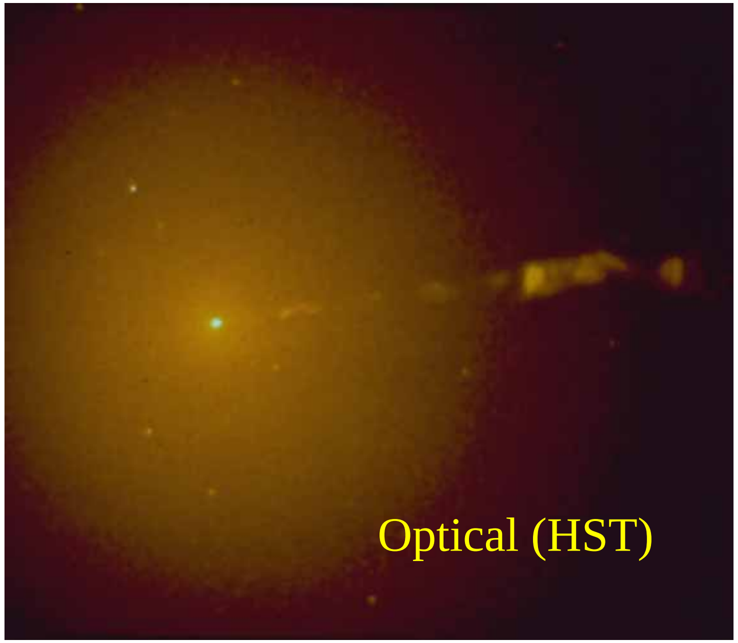
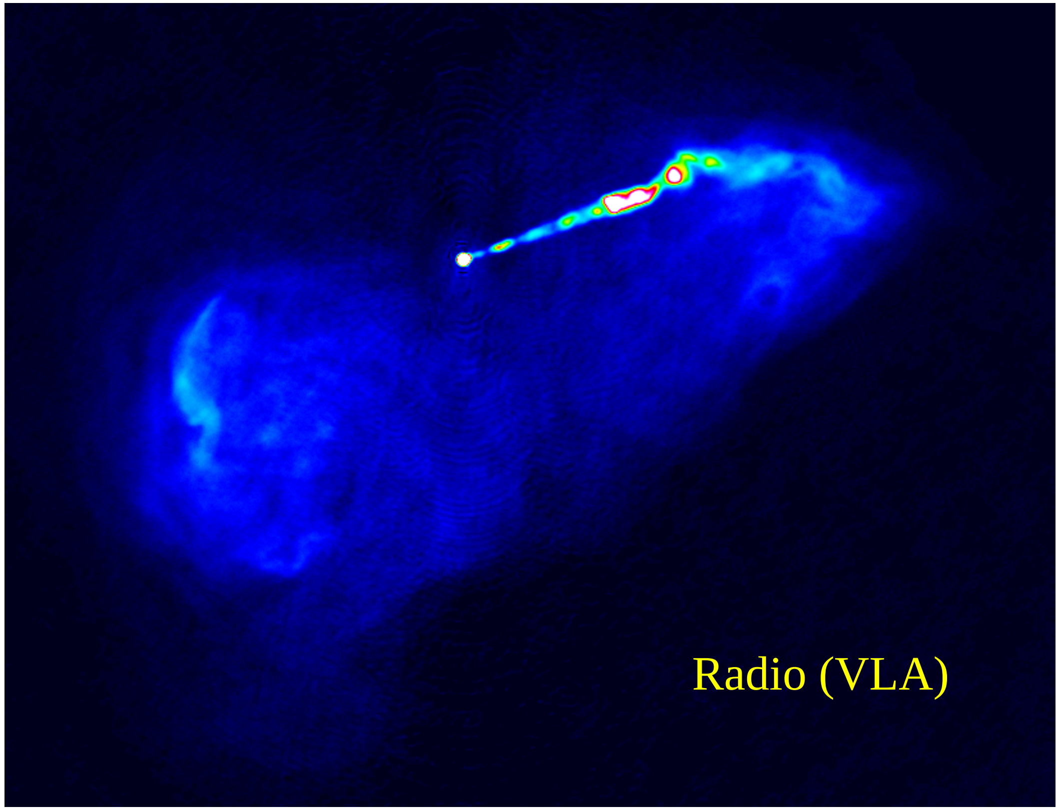
Relativistic Fluids

