Magnetised Winds

1 Critical points

1.1 Preamble

So far we have inferred the existence of one critical point – the Alfven point where the poloidal velocity equals the poloidal Alfven speed. What corresponds to the sonic point that we know exists in unmagnetised winds. As we show in the following, the sonic points expands into 2 new critical points corresponding to the fast and slow magnetoacousitc speeds.

First we derive the sonic point for a spherically symmetric flow in a different way to what we did before. Consider Bernoulli's equation in the form:

$$H(r, \rho) = \frac{1}{2}V^2 + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = E = \text{constant}$$

where V is given as a function of density and radius by mass flux conservation:

$$V = \frac{\dot{M}}{4\pi\rho r^2}$$

and the speed of sound squared is given by

$$a^2 = \gamma C \rho^{\gamma - 1}$$

A differential equation for ρ as a function of r cam be found by differentiating $H(r, \rho(r))$ wrt r, i.e.

$$H_r + H_\rho \frac{d\rho}{dr} = 0 \Rightarrow \frac{d\rho}{dr} = -\frac{H_r}{H_\rho}$$

and the critical points of this differential equation occur where

$$H_r = H_\rho = 0$$

Now

$$\begin{aligned} H_r &= V\frac{\partial V}{\partial r} + \frac{GM}{r^2} = -\frac{2\dot{M}V}{4\pi\rho r^3} + \frac{GM}{r^2} \\ &= -2\frac{V^2}{r} + \frac{GM}{r^2} = -\frac{1}{r}\left(2V^2 - \frac{GM}{r}\right) \end{aligned}$$

and

$$\begin{aligned} H_\rho &= V\frac{\partial V}{\partial \rho} + \frac{1}{\gamma - 1}\frac{\partial}{\partial \rho}a^2 \\ &= -\frac{\dot{M}V}{4\pi\rho^2 r^2} + \gamma C\rho^{\gamma - 2} \\ &= \left(-\frac{V^2}{\rho} + \frac{a^2}{\rho}\right) = -\frac{(V^2 - a^2)}{\rho} \end{aligned}$$

Hence the critical point is located at

$$V = a \qquad r = \frac{1}{2}\frac{GM}{a^2}$$

as we derived previously. This method of derivation for the location for the critical point establishes the pattern for the derivation of the critical point in an axisymmetric magnetised wind.

1.2 Axisymmetric magnetised wind

In this case, the energy equation reads

$$H = \frac{1}{2}V^2 + \frac{a^2}{\gamma - 1} - \frac{GM}{r} - \frac{\Omega r B_\phi}{4\pi\alpha} = E = \text{streamline constant}$$

Splitting the velocity into poloidal and azimuthal components,

$$V^2 = V_p^2 + V_\phi^2 = \frac{\alpha^2 B_p^2}{\rho^2} + r^2\omega^2$$

$$\text{and} \qquad \omega = \Omega \frac{\left(1 - \frac{\rho_A r_A^2}{\rho r^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)}$$

and using the previously derived expression for the azimuthal component of the magnetic field:

$$\frac{-\Omega r B_\phi}{4\pi\alpha} = \Omega^2 r_A^2 \frac{\left(1 - \frac{r^2}{r_A^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)}$$

where the streamline constants

$$r_A^2 = \frac{L}{\Omega} \qquad \text{and} \qquad \rho_A = 4\pi\alpha^2$$

define the Alfven radius and density at the Alfven radius respectively.

We now consider a given poloidal magnetic field line and consider the axisymmetric radius r as a parameter along that curve. We therefore have along a field line

$$H(r, \rho) = \frac{1}{2}\left(\frac{\alpha^2}{\rho^2}\right)B_p^2(r) + \frac{1}{2}r^2\omega^2(r, \rho) + \frac{a^2(\rho)}{\gamma - 1} - \frac{GM}{r} - \frac{\Omega r B_\phi}{4\pi\alpha}(r, \rho) =$$

This is an implicit equation for the density as a function of radius and as with spherically symmetric flow, we can determine a differential equation for $\rho(r)$ from

$$H_r + H_\rho \frac{d\rho}{dr} = 0$$

and the critical points are given by

$$H_r = H_\rho = 0$$

Equation for H_ρ.

We have

$$\frac{\partial H}{\partial \rho} = -\frac{\alpha^2 B_p^2}{\rho^3} + r^2 \omega \frac{\partial \omega}{\partial \rho} + \frac{1}{\gamma - 1} \frac{\partial}{\partial \rho} a^2 - \frac{\partial}{\partial \rho}\left(\frac{\Omega r B_\phi}{4\pi\alpha}\right)$$

The following ρ–derivatives are straightforward:

$$\frac{\partial \omega}{\partial \rho} = -\frac{\Omega \rho_A}{\rho^2} \frac{\left(1 - \frac{r_A^2}{r^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)^2}$$

$$\frac{\partial}{\partial \rho}\left(\frac{-\Omega r B_\phi}{4\pi\alpha}\right) = -\frac{\rho_A}{\rho^2} r_A^2 \Omega^2 \frac{\left(1 - \frac{r^2}{r_A^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)^2}$$

$$\frac{\partial}{\partial \rho} \frac{\gamma C \rho^{\gamma - 1}}{\gamma - 1} = \frac{a^2}{\rho}$$

The combination

$$r^2 \omega \frac{\partial \omega}{\partial \rho} + \frac{\partial}{\partial \rho}\left(\frac{-\Omega r B_\phi}{4\pi\alpha}\right) = -\Omega^2 \frac{\rho_A^2}{\rho^3} \frac{r_A^4}{r^2} \frac{\left(1 - \frac{r^2}{r_A^2}\right)^2}{\left(1 - \frac{\rho_A}{\rho}\right)^3}$$

We can now start to relate these terms to Alfven speeds etc. Consider,

$$\frac{B_\phi^2}{4\pi\rho} = \Omega^2 \frac{r_A^4}{r^2}\frac{\rho_A}{\rho}\frac{\left(1-\frac{r^2}{r_A^2}\right)^2}{\left(1-\frac{\rho_A}{\rho}\right)^2}$$

and therefore,

$$r^2\omega\frac{\partial\omega}{\partial\rho} + \frac{\partial}{\partial\rho}\left(\frac{-\Omega r B_\phi}{4\pi\alpha}\right) = -\frac{\rho_A}{\rho^2}\frac{1}{\left(1-\frac{\rho_A}{\rho}\right)} \times \frac{B_\phi^2}{4\pi\rho}$$

$$= \frac{1}{\rho\left(1-\frac{\rho}{\rho_A}\right)} \times \frac{B_\phi^2}{4\pi\rho}$$

Hence,

$$\frac{\partial H}{\partial\rho} = -\frac{\alpha^2 B_\mathrm{p}^2}{\rho^3} + \frac{1}{\rho\left(1-\frac{\rho}{\rho_A}\right)} \times \frac{B_\phi^2}{4\pi\rho} + \frac{a^2}{\rho}$$

$$= -\frac{V_\mathrm{p}^2}{\rho} + \frac{\frac{B_\phi^2}{4\pi\rho}}{\rho\left(1-\frac{\rho}{\rho_A}\right)} + \frac{a^2}{\rho}$$

As we have shown before,

$$\frac{\rho}{\rho_A} = \frac{B_\mathrm{p}^2}{4\pi\rho V_\mathrm{p}^2} = \frac{V_{\mathrm{A,p}}^2}{V_p^2}$$

so that, on multiplying $\frac{\partial H}{\partial\rho} = 0$ through by $\rho\left(1-\frac{\rho}{\rho_A}\right)$, we obtain,

$$-V_{\mathrm{p}}^2\left(1-\frac{V_{\mathrm{A,p}}^2}{V_{\mathrm{p}}^2}\right)+V_{\mathrm{A},\,\phi}^2+\left(1-\frac{V_{\mathrm{A,p}}^2}{V_{\mathrm{p}}^2}\right)a^2 \;=\; 0$$

