tau}{\partial p} \right|_s (p_2 - p_1) + \frac{1}{2} \left. \frac{\partial^2 \tau}{\partial p^2} \right|_s (p_2 - p_1)^2 \\ &\quad + O(s_2 - s_1) + O((s_2 - s_1)(p_2 - p_1))\end{aligned}\tag{53}$$

The last terms turn out to be unimportant. We now substitute into the relationship between enthalpy and pressure jumps, viz

$$h_2 - h_1 = \frac{1}{2}(p_2 - p_1)(\tau_1 + \tau_2)\tag{54}$$

writing

$$\tau_1 + \tau_2 = 2\tau_1 + (\tau_2 - \tau_1) \quad (55)$$

and putting all the above together, we obtain for this particular RH equation:

$$\begin{aligned} kT(s_2 - s_1) + \tau_1 \Delta p + \frac{1}{2} \tau_P (\Delta p)^2 + \frac{1}{6} \tau_{PP} (\Delta p)^3 \\ = \tau_1 \Delta p + \frac{1}{2} \tau_P (\Delta p)^2 + \frac{1}{4} \tau_{PP} (\Delta p)^3 \end{aligned} \quad (56)$$

Terms in Δp cancel out up until the third power, and this is why we need to keep this many terms in Δp but not in Δs . The result is

$$s_2 - s_1 = \frac{1}{12kT} \left. \frac{\partial^2 \tau}{\partial p^2} \right|_s (p_2 - p_1)^3 \quad (57)$$

Normally the quantity

$$\left. \frac{\partial^2 \tau}{\partial p^2} \right|_s > 0 \quad (58)$$

e.g. for $p = K\rho^\gamma$

$$\left. \frac{\partial^2 \tau}{\partial p^2} \right|_s = \frac{\gamma + 1}{\gamma^2} \tau p^{-2} \quad (59)$$

Thus the entropy only increases a weak shock if

$$p_2 - p_1 > 0 \quad (60)$$

The second law of thermodynamics tells us that entropy always increases so that for a weak shock

$$p_2 > p_1 \quad (61)$$

This relationship holds for a shock of arbitrary strength. However, the proof is rather involved. (See Landau & Lifshitz, Fluid Mechanics.)

5.1 Velocity of a weak shock

The mass flux is given by

$$j^2 = -\frac{\Delta p}{\Delta \tau} \approx -\frac{\Delta p}{\left. \frac{\partial \tau}{\partial p} \right|_s \Delta p} = -\frac{1}{\left. \frac{\partial \tau}{\partial p} \right|_s} \quad (62)$$

The derivative

$$\left. \frac{\partial \tau}{\partial p} \right|_s = \frac{\partial \rho^{-1}}{\partial p} = -\frac{1}{\rho^2} \frac{\partial \rho}{\partial p} = -\frac{1}{\rho^2 c_s^2} \quad (63)$$

Therefore

$$j = \rho_1 v_{x,1} = \rho_2 v_{x,2} \approx \rho_1 c_s \quad (64)$$

and to first order

$$v_{x, 1} = v_{x, 2} = c_s \quad (65)$$

i.e. the velocities of the pre- and post-shock gas are equal to the sound speed, or in other words the shock travels at the sound speed with respect to the gas on either side of the shock.

These limits are evident for the polytropic case discussed above.