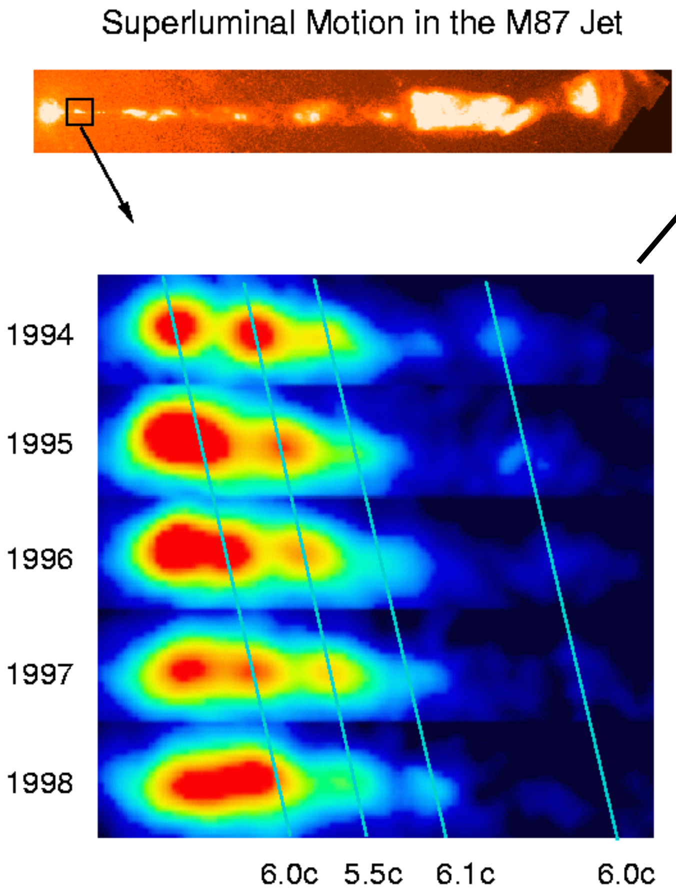
1 Relativistic fluids in astronomy



The inner part of M87



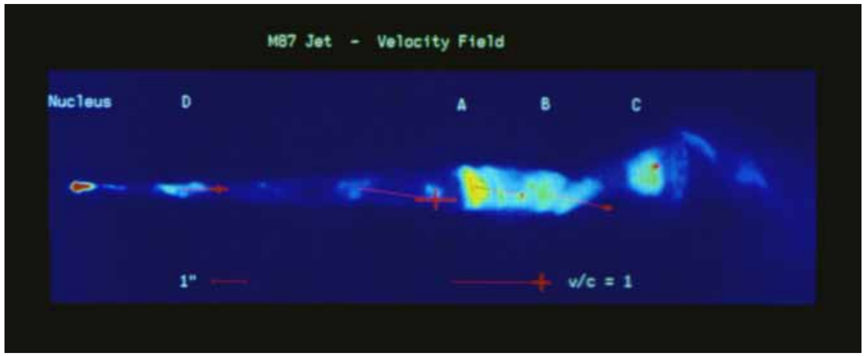
Evidence for relativistic motion



Motion of the knots in the M87 jet indicates apparent velocities in excess of the speed of light

Credits: Biretta and colleagues

Large scale component velocities are of the order of 0.5-1.0 c



2 Some relevant aspects of relativity

2.1 Four vectors and tensors

Special relativity is based on the geometry of space-time described by the metric

$$\eta_{\alpha\beta} = \mathrm{diag}\,(-1,1,1,1)$$

The interval between two points is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where

$$x^0 = ct$$

Indices

Latin go from 1 to 3; Greek from 0 to 3

Timelike and spacelike

Displacements are ***timelike*** if $ds^2 < 0$ and ***spacelike*** if $ds^2 > 0$. For timelike displacements, the proper time, τ, is given by

$$c^2 d\tau^2 = -ds^2$$

Lorentz factor

The Lorentz factor is defined by

$$\Gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Note that we use Γ for the Lorentz factor corresponding to the bulk motion of the fluid. We use γ for the Lorentz factors of the particles making up the fluid.

Since, for a timelike displacement (the displacement of a material particle)

$$\begin{aligned} c^2 d\tau^2 &= -ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= (dx^0)^2\left[1 - \left(\frac{dx^1}{dx^0}\right)^2 - \left(\frac{dx^2}{dx^0}\right)^2 - \left(\frac{dx^3}{dx^0}\right)^2\right] \\ &= c^2 dt^2\left[1 - \frac{v^2}{c^2}\right] = \Gamma^{-2} c^2 dt^2 \end{aligned}$$

then

$$c\,d\tau = \Gamma^{-1} c\,dt$$

Signature

Note the signature of the metric $(-,+,+,+)$. Other expositions of special or general relativity adopt $(+,-,-,-)$.

The main 4–vector that we deal with here is the fluid velocity,

$$u^\alpha = \frac{dx^\alpha}{c\,d\tau} = \Gamma\frac{dx^\alpha}{c\,dt} = \left(\Gamma, \Gamma\frac{v^i}{c}\right)$$

where

$$v^i = \frac{dx^i}{dt}$$

and x^i are the coordinates of a fluid element through space time.

2.2 Raising and lowering indices

Indices are raised and lowered by means of the Minkowski tensor. Some results to remember are as follows.

The inverse of the Minkowski tensor, $\eta^{\alpha\beta}$ is defined by:

$$\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

$$\eta^{\alpha\beta} = (-1, 1, 1, 1)$$

Indices are raised and lowered by the Minkowski tensor

$$v^{\alpha} = \eta^{\alpha\beta}v_{\beta} \qquad v_{\alpha} = \eta_{\alpha\beta}v^{\beta}$$

3 The stress-energy tensor

3.1 Definition

The stress–energy tensor is:

$$T^{\alpha\beta} = \begin{bmatrix} T^{00} & T^{0i} \\ T^{i0} & T^{ij} \end{bmatrix}$$

where

$$\begin{aligned} T^{00} &= \text{Energy density (incorporating rest mass energy etc.)} \\ cT^{0i} &= \text{Flux of energy in the } \alpha \text{ direction} \\ T^{i0} &= T^{0i} \\ T^{ij} &= \text{Flux of the } i \text{ component of momentum in the } j \text{ direction} \end{aligned}$$

We also have

$$\frac{1}{c}T^{i0} = \alpha \text{ component of momentum density}$$

$$= \frac{1}{c^2} \times \text{Flux of energy in the } i \text{ direction}$$

The relationship of T^{0i} to both momentum and energy

Why should we have these two different roles for the components T^{0i}? Remember that in special relativity, mass and energy are related by

$$E = mc^2$$

Therefore a flux of energy is equivalent to c^2 times a flux of mass. But the mass flux is ρv^i and this is just the momentum density. Hence

$$\text{Energy flux} = c^2 \times \text{Momentum density}$$

and

$$\text{Momentum density} = \frac{1}{c^2} \times \text{Energy flux}$$

3.2 Stress energy tensor for a perfect fluid

The above characterisation of the stress-energy tensor is valid in general. For a perfect fluid in which the pressure is isotropic and normal to any surface we develop an expression for the stress-energy tensor as follows.