On multiplying through by V_{p}^2,

$${}^{4}_{\mathrm{p}}-(V_{\mathrm{A,p}}^2+V_{\mathrm{A},\,\phi}^2+a^2)V_p^2+V_{\mathrm{A,p}}^2a^2 \;=\; 0$$

Now, the total Alfven speed, V_{A}, is given by

$$V_{\mathrm{A}}^2 \;=\; V_{\mathrm{A,p}}^2+V_{\mathrm{A},\,\phi}^2$$

and the equation for the critical poloidal velocity is

$$V_{\mathrm{p}}^4-(V_A^2+a^2)V_{\mathrm{p}}^2+V_{\mathrm{A,p}}^2a^2 \;=\; 0$$

i.e.

$$V_{\mathrm{p}}^2 \;=\; \frac{1}{2}(V_A^2+a^2)\pm\frac{1}{2}[(V_A^2+a^2)^2-4V_{\mathrm{A,p}}^2a^2]^{1/2}$$

This is the equation for slow and fast magnetoacoustic waves along the poloidal field lines, i.e. $V_{\mathrm{A}}\cos\psi \;=\; V_{\mathrm{A,p}}$. Hence a magnetised wind has *three* critical points, one corresponding to the Alfven point and the other two corresponding to the fast and slow magnetoacoustic speeds.

2 Limiting cases of critical points

2.1 Weak magnetic field ($V_A \ll a$)

This would be the case for a wind which is primarily thermally driven but which has some magnetic field.

Writing

$$V_{\mathrm{p}}^2 \;=\; \frac{1}{2}(V_A^2 + a^2) \pm a^2\left[\left(1 + \frac{V_{\mathrm{A}}^2}{a^2}\right)^2 - 4\frac{V_{\mathrm{A,p}}^2}{a^2}\right]^{1/2}$$

and expanding the square root in powers of $\dfrac{V_{\mathrm{A}}^2}{a^2}$ gives

$$\begin{aligned} V_{\mathrm{p}}^2 \;&=\; \frac{1}{2}(V_{\mathrm{A}}^2 + a^2) \pm \frac{1}{2}a^2\left[1 + \frac{V_{\mathrm{A}}^2}{a^2} - 2\frac{V_{\mathrm{A,p}}^2}{a^2}\right] \\ &=\; V_{\mathrm{A,p}}^2,\, V_{\mathrm{A}}^2 - V_{\mathrm{A,p}}^2 + a^2 \\ &=\; V_{\mathrm{A,p}},\, V_{\mathrm{A},\phi}^2 + a^2 \end{aligned}$$

as the two values of the slow and fast magnetoacoustic speeds. The slow speed is therefore (to first order) the same as the poloidal Alfven speed and the fast magnetoacoustic speed is approximately $(V_{\mathrm{A},\phi}^2 + a^2)^{1/2}$, i.e. close to the sound speed. Therefore the slow magnetoacoustic point and the Alfven point are very close.

3 Strong magnetic field ($V_{\mathrm{A}} \gg a$)

For this case we put

$$_{\mathrm{p}}^{2} \;=\; \frac{1}{2}(V_{\mathrm{A}}^2 + a^2) \pm \frac{1}{2}V_{\mathrm{A}}^2\left[\left(1 + \frac{a^2}{V_{\mathrm{A}}^2}\right)^2 - \frac{4V_{\mathrm{A,p}}^2 a^2}{V_{\mathrm{A}}^4}\right]^{1/2}$$

and expand the square root in powers of $\dfrac{V_{\mathrm{A}}^2}{a^2}$ to obtain:

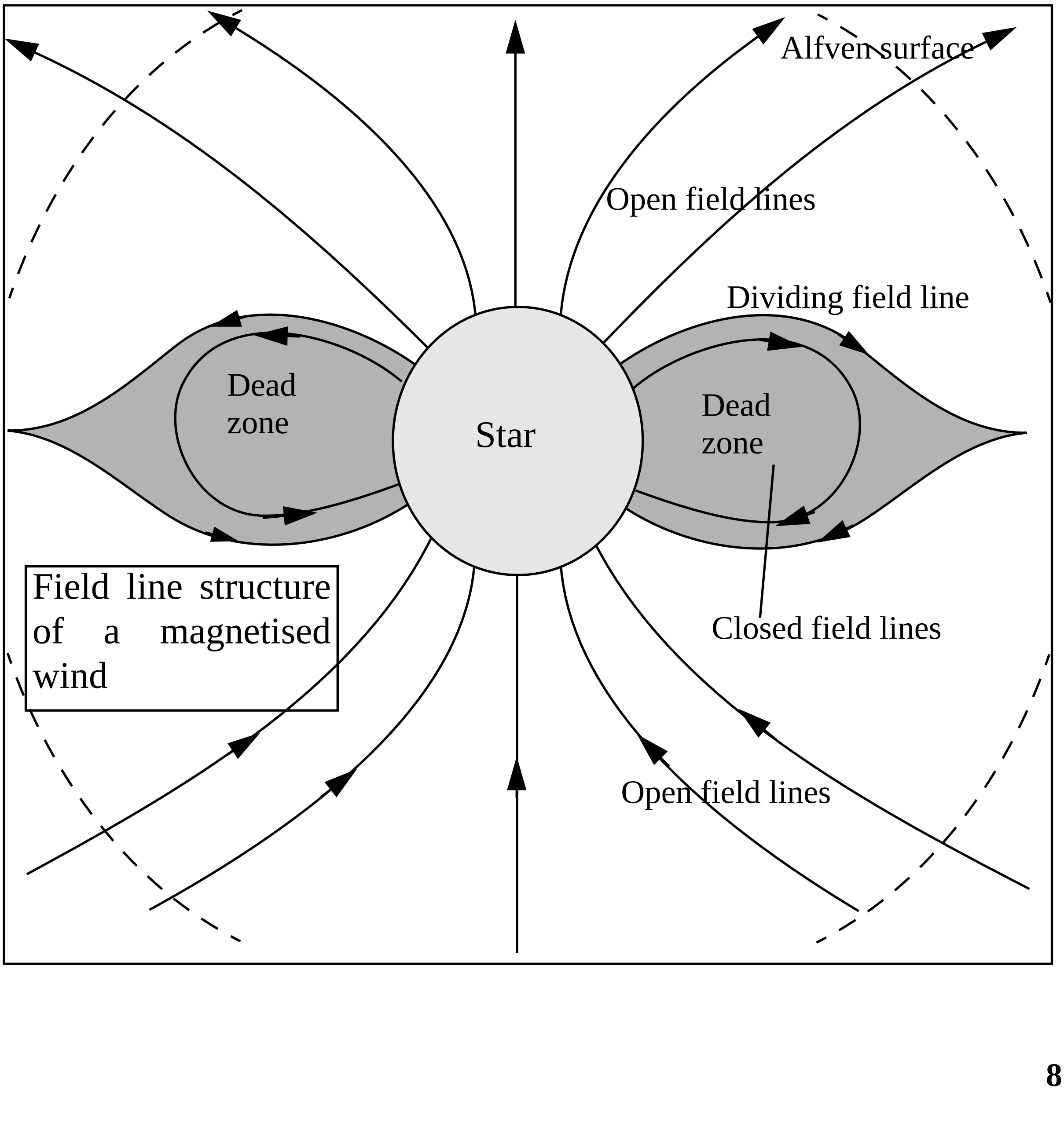
$$\begin{aligned} V_{\mathrm{p}}^2 \;&=\; \frac{V_{\mathrm{A,p}}^2}{V_{\mathrm{A}}^2}a^2,\, V_A^2 + a^2\left(1 - \frac{V_{\mathrm{A,p}}^2}{V_{\mathrm{A}}^2}\right) \\ &=\; \frac{V_{\mathrm{A,p}}^2}{V_{\mathrm{A}}^2}a^2,\, V_{\mathrm{A}}^2 + a^2\frac{V_{\mathrm{A},\phi}^2}{V_{\mathrm{A}}^2} \end{aligned}$$

In this case, the slow magnetoacoustic speed is of order the sound speed and the fast magnetoacoustic speed is close to the Alfven speed.

4 Rapid rotator

4.1 Introduction

Some of the most interesting effects of magnetised winds occur for so-called "rapid" or "fast" rotators. The precise definition is given below. Essentially these are stars for which the combined effect of rotation and magnetic field overwhelms all other forces. In general the field-line structure of a magnetised wind is as given in the following diagram.



Field line structure of a magnetised wind

One of the features of magnetised winds is the "dead-zone" indicated in the figure. More about this later.

The starting point in the analysis is Bernoulli's equation:

$$\frac{1}{2}V_p^2 + \frac{1}{2}V_\phi^2 + \frac{a^2}{\gamma - 1} - \frac{GM}{r} - \frac{\Omega r B_\phi}{4\pi\alpha} = E = \text{total specific energy}$$

where

$$_{\phi} = r\Omega\frac{\left(1 - \frac{\rho_A r_A^2}{\rho r^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)} \qquad V_\phi^2 = r^2\Omega^2\frac{\left(1 - \frac{\rho_A r_A^2}{\rho r^2}\right)^2}{\left(1 - \frac{\rho_A}{\rho}\right)^2}$$

$$\frac{-\Omega r B_\phi}{4\pi\alpha} = r_A^2\Omega^2\frac{\left(1 - \frac{r^2}{r_A^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)}$$

4.2 Evaluation of streamline constants

So far, although we have given a physical interpretation of the meaning of the streamline constants, we have not evaluated these in any given physical situation. We are now going to evaluate these for the case of a "rapid rotator".

Evaluation of Ω

Evaluate Ω near surface of star:

$$\omega = \left.\frac{V_\phi}{r}\right|_{\text{surface}} = \Omega\frac{\left(1 - \dfrac{\rho_A r_A^2}{\rho_0 r_0^2}\right)}{\left(1 - \dfrac{\rho_A}{\rho_0}\right)}$$

where $r = r_0$ is the value of the axisymmetric radial coordinate at the surface of the star.

$$r = R_0 \sin\theta$$

where R_0 is the value of the spherically symmetric radial coordinate and θ is the colatitude (spherical polar angle).

Assume that the Alfven surface is so far out that the density at the Alfven point:

$$\rho_A \ll \rho_{\text{surface}}$$

We can also show, plausibly, that

$$\frac{\rho_A r_A^2}{\rho_0 r_0^2} \ll 1$$

along open field lines, even though $r_A \gg r_0$.

We assume that the open field lines are approximately spherically symmetric. In such a flow

$$\rho V R^2 = \frac{\rho V r^2}{\sin^2\theta} = \dot{M} = \text{Mass flux per steradian}$$

$$\Rightarrow \rho V r^2 = \dot{M}\sin^2\theta$$

Since $\sin^2\theta \approx \text{constant}$ along a streamline, then

$$\rho r^2 \propto \frac{1}{V} \Rightarrow \frac{\rho_A r_A^2}{\rho_0 r_0^2} = \frac{V_0}{V_A}$$

so that of the velocity starts off low and accelerates out to the Alfven point, then

$$\frac{\rho_A r_A^2}{\rho_0 r_0^2} \ll 1$$

This constitutes a plausibility argument that this condition is satisfied. Except in pathological circumstances, this argument should be valid. Hence the equation for the angular velocity at the surface of the star becomes:

$$\omega_0 \approx \Omega$$

that is the field-line pattern rotates with the angular velocity of the star.

The streamline constant E

In view of the above, the term

$$-\frac{\Omega r B_\phi}{4\pi\alpha} = r_A^2 \Omega^2 \frac{\left(1 - \frac{r^2}{r_A^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)} \approx r_A^2 \Omega^2$$

so that the streamline constant

$$E \approx \frac{1}{2} V_{p,\,0}^2 + \frac{1}{2} r_0^2 \Omega^2 + \frac{a_0^2}{\gamma - 1} + r_A^2 \Omega^2$$

Definition of a rapid rotator

We (finally) define a rapid rotator to be one in which

$$r_A \Omega \gg (V_{p,0}, r_0 \Omega, a_0)$$

Thus the definition of a rapid rotator simultaneously involves a strong magnetic field ($r_A \gg r_0$) and rapid rotation ($\Omega \gg \frac{(V_{p,0}, a_0)}{r_A}$)

When this condition is satisfied, the streamline constant

$$E \cong r_A^2 \Omega^2$$

4.3 Solution along open field lines

With the above streamline constants, Bernoulli's equation becomes:

$$\frac{1}{2}V_p^2 + \frac{1}{2}r^2\Omega^2 \frac{\left(1 - \frac{\rho_A r_A^2}{\rho r^2}\right)^2}{\left(1 - \frac{\rho_A}{\rho}\right)^2} + r_A^2 \Omega^2 \frac{\left(1 - \frac{r^2}{r_A^2}\right)}{\left(1 - \frac{\rho_A}{\rho}\right)} = E = r_A^2 \Omega^2$$

This equation describes flow from the surface of the star, out through the Alfven point ($\frac{\rho}{\rho_A} = \frac{V}{V_A} = \frac{r}{r_A} = 1$)

We assume that the flow is approximately spherically symmetric beyond the Alfven surface, so that, putting V_A as the poloidal velocity ($=$ poloidal Alfven speed) at the Alfven point, we have, as before:

$$\rho \;=\; \frac{\dot{M}\sin^2\theta}{V_p r^2} \qquad\qquad \rho r^2 \;=\; \frac{\dot{M}\sin^2\theta}{V_p}$$

$$\frac{\rho_A r_A^2}{\rho r^2} \;=\; \frac{V_p}{V_A} \qquad\qquad \frac{\rho}{\rho_A} \;=\; \frac{V_A}{V}\frac{r_A^2}{r^2}$$

Terminal velocity

One of the important characteristics of such a wind that we wish to find is the terminal velocity, or the velocity at infinity. We therefore look for a solution such that

$$\frac{\rho}{\rho_A} \to 0 \qquad \text{as} \qquad V_p \to V_\infty$$

In view of these limits we write Bernoulli's equation as

$$\underset{T_1}{\frac{1}{2}V_p^2} + \underset{T_2}{\frac{1}{2}r^2\Omega^2\frac{\left(\frac{\rho}{\rho_A} - \frac{r_A^2}{r^2}\right)^2}{\left(\frac{\rho}{\rho_A} - 1\right)^2}} + \underset{T_3}{r_A^2\Omega^2\frac{\left(\frac{\rho}{\rho_A} - \frac{\rho r^2}{\rho_A r_A^2}\right)}{\left(\frac{\rho}{\rho_A} - 1\right)}} \;=\; r_A^2\Omega^2$$