Consider the rest-frame of the fluid. This is defined as that frame in which

$$u^\alpha = (1, 0, 0, 0)$$

In this frame,

$$\begin{aligned} \text{Energy density} &= e \\ \text{Energy flux} &= 0 \\ \text{Momentum flux density} &= p\delta^{ij} \end{aligned}$$

Hence,

$$T^{\alpha\beta} = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} = \begin{bmatrix} e+p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

This can be covariantly expressed as

$$T^{\alpha\beta} = (e+p)u^{\alpha}u^{\beta} + p\eta^{\alpha\beta}$$

Relativistic enthalpy

The quantity

$$w = e + p$$

is called the relativistic enthalpy in analogy to the corresponding non-relativistic expression

$$\rho h = \varepsilon + p$$

Note that the relativistic expression contains the rest-mass energy. Often we expression the relativistic enthalpy in the form:

$$w = \rho c^2 + \varepsilon + p$$

where ρc^2 is the rest-mass energy density and $\varepsilon = e - \rho c^2$ is the internal energy.

3.3 Expressions for the components of $T^{\alpha\beta}$ in an arbitrary frame

The various components of the stress-energy tensor can then be expressed as:

$$T^{00} = (e+p)u^0u^0 - p = \Gamma^2(e+p) - p$$

$$T^{0i} = (e+p)\Gamma^2\frac{v^i}{c} \Rightarrow cT^{0i} = (e+p)\Gamma^2 v^i \qquad \frac{1}{c}T^{0i} = \left(\frac{e+p}{c^2}\right)\Gamma^2 v^i$$

$$T^{ij} = (e+p)\Gamma^2\frac{v^i v^j}{c^2} + p\delta^{ij}$$

4 The relativistic fluid equations

4.1 Energy and momentum equations

The relativistic fluid equations are

$$T^{\alpha\beta}{}_{,\beta} \equiv \frac{\partial T^{\alpha\beta}}{\partial x_\beta} = 0$$

To see why these are the equations, let us examine the different equations corresponding to $\alpha = 0$ and $\alpha = i$.

Time component

$$\alpha = 0 \Rightarrow T^{00}{}_{,0} + T^{0j}{}_{,j} = 0$$

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0j}}{\partial x_j} = 0$$

$$\Rightarrow \frac{\partial T^{00}}{\partial t} + \frac{\partial (cT^{0j})}{\partial x_j} = 0$$

In words:

$$\frac{\partial(\text{Energy density})}{\partial t} + \text{divergence}(\text{Energy flux}) = 0$$

Spatial component

$$\alpha = i \Rightarrow T^{i0}{}_{,0} + T^{ij}{}_{,j} = 0$$

$$\frac{1}{c}\frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x_j} = 0$$

$$\frac{\partial(c^{-1}T^{i0})}{\partial t} + \frac{\partial T^{ij}}{\partial x_j} = 0$$

In words:

$$\frac{\partial(\text{Momentum density})}{\partial t} + \text{divergence}(\text{Momentum flux density}) = 0$$

These are the appropriate conservation laws for energy and momentum.

4.2 The conservation of number density

Assuming that our fluid consists of particles that do not transform into other particles, then their number is conserved. Let

$$n = \text{Number of particles per unit proper volume}$$

i.e. n is the number density of particles in the rest frame of the fluid. The conservation law for number is

$$\frac{\partial(nu^\alpha)}{\partial x^\alpha} = 0 \Rightarrow \frac{1}{c}\frac{\partial(\Gamma n)}{\partial t} + \frac{\partial(\Gamma n v^i/c)}{\partial x_i}$$

$$\Rightarrow \frac{\partial(\Gamma n)}{\partial t} + \frac{\partial(\Gamma n v^i)}{\partial x_i} = 0$$

This makes sense since the number density in an arbitrary frame is Γn.

Rest mass density

Let m be the average rest mass of particles in the fluid. The rest-mass energy density in the rest frame is nmc^2. The rest-mass energy density in an arbitrary frame is

$$\rho c^2 = \Gamma n m c^2$$

5 Non-relativistic limit

The non-relativistic limit is instructive since it confirms

- That we have the right equations
- Clarifies the relationship between energy flux and momentum density

5.1 Nonrelativistic limits of the components of the energy-momentum tensor

In proceeding to the non-relativistic limit, we express the energy density in the form:

$$e = nmc^2 + \varepsilon$$

where ε is the internal energy density.

The T^{00} component

$$
\begin{aligned}
T^{00} &= \Gamma^2(e+p) - p \\
&= \Gamma^2(nmc^2 + \varepsilon + p) - p \\
&= \Gamma(\Gamma nmc^2) + \Gamma^2(\varepsilon + p) - p
\end{aligned}
$$

Now

$$
\begin{aligned}
\Gamma &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \\
\Gamma^2 &= 1 + \frac{v^2}{c^2}
\end{aligned}
$$

So

$$
T^{00} \approx \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\rho c^2 + (\varepsilon + p) - p = \rho c^2 + \left(\frac{1}{2}\rho v^2 + \varepsilon\right)
$$

So we can see that the energy density in an arbitrary frame, breaks up into the rest mass energy density plus the kinetic energy density plus the internal energy density.