As $r \to \infty$

$$_1 \to \frac{1}{2}V_\infty^2$$

$$_2 \to r^2\Omega^2\frac{V_A^2}{V^2}\frac{r_A^4}{r^4} \propto \frac{1}{r^2}$$

$$_3 \to r_A^2\Omega^2\frac{\left(\frac{V_A}{V}\frac{r_A^2}{r^2} - \frac{V_A}{V_p}\right)}{(-1)} \;=\; r_A^2\Omega^2\frac{V_A}{V_p}$$

Hence, Bernoulli's equation at infinity becomes:

$$\frac{1}{2}V_\infty^2 + r_A^2\Omega^2\frac{V_A}{V_\infty} = r_A^2\Omega^2$$

Divide through by $r_A^2\Omega^2$

$$\frac{1}{2}\left(\frac{V_\infty}{r_A\Omega}\right)^2 + \frac{V_A}{V_p} = 1$$

$$\frac{1}{2}\left(\frac{V_A}{r_A\Omega}\right)^2\left(\frac{V_\infty}{V_A}\right)^2 + \left(\frac{V_\infty}{V_A}\right)^{-1} = 1$$

Write

$$\alpha = \frac{V_A}{r_A\Omega} \qquad u = \frac{V_\infty}{V_A}$$

and multiply through by u, then

$$f(u) = \frac{1}{2}\alpha^2u^3 - u + 1 = 0.$$

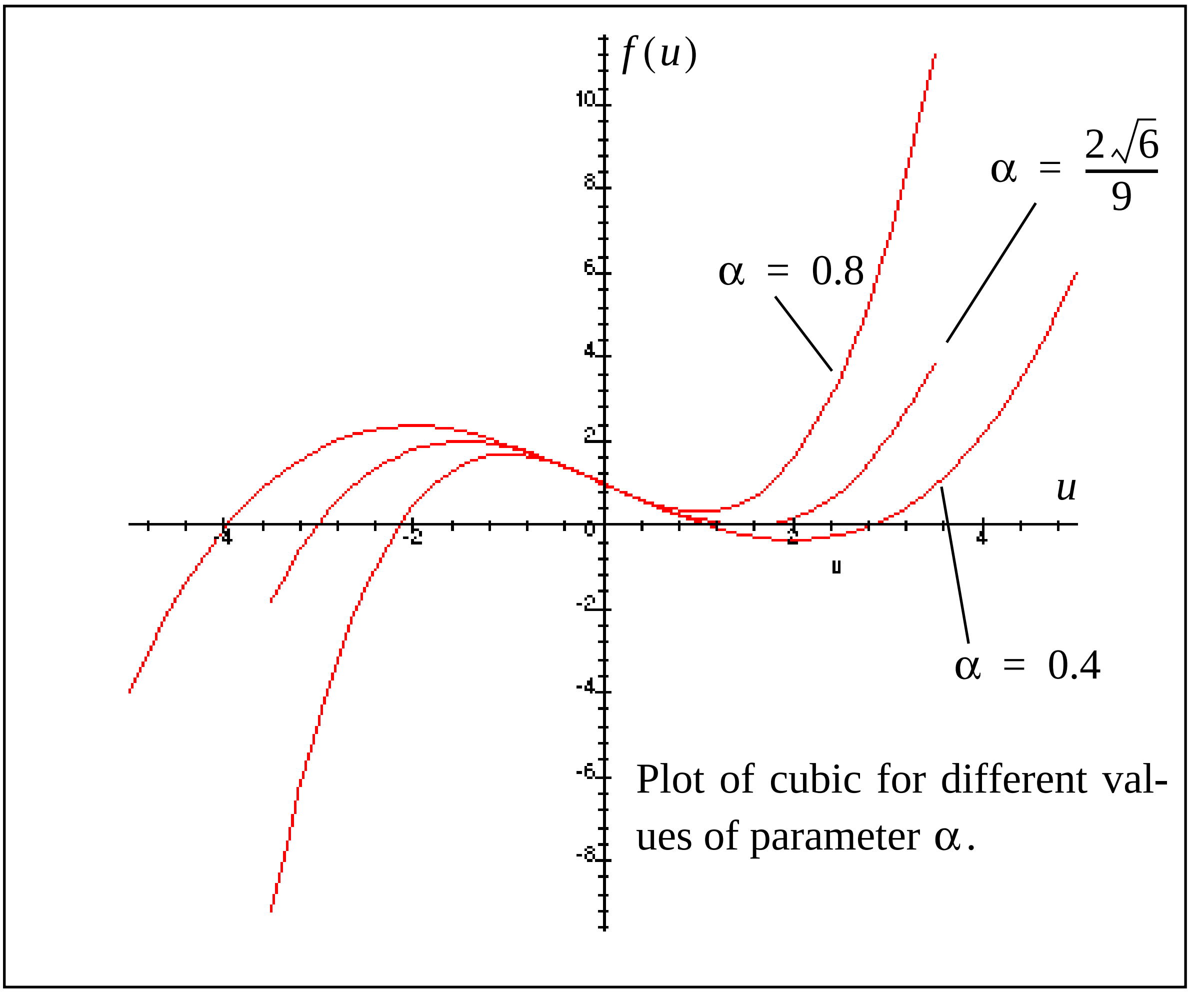
Condition for unique physical solution

The solution for the terminal velocity depends upon the roots of the cubic. The possibilities are

- one real root, two conjugate complex
- three real roots

When we have more than one positive real root, the solution for the terminal velocity is not unique, and it would seem, unphysical. It is therefore important to investigate the possible roots of this cubic.

The following plots show what $f(u)$ is like.



Plot of cubic for different values of parameter α.

For values of $\alpha < \frac{2\sqrt{6}}{9}$, the cubic has 2 positive real roots and one negative real root; for $\alpha = \frac{2\sqrt{6}}{9} \approx 0.544$, the cubic has one repeated real root and one negative real root; for $\alpha > \frac{2\sqrt{6}}{9}$, the cubic has 2 complex conjugate roots and one negative real root.

The critical value of α

The critical value of α comes from the condition that the cubic have 2 equal real roots. This follows from:

$$f(\alpha, u) = \frac{1}{2}\alpha^2 u^3 - u + 1 = 0$$

and $$\frac{df}{du}(\alpha, u) = \frac{3}{2}\alpha^2 u^2 - 1 = 0$$

We can easily solve the 2 equations

$$3f(\alpha, u) = \frac{3}{2}\alpha^2 u^3 - 3u + 3 = 0$$

and $$u\frac{df}{du}(\alpha, u) = \frac{3}{2}\alpha^2 u^3 - u = 0$$

to obtain

$$2u - 3 = 0 \Rightarrow u = \frac{3}{2}$$

and

$$f\left(\alpha, \frac{3}{2}\right) = \frac{1}{2}\alpha^2\left(\frac{3}{2}\right)^3 - \frac{3}{2} + 1 = 0 \Rightarrow \alpha = \frac{2\sqrt{6}}{9}.$$

Hence the solution with one real root, satisfies

$$\alpha = \frac{V_A}{r_A\Omega} = \frac{2\sqrt{6}}{9}$$

$$\frac{V_\infty}{V_A} = \frac{3}{2} \Rightarrow V_\infty = \frac{\sqrt{6}}{3}r_A\Omega$$

This analysis shows that there will only be a unique solution when there is a real repeated root and that it is given by the above solution. The intriguing point about this solution is that we have determined both the velocity at the Alfven point and the terminal velocity from one equation.

5 Estimation of Alfven Radius

The following estimate of the Alfven radius is based upon assuming a spherically symmetric flow along an open field line from the surface of the star outwards. This gives a good indication of how the Alfven radius depends upon the strength of the magnetic field.