The components T^{0i}

$$
\begin{aligned}
T^{0i} &= (e+p)\Gamma^2\frac{v^i}{c} = (nmc^2+\varepsilon+p)\Gamma^2\frac{v^i}{c} \\
&= \Gamma\rho c^2\frac{v^i}{c} + \Gamma^2(\varepsilon+p)\frac{v^i}{c} \\
&\approx \left(1+\frac{1}{2}\frac{v^2}{c^2}\right)\rho c^2\frac{v^i}{c} + \left(1+\frac{v^2}{c^2}\right)(\varepsilon+p)\frac{v^i}{c} \\
&= \rho c^2\frac{v^i}{c} + \frac{1}{2}\rho v^2\frac{v^i}{c} + (\varepsilon+p)\frac{v^i}{c}
\end{aligned}
$$

$$
\text{Momentum density} = \frac{1}{c}T^{0i} = \rho v^i + \left[\frac{1}{2}\rho\left(\frac{v}{c}\right)^2 v^i + \frac{(\varepsilon+p)}{c^2}v^i\right]
$$

$$
\text{Energy flux density} = cT^{0i} = \rho c^2 v^i + \left[\frac{1}{2}\rho v^2 v^i + (\varepsilon+p)v^i\right]
$$

The components T^{ij}

$$T^{ij} = (e+p)\Gamma^2 \frac{v^i v^j}{c^2} + p\delta^{ij} = (nmc^2 + \varepsilon + p)\Gamma^2 \frac{v^i v^j}{c^2} + p\delta^{ij}$$

$$\approx \Gamma \rho c^2 \frac{v^i v^j}{c^2} + \Gamma^2(\varepsilon + p)\frac{v^i v^j}{c^2} + p\delta^{ij}$$

$$= [\rho v^i v^j + p\delta^{ij}] + (\varepsilon + p)\frac{v^i v^j}{c^2} + \text{Higher order in } \frac{v^2}{c^2}$$

The term Γnmv^i

This originates from the continuity equation for the number of particles.

$$\Gamma nmu^i = \rho v^i$$

5.2 Non-relativistic equations

Continuity equation

The conservation equation for rest-mass

$$\frac{\partial(\Gamma nm)}{\partial t} + \frac{\partial(\Gamma nmv^i)}{\partial x_i} = 0$$

is, without approximation:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} = 0$$

Momentum equation

The relativistic form

$$\frac{\partial(c^{-1}T^{i0})}{\partial t} + \frac{\partial T^{ij}}{\partial x_j} = 0$$

becomes

$$\frac{\partial\left(\rho v^i + \left[\frac{1}{2}\rho\left(\frac{v}{c}\right)^2 v^i + \frac{(\varepsilon + p)}{c^2} v^i\right]\right)}{\partial t} + \frac{\partial\left([\rho v^i v^j + p\delta^{ij}] + (\varepsilon + p)\frac{v^i v^j}{c^2}\right)}{\partial x^j} = 0$$

The leading terms are the usual non-relativistic terms, the others are of order $(v/c)^2$ by comparison. Hence, the momentum equations become:

$$\frac{\partial(\rho v^i)}{\partial t} + \frac{\partial[\rho v^i v^j + p\delta^{ij}]}{\partial x^j} = 0$$

The energy equation

The relativistic form is:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial(cT^{0j})}{\partial x_j} = 0$$

With the non-relativistic expressions:

$$\frac{\partial\left[\rho c^2 + \left(\frac{1}{2}\rho v^2 + \varepsilon\right)\right]}{\partial t} + \frac{\partial\left[\rho c^2 v^j + \left[\frac{1}{2}\rho v^2 v^j + (\varepsilon + p)v^j\right]\right]}{\partial x^j} = 0$$

In this case we need to go beyond the zeroth order expression since this is

$$\frac{\partial(\rho c^2)}{\partial t} + \frac{\partial(\rho c^2 v^j)}{\partial x^j} = 0$$

i.e. the continuity equation for rest mass energy density.

The next order in $(v/c)^2$ gives:

$$\frac{\partial\left(\frac{1}{2}\rho v^2 + \varepsilon\right)}{\partial t} + \frac{\partial\left[\frac{1}{2}\rho v^2 v^j + (\varepsilon + p)v^j\right]}{\partial x^j} = 0$$

which is the non-relativistic form of the energy equation.

Note that both the momentum equation and the energy equation have involved the same term T^{0i}. It is the different contributions from terms of different orders in $(v/c)^2$ which have given rise to the different contributions to both the energy and momentum equations.

6 Equations of state

6.1 Ultrarelativistic particles and the distribution function

Often when we have relativistic motion, the internal composition of the fluid is relativistic, as in the case of the M87 jet. Most extragalactic jets are made up of ultrarelativistic particles with Lorentz factors much greater than unity. In this case the energy of a particle is the speed of light times the momentum, i.e.

$$E = cp$$

as it is for photons.

To derive the equation of state, we use a distribution function $f(\boldsymbol{p})$ so that

$$\begin{aligned} &\text{Number density of particles in volume } d^3p \text{ of momentum space} \\ &= f(\boldsymbol{p})d^3p \end{aligned}$$

For the so-called *perfect fluids* we are dealing with here, the distribution function is isotropic, i.e.

$$f(\boldsymbol{p}) = f(p)$$

and depends only only on the magnitude of the momentum.

6.2 The energy density and the pressure

The energy density is

$$e = \int_{p-\text{space}} E f(\boldsymbol{p}) d^3 p = 4\pi \int_0^\infty E f(p) p^2 dp = 4\pi c \int_0^\infty p^3 f(p) dp$$

The pressure tensor (equivalently, the stress tensor in the rest frame) is

$$T^{ij} = \int_{p-\text{space}} p^i v^j f(\boldsymbol{p}) d^3 p = 4\pi \int_0^\infty \frac{p^i p^j}{m} f(p) p^2 dp$$

Here we take $m = \gamma m_0$ to be the mass of the particle, γ to be the Lorentz factor and m_0 to be the rest-mass energy.

The isotropy of the distribution function guarantees the isotropy of the stress tensor. Hence

$$T^{ij} = p\delta^{ij}$$

and the value of p can be estimated from the trace, i.e.

$$3p = T^i{}_i = 4\pi\int_0^\infty \frac{p^2}{m} f(p)p^2 dp$$

Now

$$m = \frac{E}{c^2} = \frac{p}{c}$$

Hence

$$3p = 4\pi c\int_0^\infty pf(p)p^2 dp = e$$

$$\Rightarrow p = \frac{1}{3}e$$

This is of course is also true for a photon gas where $E = cp$ is exact.

7 The relativistic fluid equations

7.1 The fluid equations in terms of the 4-velocity

Let us now consider the fully relativistic equations derived from the divergence of the stress-energy tensor, i.e.

$$T^{\alpha\beta}{}_{,\beta} = 0 \qquad \text{where} \qquad T^{\alpha\beta} = wu^{\alpha}u^{\beta} + p\eta^{\alpha\beta}$$

The divergence is:

$$T^{\alpha\beta}{}_{,\beta} = u^{\alpha}(wu^{\beta})_{,\beta} + wu^{\beta}u^{\alpha}{}_{,\beta} + p^{,\alpha}$$

(Note that $p^{,\alpha} = \eta^{\alpha\beta}p_{,\beta}$)

This has a form similar to that of the non-relativistic equations. However, the term $(wu^{\beta})_{,\beta}$ is a bit anomalous. We deal with this as follows.

Take the scalar product of this equation with u^{α}. Then,

$$u_{\alpha}u^{\alpha}(wu^{\beta})_{,\beta} + wu^{\beta}u_{\alpha}u^{\alpha}{}_{,\beta} + u_{\alpha}p^{,\alpha}$$

Since

$$u^{\alpha}u_{\alpha} = -1$$

then

$$u^{\alpha}{}_{,\beta}u_{\alpha} + u^{\alpha}u_{\alpha,\beta} = 2u^{\alpha}u_{\alpha,\beta} = 0$$

Hence,

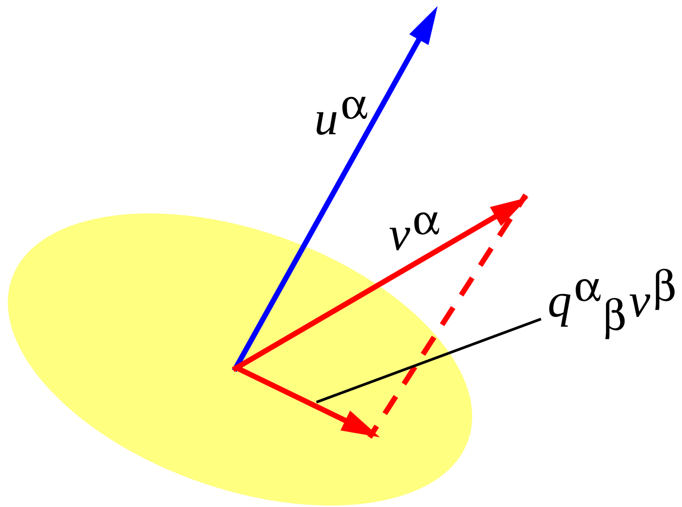
$$-(wu^{\beta})_{,\beta} + u_{\alpha}p^{,\alpha} = 0$$

i.e.

$$(wu^{\beta})_{,\beta} = u_{\beta}p^{,\beta}$$

The equations of motion of the fluid then become:

$$wu^{\beta}u^{\alpha}{}_{,\beta} + p^{,\alpha} + u^{\alpha}u_{\beta}p^{,\beta} = 0$$



Projection operator

The terms involving the pressure can be written:

$$p^{,\alpha} + u^\alpha u_\beta p^{,\beta} = (\delta^\alpha{}_\beta + u^\alpha u_\beta) p^{,\beta}$$

The tensor

$$q^\alpha{}_\beta = \delta^\alpha{}_\beta + u^\alpha u_\beta$$

is the projection operator. Every vector contracted with $q^\alpha{}_\beta$ is othorgonal to u^α.

Final form of fluid equations

Using the projection operator, we have,

$$w u^\beta u^\alpha{}_{,\beta} = -q^{\alpha\beta} p_{,\beta}$$

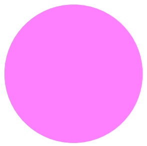
The left hand side represents the relativistic generalisation of the non-relativistic differentiation following the motion of the three–velocity,

$$\rho \frac{\partial v^i}{\partial t} + \rho v^j \frac{\partial v^i}{\partial x^j}$$

The right hand side is the negative of the pressure gradient perpendicular to the 4–velocity. Since the scalar product of these equations with u^α now leads to an identity, the above four equations are equivalent to three.

7.2 The entropy equation in relativistic fluid dynamics

The second law of thermodynamics



Comoving volume with fixed number N of baryons

σ = Entropy per baryon

n = baryon density

e = Energy density

p = pressure

T = temperature in energy units

The equation of state of all fluids, including relativistic fluids has other consequences that can be derived by consideration of the second law of thermodynamics.

Consider the second law of thermodynamics in the form

$$TdS = dU + pdV$$

where, T is the temperature (in energy units), U is the internal energy and V the volume of a comoving element of fluid which contains a fixed number, N of baryons. This equations applies in the rest frame of the fluid.

Why concentrate on baryons? In most fluids, except under the most extreme of conditions, the baryon number is conserved. For example, even when nuclear reactions are being considered, the number of baryons is conserved.

Also, the rest-mass energy of such a volume is constant so that we can replace the internal energy with the total energy, E. This appropriate for a relativistic approach. The entropy equation then becomes

$$TdS = dE + pdV$$

In terms of the variables in the above diagram

$$S = N\sigma \qquad E = N \times \text{energy per baryon} = N\frac{e}{n} \qquad V = \frac{N}{n}$$

Hence:

$$Td(N\sigma) = d\left(N\frac{e}{n}\right) + pd\left(\frac{N}{n}\right)$$

$$\rightarrow Td\sigma = d\left(\frac{e}{n}\right) + pd\left(\frac{1}{n}\right)$$

The equivalent equation in non-relativistic fluid dynamics is

$$Td\sigma = d\left(\frac{\varepsilon}{\rho}\right) + pd\left(\frac{1}{\rho}\right)$$

In this case σ is the entropy per unit *mass*.