Consider spherically symmetric flow and the following parameter:

$$\frac{\Phi^2}{4\pi\dot{M}_w}$$

where Φ is the magnetic flux per steradian and $\dot{M}_w$ is the mass flux per steradian. We have

$$\frac{\Phi^2}{4\pi\dot{M}_w} = \frac{(BR^2)^2}{4\pi\rho V_p R^2} = R^2 \times \frac{B^2}{4\pi\rho V_p} = R^2 \frac{V_A^2}{V_p}$$

Both Φ and $\dot{M}_w$ are invariants along a streamline, so that this parameter can be evaluated at the Alfven point where $V_p = V_A$. Hence

$$\frac{\Phi^2}{4\pi\dot{M}_w} = R_A^2 V_A = \frac{2\sqrt{6}}{9} R_A^3 \Omega$$

Now let us evaluate this parameter at the surface of the star:

$$\frac{\Phi^2}{4\pi\dot{M}_w} = \frac{B_0 R_0^2}{4\pi\rho_0 V_0} = \frac{V_{A,0}^2}{V_0} R_0^2$$

Hence,

$$\frac{V_{A,0}^2}{V_0}R_0^2 = \frac{2\sqrt{6}}{9}R_A^3\Omega$$

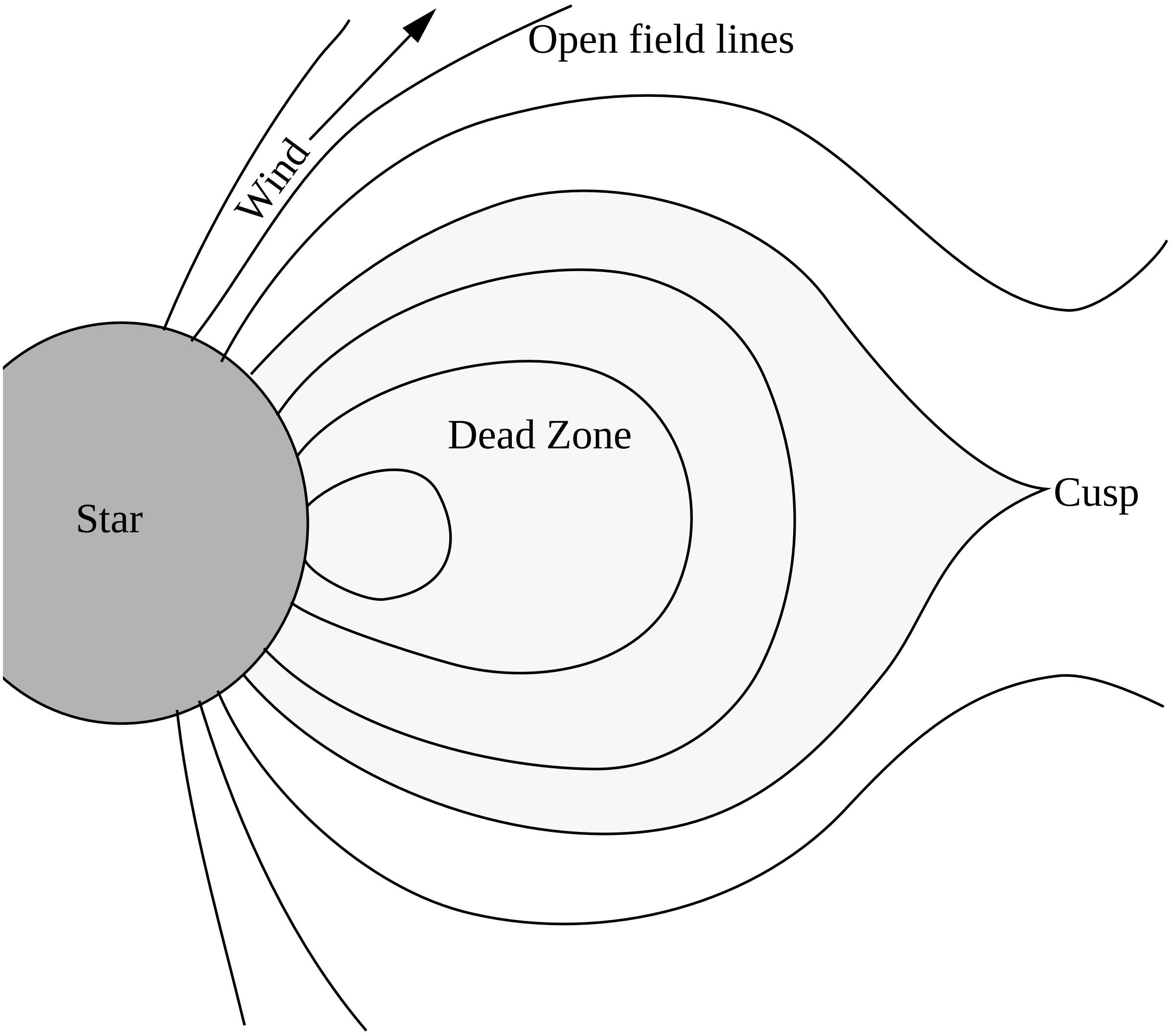
$$\Rightarrow\left(\frac{R_A}{R_0}\right)^3 = \frac{9}{2\sqrt{6}}\frac{V_{A,0}^2}{V_0(R_0\Omega)} = \frac{9}{2\sqrt{6}}\frac{V_{A,0}^2}{V_0V_{\mathrm{rot}}}$$

Normally the initial poloidal speed would be of order the sound speed. Hence the condition for a large Alfven radius is that the Alfven speed at the surface of the star satisfies:

$$\frac{V_{A,0}^2}{c_0V_{\mathrm{rot}}} \gg 1$$

6 The Dead Zone

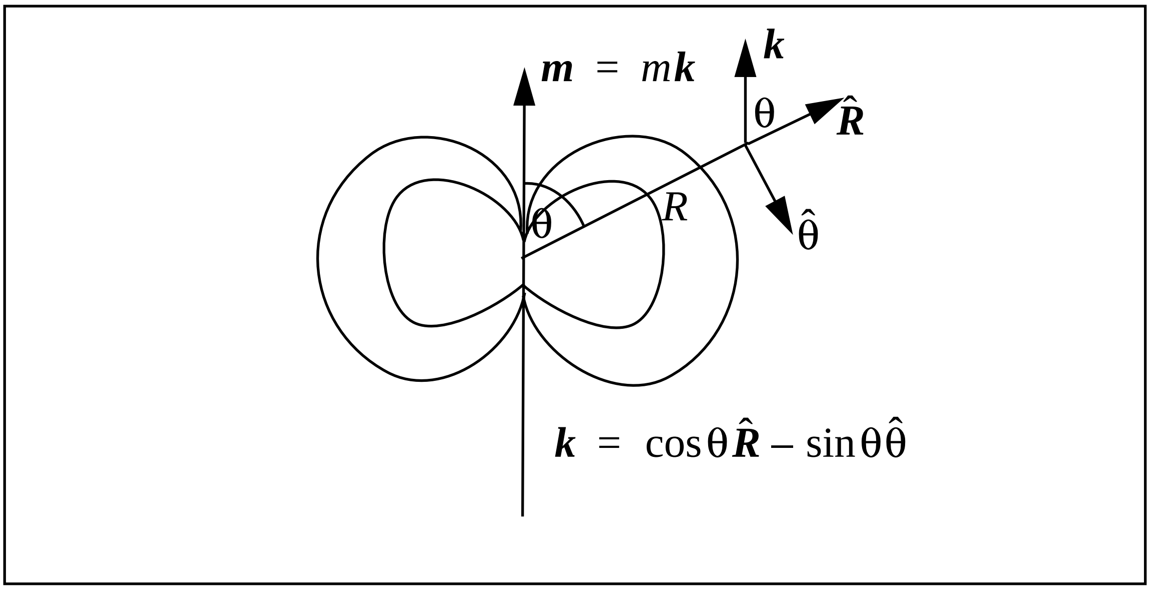
See L. Mestel & H.C. Spruit (MNRAS, **226**, 57)



The dead zone results from the equilibrium between the magnetic force which holds in the plasma and the centrifugal and pressure forces which tend to make the plasma flow outwards.