Entropy and enthalpy

We have

$$Td\sigma = d\left(\frac{e}{n}\right) + pd\left(\frac{1}{n}\right) = \frac{1}{n}de - \frac{(e+p)}{n^2}dn = \frac{1}{n}de - \frac{w}{n^2}dn$$

We can also express this equation in terms of the differential of the enthalpy, viz.

$$Td\sigma = d\left(\frac{e+p}{n}\right) + \frac{e}{n^2}dn - \frac{dp}{n} + \frac{p}{n^2}dn - \frac{(e+p)}{n^2}dn$$

$$= d\left(\frac{w}{n}\right) - \frac{dp}{n}$$

The constancy of entropy along a streamline

Let us return to the equation that we derived from the scalar product of the fluid equations with u^α, i.e.

$$(wu^\alpha)_{,\alpha} = u^\alpha p_{,\alpha}$$

We can write this as

$$wu^\alpha{}_{,\alpha} + u^\alpha w_{,\alpha} = u^\alpha p_{,\alpha}$$

Now use the conservation of baryon density

$$(nu^\alpha)_{,\alpha} = u^\alpha n_{,\alpha} + nu^\alpha{}_{,\alpha} = 0$$

$$\Rightarrow u^\alpha{}_{,\alpha} = -\frac{1}{n}u^\alpha n_{,\alpha}$$

Therefore,

$$wu^{\alpha}{}_{,\alpha} + u^{\alpha}w_{,\alpha} = u^{\alpha}w_{,\alpha} - \frac{1}{n}u^{\alpha}n_{,\alpha} = nu^{\alpha}\left(\frac{w}{n}\right)_{,\alpha} = u^{\alpha}p_{\alpha}$$

$$\Rightarrow u^{\alpha}\left(\frac{w}{n}\right)_{,\alpha} - \frac{1}{n}u^{\alpha}p_{,\alpha} = 0$$

The operator

$$u^{\alpha}\frac{\partial}{\partial x^{\alpha}} = \frac{d}{ds}$$

i.e. differentiation along a world line. Hence the above equation for the relativistic enthalpy is

$$\frac{d}{ds}\left(\frac{w}{n}\right) - \frac{1}{n}\frac{dp}{ds} = 0$$

Consider the equation for the differential of the entropy

$$T d\sigma = d\left(\frac{w}{n}\right) - \frac{dp}{n}$$

$$\Rightarrow T\frac{d\sigma}{ds} = \frac{d}{ds}\left(\frac{w}{n}\right) - \frac{1}{n}\frac{dp}{ds} = 0$$

i.e. entropy is constant along a streamline.

8 The speed of sound

8.1 Sound waves from perturbation of the relativistic fluid equations

The speed of sound in a relativistic gas is of interest. This is best derived directly from perturbation of

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} = 0$$

We take as the unperturbed state,

$$v^i = 0 \qquad p = p_0 \qquad e = e_0 \qquad w = w_0 = e_0 + p_0$$

and the perturbed variables are

$$v^i = v_1^i \qquad p = p_0 + p_1 \qquad e = e_0 + e_1 \qquad w = w_0 + (e_1 + p_1)$$

We expand to first order in these quantities.

The perturbation to the Lorentz factor is given by:

$$\Gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} = 1 + O_2$$

The perturbations to the stress-energy tensor are give by:

$$T^{00} = \Gamma^2(e+p) - p \approx e_0 + e_1$$

$$T^{0i} = \Gamma w \frac{v^i}{c} \approx (w_0 + w_1)\frac{v_1^i}{c} = w_0 \frac{v_1^i}{c}$$

$$T^{ij} = \Gamma^2 w \frac{v^i v^j}{c^2} + p\delta^{ij} \approx p_1 \delta^{ij}$$

The perturbation equations are,

$$\alpha = 0 \qquad \frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{0i}}{\partial x^i} \approx \frac{1}{c}\frac{\partial(e_0 + e_1)}{\partial t} + \frac{\partial\left(w_0 \frac{v^j}{c}\right)}{\partial x^j} = 0$$

$$\Rightarrow \frac{\partial e_1}{\partial t} + w_0 \frac{\partial v^j}{\partial x^j} = 0$$

and

$$\alpha = i \qquad \frac{\partial T^{i0}}{\partial x^0} + \frac{\partial T^{ij}}{\partial x^j} \approx \frac{1}{c}\frac{\partial\left(w_0 \frac{v^i}{c}\right)}{\partial t} + \frac{\partial((p_0 + p_1)\delta^{ij})}{\partial x^j}$$

$$\Rightarrow \frac{w_0}{c^2}\frac{\partial v^i}{\partial t} + \frac{\partial p}{\partial x^i} = 0$$

Hence the two perturbation equations are:

$$\frac{\partial e_1}{\partial t} + w_0 \frac{\partial v^j}{\partial x^j} = 0$$

$$\frac{w_0}{c^2}\frac{\partial v^i}{\partial t} + \frac{\partial p}{\partial x^i} = 0$$

Take the divergence of the second equation:

$$\frac{w_0}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial v^i}{\partial x^i}\right) + \frac{\partial^2 p}{\partial x^i \partial x^i} = 0$$

From the first equation

$$w_0 \frac{\partial}{\partial t}\left(\frac{\partial v^i}{\partial x^i}\right) = -\frac{\partial^2 e_1}{\partial t^2}$$

Therefore,

$$\frac{\partial^2 p_1}{\partial x^i \partial x^i} - \frac{1}{c^2}\frac{\partial^2 e_1}{\partial t^2} = 0$$

Now relate the perturbations to the pressure and energy density via the equation of state, which we represent in the form:

$$p = p(e, \sigma)$$

Hence,

$$p_1 = \left(\frac{\partial p}{\partial e}\right)_\sigma e_1 + \left(\frac{\partial p}{\partial \sigma}\right)_e \sigma_1$$

For adiabatic perturbations:

$$p_1 = \left(\frac{\partial p}{\partial e}\right)_\sigma e_1$$

and the wave equation for the pressure perturbations becomes:

$$\frac{\partial^2 e_1}{\partial x^i \partial x^i} - \frac{1}{c^2(\partial p / \partial e)} \frac{\partial^2 e_1}{\partial t^2} = 0$$

This represents sound waves travelling with a speed:

$$c_s^2 = c^2 \left(\frac{\partial p}{\partial e}\right)_\sigma$$

8.2 Sound speed for an ultrarelativistic equation of state

If

$$p = \frac{1}{3} e$$

then

$$\frac{\partial p}{\partial e} = \frac{1}{3} \Rightarrow c_s^2 = \frac{1}{3}c^2 \Rightarrow c_s = \frac{c}{\sqrt{3}}$$

This spped is quite fast, $c_s = 0.577c$.

9 Relativistic shocks

9.1 The junction conditions

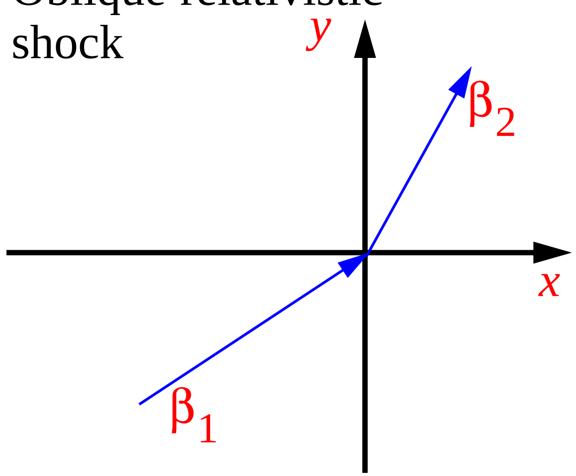
The junction conditions for relativistic shocks follow from the same approach as for non-relativistic fluid dynamics, i.e. conservation of rest-mass or number density, conservation of momentum and conservation of energy across the shock.

In the following and in much of relativistic fluid dynamics, we use

$$\beta^i = \frac{v^i}{c} \qquad \beta = \frac{v}{c}$$

$$0 < \beta < 1$$

Oblique relativistic shock



The relativistic Rankine-Hugoniot relations are derived from continuity of number, momentum and energy across the shock normal.

Number

$$\Gamma_1 n_1 \beta_{1x} = \Gamma_2 n_2 \beta_{2x}$$

Momentum (x-component)

$$T_1^{xx} = T_2^{xx}$$

$$w_1 \Gamma_1^2 \beta_{1x}^2 + p_1 = w_2 \Gamma_2^2 \beta_{2x}^2 + p_2$$

Momentum (y-component)

$$T_1^{yx} = T_2^{yx}$$

$$w_1 \Gamma_1^2 \beta_{1x} \beta_{1y} = w_2 \Gamma_2^2 \beta_{2x} \beta_{2y}$$

Energy

$$w_1 \Gamma_1^2 \beta_{1x} = w_2 \Gamma_2^2 \beta_{2x}$$

Summary of relativistic junction conditions

$$\Gamma_1 n_1 \beta_{1x} = \Gamma_2 n_2 \beta_{2x}$$

$$w_1 \Gamma_1^2 \beta_{1x}^2 + p_1 = w_2 \Gamma_2^2 \beta_{2x}^2 + p_2$$

$$w_1 \Gamma_1^2 \beta_{1x} \beta_{1y} = w_2 \Gamma_2^2 \beta_{2x} \beta_{2y}$$

$$w_1 \Gamma_1^2 \beta_{1x} = w_2 \Gamma_2^2 \beta_{2x}$$

9.2 Solution of the relativistic junction conditions

Dividing the third equation by the fourth:

$$\beta_{1y} = \beta_{2y} = \beta_y$$

i.e. the component of velocity normal to the shock is unchanged, as in non-relativistic shocks.

Velocity

Equations (2) and (4) can be solved to give the velocity on both sides of the shock in terms of the hydrodynamic variables.

Parametrisation of the velocity and Lorentz factor

$$\Gamma^{-2} = 1 - \beta_x^2 - \beta_y^2$$

Put

$$\beta_x = (1 - \beta_y^2)^{1/2} \tanh \phi$$

$$\Rightarrow \Gamma^{-2} = (1 - \beta_y^2)(1 - \tanh^2 \phi) = (1 - \beta_y^2) \operatorname{sech}^2 \phi$$

$$\Rightarrow \Gamma = (1 - \beta_y^2)^{-1/2} \cosh \phi$$

Since, β_y is the same for both sides of the shock, then, we have

$$\beta_{1x} = (1 - \beta_y^2)^{1/2} \tanh \phi_1 \qquad \Gamma_1 = (1 - \beta_y^2)^{-1/2} \cosh \phi_1$$

$$\beta_{2x} = (1 - \beta_y^2)^{1/2} \tanh \phi_2 \qquad \Gamma_2 = (1 - \beta_y^2)^{-1/2} \cosh \phi_2$$

We make these substitutions into

$$w_1\Gamma_1^2\beta_{1x}^2 + p_1 = w_2\Gamma_2^2\beta_{2x}^2 + p_2$$
$$w_1\Gamma_1^2\beta_{1x} = w_2\Gamma_2^2\beta_{2x}$$

to obtain

$$w_1\sinh^2\phi_1 + p_1 = w_2\sinh^2\phi_2 + p_2$$
$$w_1\sinh\phi_1\cosh\phi_1 = w_2\sinh\phi_2\cosh\phi_2$$

Velocities

The solutions for the velocities are algebraically long but straightforward and are derived in the appendix:

$$\beta_{1x} = (1-\beta_y^2)^{1/2}\tanh\phi_1 = (1-\beta_y^2)^{1/2}\left[\frac{(p_2-p_1)(e_2+p_1)}{(e_2-e_1)(e_1+p_2)}\right]^{1/2}$$

$$\beta_{2x} = (1-\beta_y^2)^{1/2}\tanh\phi_2 = (1-\beta_y^2)^{1/2}\left[\frac{(p_2-p_1)(e_1+p_2)}{(e_2-e_1)(e_2+p_1)}\right]^{1/2}$$