6.1 Preliminaries: Dipole field

We shall approximate the field close to the star as a dipole. This is justifiable on the grounds that the field configuration resulting from an arbitrary distribution of currents is normally dominated by the dipole term.



The magnetic field of a dipole, in spherical polar coordinates, is given by:

$$
\begin{aligned}
B &= \frac{[3\hat{\boldsymbol{R}}(\hat{\boldsymbol{R}} \cdot \hat{\boldsymbol{m}}) - \boldsymbol{m}]}{R^3} = \frac{3\hat{\boldsymbol{R}}(m\cos\theta) - m\boldsymbol{k}}{r^3} \\
&= \frac{m[3\cos\theta\hat{\boldsymbol{R}} - \boldsymbol{k}]}{R^3} \\
&= \frac{m}{R^3}[2\cos\theta\hat{\boldsymbol{R}} + \sin\theta\hat{\theta}]
\end{aligned}
$$

where $\boldsymbol{k}$ is the unit vector in the direction of the dipole, $\boldsymbol{m} = m\boldsymbol{k}$ is the dipole moment and R is the spherical polar radial coordinate.

Equations of dipole field lines

The equations of dipole field lines are determined the tangent vector to the field lines. i.e.

$$\frac{dR}{du}\hat{\boldsymbol{R}} + R\frac{d\theta}{du}\hat{\theta} = \frac{m}{R^3}[2\cos\theta\hat{\boldsymbol{R}} + \sin\theta\hat{\theta}]$$

Thus the equations of the field lines are determined from:

$$\frac{dR}{du} = \frac{2m\cos\theta}{R^3} \qquad R\frac{d\theta}{du} = \frac{m\sin\theta}{R^3}$$

Dividing one equation by the other:

$$\begin{aligned}\frac{1}{R}\frac{dR}{d\theta} &= 2\frac{\cos\theta}{\sin\theta}\\ \Rightarrow \ln R &= 2\ln\sin\theta + C\\ \Rightarrow R &= A\sin^2\theta\end{aligned}$$

The equation for the dipole field lines can be expressed in any one of a number of different forms. For example, suppose a dipole field line emerges from the surface of a star, $R = R_0$ at $\theta = \theta_0$, then

$$A = \frac{R_0}{\sin^2\theta_0} \Rightarrow \frac{R}{R_0} = \frac{\sin^2\theta}{\sin^2\theta_0}$$

Magnitude of a dipole field

$$\begin{aligned}B^2 &= \frac{m^2}{R^6}[4\cos^2\theta + \sin^2\theta] = \frac{m^2}{R^6}[4\cos^2\theta + 1 - \cos^2\theta]\\ &= \frac{m^2}{R^6}[1 + 3\cos^2\theta]\\ \Rightarrow B &= \frac{m}{R^3}[1 + 3\cos^2\theta]^{1/2}\end{aligned}$$

6.2 Equations of motion in a rotating frame

The equations we have considered so far are expressed in an inertial frame, i.e. one in which the relationship between force and acceleration is given by $\boldsymbol{F} = m\boldsymbol{a}$. Often, we like to consider equations in a frame which is rotating wrt an inertial frame, e.g. in the present context, the frame of reference of a rotating star. Therefore we need to relate time derivatives in the inertial frame to derivatives in the rotating frame.

The components of *any* vector in the inertial frame are related to the components in the rotating frame by:

$$\underbrace{v_i}_{\text{Inertial frame}} = a_{ji}(t)\underbrace{v_j'}_{\text{Rotating frame}}$$

where a_{ij} is the transformation matrix relating components in the rotating frame to components in the inertial frame, i.e.

$$x_i' = a_{ij}x_j$$

$$\text{where} \qquad a_{ij}a_{ik} = \delta_{jk}$$

Let the 2 frames coincide at time t and consider the relationship between components at time $t + \delta t$. Then,

$$v_i(t+\delta t) = v_i(t) + \delta v_i = \overbrace{(\delta_{ji} + \Omega_{ji}\delta t)}^{a_{ji}}\left(v_j'(t) + \frac{dv_j'}{dt}\delta t\right)$$

$$= \left(v_i'(t) + \frac{dv_i'}{dt}\delta t + \Omega_{ji}v_j'(t)\delta t\right)$$

From the orthogonality property of a_{ij}, we have

$$\begin{aligned}(\delta_{ij} + \Omega_{ij}\delta t)(\delta_{ik} + \Omega_{ki}\delta t) &= \delta_{jk} \\ \delta_{jk} + (\Omega_{jk} + \Omega_{kj})\delta t &= 0 \\ \Rightarrow \Omega_{jk} + \Omega_{kj} &= 0 \\ \Omega_{kj} &= -\Omega_{jk}\end{aligned}$$

We define the instantaneous angular velocity of the rotating frame by

$$\begin{aligned}\Omega_i &= \frac{1}{2}\varepsilon_{ijk}\Omega_{jk} \\ \Rightarrow \Omega_{jk} &= \varepsilon_{jki}\Omega_i\end{aligned}$$

Hence, the change in the vector v_i is given by

$$\delta v_i = \frac{\partial v_i}{\partial t}\delta t + \varepsilon_{ijk}\Omega_j v_k \delta t$$

where the partial time derivative refers to differentiation wrt t in the rotating frame.

The rate of change of v_i in the non-rotating frame is therefore,

$$\begin{aligned}\frac{dv_i}{dt} &= \frac{\partial v_i}{\partial t} + \varepsilon_{ijk}\Omega_j v_k \\ \frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\Omega}(t) \times \mathbf{v}\end{aligned}$$

Note that the angular velocity can be a function of time, although the usual application of this relation is for cases when Ω is constant.

This relation can be applied to any vector. Applying to the position vector:

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial t} + \Omega \times \boldsymbol{r},$$

i.e. the velocity in the inertial frame is

$$\boldsymbol{v}_{\text{inertial}} = \boldsymbol{v}_{\text{rot}} + \Omega \times \boldsymbol{r}$$

This relation justifies calling Ω the angular velocity; when a point is stationary in the rotating frame ($\frac{\partial \boldsymbol{r}}{\partial t} = \boldsymbol{0}$), the velocity is given by $\Omega \times \boldsymbol{r}$.

Now applying the same relation to the velocity vector

$$\begin{aligned}
\frac{d\boldsymbol{v}}{dt} &= \frac{\partial \boldsymbol{v}}{\partial t} + \Omega \times \boldsymbol{v} \\
&= \frac{\partial \boldsymbol{v}}{\partial t} + \Omega \times \left(\frac{\partial \boldsymbol{r}}{\partial t} + \Omega \times \boldsymbol{r}\right) \\
&= \frac{\partial \boldsymbol{v}}{\partial t} + \underbrace{\boxed{\Omega \times \frac{\partial \boldsymbol{r}}{\partial t}}}_{\text{Coriolis acceleration}} + \underbrace{\boxed{\Omega \times (\Omega \times \boldsymbol{r})}}_{\text{Centrifugal acceleration}}
\end{aligned}$$

Acceleration in inertial frame = Acceleration in rotating frame
+ Coriolis acceleration + Centrifugal acceleration

The Coriolis and centrifugal forces are “fictitious” forces resulting from the fact that the rotating reference frame is non-inertial.