9.3 Ultrarelativistic equation of state

For an ultrarelativistic equation of state,

$$p = \frac{1}{3}e$$

the above equations become:

$$\beta_{1x} = (1-\beta_y^2)^{1/2}\left[\frac{1}{3}\frac{3p_2+p_1}{3p_1+p_2}\right]^{1/2} \qquad \beta_{2x} = (1-\beta_y^2)^{1/2}\left[\frac{1}{3}\frac{3p_1+p_2}{3p_2+p_1}\right]^{1/2}$$

9.4 Normal shocks

Mainly for convenience, let us now concentrate on normal shocks in which $\beta_y = 0$. There are a couple of interesting results here

Product of velocities

$$\beta_{1x}\beta_{2x} = \frac{1}{3}$$

Weak shock

In a weak shock $p_1 \approx p_2$. This implies that:

$$\beta_{1x} \approx \frac{1}{\sqrt{3}} \qquad \beta_{2x} = \frac{1}{\sqrt{3}}$$

i.e. the velocities, before and after the shock are equal to the sound speed.

Strong shock

$$\frac{p_2}{p_1} \to \infty \Rightarrow \beta_{1x} \to 1 \qquad \beta_{2x} \to \frac{1}{3}$$

The densities on either side of the shock are related by:

$$\Gamma_1 n_1 \beta_{1x} = \Gamma_2 n_2 \beta_{2x}$$

Since $\tilde{n} = \Gamma n$ is the lab frame density then

$$\tilde{n}_1 \beta_{1x} = \tilde{n}_2 \beta_{2x} \Rightarrow \tilde{n}_2 = \frac{1}{3} \tilde{n}_1$$

for a strong shock.

Important points

The points to note here, are

- In a relativistic fluid, a weak shock can travel quite fast, essentially because the benchmark speed, the sound speed, is fast.

- The strong shock limit has quite different velocity and density solutions than that in a non-relativistic fluid.

10 Appendix

10.1 Solution of shock equations

$$\begin{aligned} w_1 \sinh^2\phi_1 + p_1 &= w_2 \sinh^2\phi_2 + p_2 \\ w_1 \sinh\phi_1 \cosh\phi_1 &= w_2 \sinh\phi_2 \cosh\phi_2 \end{aligned}$$

Taking the parameters, $w_{1,2}$ and $p_{1,2}$ as given, we now have to solve for the two unknowns ϕ_1 and ϕ_2.

Let $x_1 = \cosh^2\phi_1 \qquad x_2 = \cosh^2\phi_2$ and square the second of the above equations:

$$w_1(x_1 - 1) + p_1 = w_2(x_2 - 1) + p_2$$
$$w_1^2(x_1 - 1)x_1 = w_2^2(x_2 - 1)x_2$$

In the first of the above equations $p - w = -e$ so that these two equations are:

$$w_1 x_1 - e_1 = w_2 x_2 - e_2$$
$$w_1^2(x_1 - 1)x_1 = w_2^2(x_2 - 1)x_2$$

The aim of the following is use these two equations to obtain linear equations for x_1 and x_2. To do so we square the first and subtract from the second.

$$w_1^2 x_1^2 - 2e_1 w_1 x_1 + e_1^2 = w_2^2 x_2^2 - 2e_2 w_2 x_2 + e_2^2$$

$$w_1^2 x_1^2 - w_1^2 x_1 = w_2^2 x_2^2 - w_2^2 x_2$$

$$\text{Subtract} \Rightarrow (w_1 - 2e_1) w_1 x_1 - e_1^2 = (w_2 - 2e_2) w_{2x_2} - e_2^2$$

Use $w_1 - 2e_1 = p_1 - e_1$ and the other previous linear equation, then

$$w_1 x_1 - e_1 = w_2 x_2 - e_2$$

$$(p_1 - e_1) w_1 x_1 - e_1^2 = (p_2 - e_2) w_{2x_2} - e_2^2$$

In matrix form:

$$\begin{bmatrix} w_1 & -w_2 \\ (e_1 - p_1) w_1 & -(e_2 - p_2) w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e_2 - e_1 \\ -(e_2^2 - e_1^2) \end{bmatrix}$$

The solutions are (using Maple)

$$x_1 = \cosh^2\phi_1 = \frac{(e_2 - e_1)(e_1 + p_2)}{(e_1 + p_1)(p_1 + e_2 - p_2 - e_1)}$$

$$x_2 = \cosh^2\phi_2 = \frac{(e_2 - e_1)(e_2 + p_1)}{(e_2 + p_2)(p_1 + e_2 - p_2 - e_1)}$$

Since $\tanh^2\phi = 1 - \mathrm{sech}^2\phi = 1 - \frac{1}{x}$ then

$$\tanh^2\phi_1 = \frac{(p_2 - p_1)(e_2 + p_1)}{(e_2 - e_1)(e_1 + p_2)} \qquad \tanh^2\phi_2 = \frac{(p_2 - p_1)(e_1 + p_2)}{(e_2 - e_1)(e_2 + p_1)}$$

Velocities

We now have the solutions for the velocities:

$$\beta_{1x} = (1-\beta_y^2)^{1/2}\tanh\phi_1 = (1-\beta_y^2)^{1/2}\left[\frac{(p_2-p_1)(e_2+p_1)}{(e_2-e_1)(e_1+p_2)}\right]^{1/2}$$

$$\beta_{2x} = (1-\beta_y^2)^{1/2}\tanh\phi_2 = (1-\beta_y^2)^{1/2}\left[\frac{(p_2-p_1)(e_1+p_2)}{(e_2-e_1)(e_2+p_1)}\right]^{1/2}$$