Application to the momentum equations

When the angular velocity is fixed in direction (say the z–direction) the term

$$\Omega \times (\Omega \times \boldsymbol{r}) \;=\; (-r\Omega^2)\hat{\boldsymbol{r}}$$

where the $\hat{\boldsymbol{r}}$ refers to axisymmetric coordinates, and we can write the momentum equations in the form

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho\Omega \times \boldsymbol{V} + \rho(\boldsymbol{V} \cdot \nabla\boldsymbol{V}) \;=\; -\nabla P + \rho r\Omega^2\hat{r} + \frac{1}{4\pi}(\nabla\times\boldsymbol{B}) \times \boldsymbol{B} - \rho\nabla\psi$$

We can write the centrifugal term as

$$\rho r\Omega^2 \;=\; \rho\frac{\partial}{\partial r}\left(\frac{1}{2}r^2\Omega^2\right) \;=\; \rho\nabla\left(\frac{1}{2}r^2\Omega^2\right) \;=\; -\rho\nabla\psi_{\mathrm{cent}}$$

where

$$\psi_{\mathrm{cent}} \;=\; -\frac{1}{2}r^2\Omega^2 \;=\; -\frac{1}{2}(\omega_0 R)^2\sin^2\theta$$

is the centrifugal potential and ω_0 is the angular rotation rate of the star. Thus the total potential of a point mass is, in the rotating frame,

$$\psi_{\mathrm{tot}} = \frac{-GM}{R} - \frac{1}{2}(\omega_0 R)^2\sin^2\theta$$

and the momentum equations are:

$$\rho\frac{\partial \boldsymbol{V}}{\partial t} + \rho\Omega \times \boldsymbol{V} + \rho(\boldsymbol{V} \cdot \nabla\boldsymbol{V}) \;=\; -\nabla P + \frac{1}{4\pi}(\nabla\times\boldsymbol{B}) \times \boldsymbol{B} - \rho\nabla\psi_{\mathrm{tot}}$$

6.3 Magnetostatic balance in dipole field

When $\boldsymbol{V} \;=\; 0$ the static balance in the magnetic field is given by:

$$-\nabla P - \rho\nabla\psi_{\mathrm{tot}} + \frac{1}{4\pi}(\nabla\times\boldsymbol{B}) \times \boldsymbol{B} \;=\; \boldsymbol{0}$$

Take the scalar product of this equation with the unit vector along the direction of the magnetic field and use

$$\begin{aligned} \nabla P \cdot \hat{\boldsymbol{b}} &= \text{Directional derivative of } P \text{ along } \boldsymbol{B} \\ &= \frac{dP}{ds} \end{aligned}$$

where s is the arc length along $\boldsymbol{B}$. The magnetostatic equation becomes

$$\frac{dP}{ds} + \rho \frac{d\psi_{\text{tot}}}{ds} = 0$$

We assume that the gas in the dead zone is isothermal so that

$$P = \frac{\rho k T}{\mu m_p}$$

with T constant. The solution of this equation goes the same as for iso-thermal atmospheres:

$$\begin{aligned} \frac{kT}{\mu m_p} \frac{1}{\rho} \frac{d\rho}{ds} &= -\frac{d\psi_{\text{tot}}}{ds} \\ \Rightarrow \frac{kT}{\mu m_p} \ln \rho &= -\psi_{\text{tot}} + \text{constant} \end{aligned}$$

and we specify the constant by taking the density $\rho = \rho_0$ at the base of the dipolar field line where $R = R_0$, i.e. the photosphere-corona boundary. Hence,

$$\begin{aligned} \frac{kT}{\mu m_p} \ln \frac{\rho}{\rho_0} &= -(\psi_{\text{tot}} - \psi_{\text{tot, 0}}) \\ \Rightarrow \frac{\rho}{\rho_0} &= \exp\left[-\frac{\mu m_p}{kT}(\psi_{\text{tot}} - \psi_{\text{tot,0}})\right] \end{aligned}$$

Now,

$$\psi - \psi_0 = -\frac{GM}{R} + \frac{GM}{R_0} - \frac{1}{2}\omega_0^2(R^2\sin^2\theta - R_0^2\sin^2\theta_0)$$

$$-\frac{\mu m_p}{kT}(\psi - \psi_0) = \frac{\mu m_p}{kT}\left(\frac{GM}{R} - \frac{GM}{R_0}\right) + \frac{1}{2}\frac{\omega_0^2\mu m_p}{kT}(R^2\sin^2\theta - R_0^2\sin^2\theta_0)$$

$$= \left(-\frac{GM}{R_0}\right)\left(\frac{\mu m_p}{kT}\right)\left[1 - \left(\frac{R_0}{R}\right)\right]$$

$$+ \frac{1}{2}\frac{(\omega_0 R_o)^2}{(kT/\mu m_p)}\left[\left(\frac{R}{R_0}\right)^2\sin^2\theta - \sin^2\theta_0\right]$$

$$= -\frac{GM}{Ra_0^2}\left[1 - \left(\frac{R_0}{R}\right)\right] + \frac{1}{2}\frac{(\omega_0 R_o)^2}{a_0^2}\left[\left(\frac{R}{R_0}\right)^2\sin^2\theta - \sin^2\theta_0\right]$$

where $a_0 = \left(\frac{kT}{\mu m_p}\right)^{1/2}$ is the isothermal sound speed in the dead zone.

Hence the expression for the density along a field line is:

$$\frac{\rho}{\rho_0} = \exp\left\{\left(-\frac{GM}{R_0 a_0^2}\right)\left[1 - \left(\frac{R_0}{R}\right)\right]\right\}$$

$$\times \exp\left\{\frac{1}{2}\frac{(\omega_0 R_0)^2}{a_0^2}\left[\left(\frac{R}{R_0}\right)^2\sin^2\theta - \sin^2\theta_0\right]\right\}$$

and the gas pressure associated with this density is given by

$$P = \rho a_0^2 = \rho_0 a_0^2 \times \left(\frac{\rho}{\rho_0}\right)$$

$$= \rho_0 a_0^2 \exp\left\{\left(-\frac{GM}{R_0 a_0^2}\right)\left[1 - \left(\frac{R_0}{R}\right)\right]\right\}$$

$$\times \exp\left\{\frac{1}{2}\frac{(\omega_0 R_o)^2}{a_0^2}\left[\left(\frac{R}{R_0}\right)^2 \sin^2\theta - \sin^2\theta_0\right]\right\}$$

So far all of this is exact with no approximations. We now consider the condition at the cusp point which defines the transition from the dead zone to the wind zone on the equator.

At the cusp we have gas in the dead zone in equilibrium with gas in the outflowing wind. Pressure balance tells us that:

$$\left[P + \frac{B_p^2}{8\pi}\right]_{\text{dead zone}} = \left[P + \frac{B_p^2}{8\pi} + \frac{B_\phi^2}{8\pi}\right]_{\text{wind}}$$

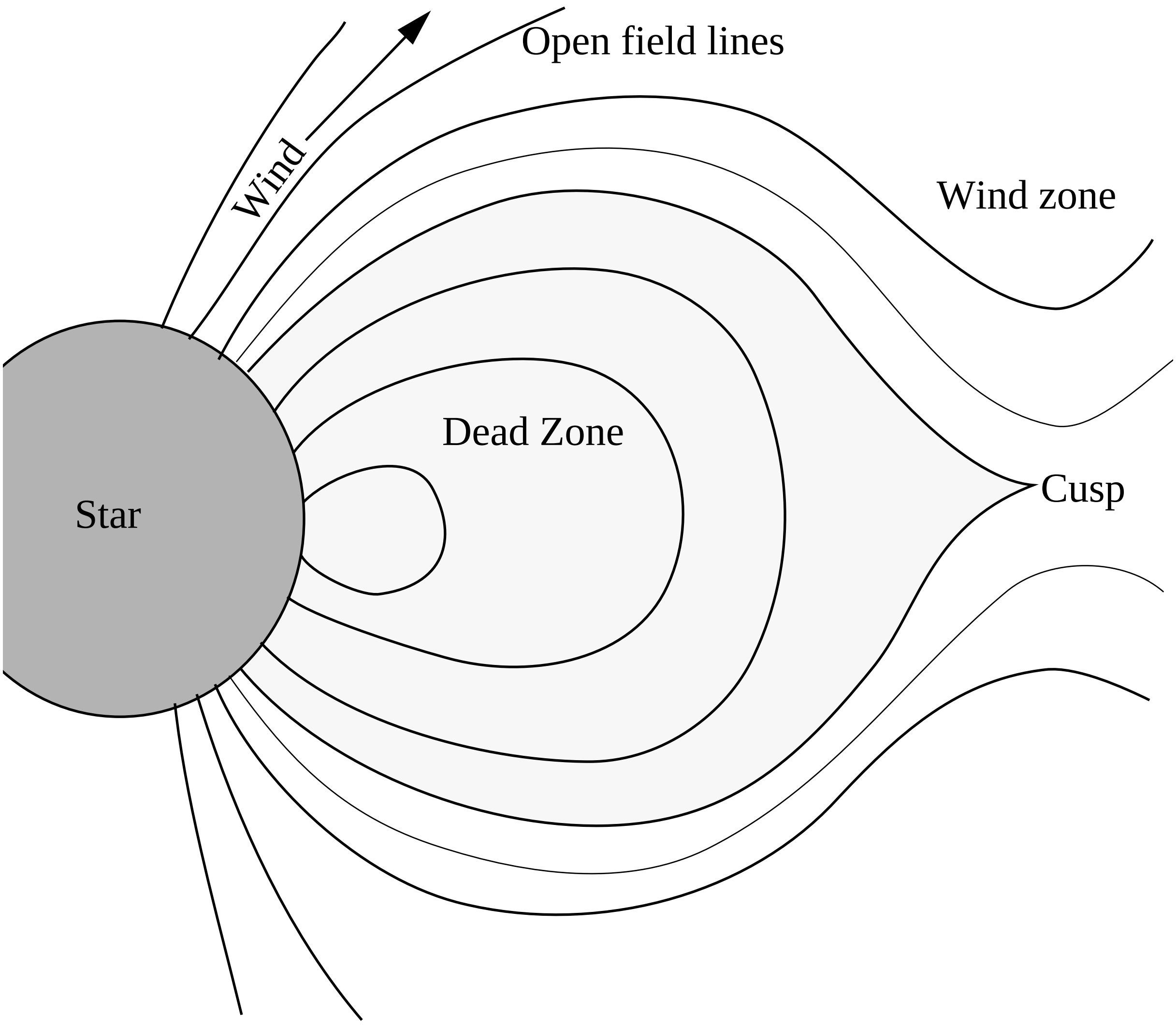
Note that there is no toroidal field in the dead zone because there is no gas motion to twist the magnetic field in the azimuthal direction. Moreover, at the cusp point. we have field lines converging into a point of zero area, so that the poloidal field there is zero.

We now introduce 2 approximations:

1. Outside the cusp point the toroidal field is small compared to the poloidal field.

2. The poloidal field in the wind zone just outside the cusp is given approximately by the dipole field, i.e. the cusp represents the transition between the $\frac{1}{R^3}$ dipolar field and the $\frac{1}{R^2}$ wind zone field.

With these approximations the pressure balance at the cusp becomes



$$P_{\text{dead zone}} \approx \frac{B^2_{\text{dipole}}\left(R, \frac{\pi}{2}\right)}{8\pi}$$

This gives an equation for the location of the cusp. Instead of using the dipole moment m as the parameter describing the dipole field, we use the magnitude of the magnetic field on the equator:

$$B_0 = \frac{m}{R_0^3} \Rightarrow m = B_0 R_0^3$$

and the pressure of the dipole field on the equator is given by:

$$\frac{B_p^2}{8\pi} = \frac{B_0^2}{8\pi}\left(\frac{R_0}{R}\right)^6$$

Hence the equation for pressure balance at the cusp is given by:

$$\frac{B_0^2}{8\pi}\left(\frac{R_0}{R}\right)^6 = \rho_0 a_0^2 \exp\left\{\left(-\frac{GM}{R_0 a_0^2}\right)\left[1-\left(\frac{R_0}{R}\right)\right]\right\}$$
$$\times \exp\left\{\frac{1}{2}\frac{(\omega_0 R_o)^2}{a_0^2}\left[\left(\frac{R}{R_0}\right)^2\sin^2\theta - \sin^2\theta_0\right]\right\}$$

We put this equation into non-dimensional form by dividing through by $\frac{B_0^2}{8\pi}$ and also by utilising the equation for a dipole field line

$$\frac{R}{R_0} = \frac{\sin^2\theta}{\sin^2\theta_0} \Rightarrow \sin^2\theta_0 = \sin^2\theta \times \left(\frac{R_0}{R}\right)$$

Also $\theta = \frac{\pi}{2}$ simplifies matters so that $\sin\theta = 1$.

The final equation locating the cusp is:

$$\left(\frac{R_0}{R}\right)^6 = \frac{8\pi\rho_0 a_0^2}{B_0^2}\exp\left\{\left(-\frac{GM}{R_0 a_0^2}\right)\left[1-\left(\frac{R_0}{R}\right)\right]\right\}$$
$$\times \exp\left\{\frac{1}{2}\frac{(\omega_0 R_0)^2}{a_0^2}\left[\left(\frac{R}{R_0}\right)^2 - \left(\frac{R_0}{R}\right)\right]\right\}$$

This is a transcendental equation for $\frac{R}{R_0}$ with parameters:

$$\alpha = \frac{GM}{R_0 a_0^2} \qquad \beta = \frac{8\pi\rho_0 a_0^2}{B_0^2} \qquad \gamma = \frac{(\omega_0 R_0)^2}{a_0^2}$$

6.4 Solution for the cusp point for typical parameters

Consider first the sun:

$$\text{Coronal isothermal sound speed}^2 = a_0^2 = \frac{kT}{\mu m_p} = \frac{k \times 2 \times 10^6}{\mu m_p}$$

$$= 2.7 \times 10^{14}\,\mathrm{cm^2/sec^2}$$

$$\Rightarrow a_0 = 1.6 \times 10^7\ \mathrm{cm/s}$$

$$\alpha = \frac{GM}{R_0 a_0^2} = \frac{G \times 2.0 \times 10^{33}}{6.96 \times 10^{10} \times 2.7 \times 10^{14}} = 7.1$$

Mestel and Spruit consider 2 values for the solar value of

$$\beta = \frac{8\pi\rho a_0^2}{B^2} = \frac{1}{4}, \frac{1}{60} = 0.25, 0.017$$

We shall take $\beta = 0.25$ for the purposes of illustration.

To estimate $\gamma = \dfrac{(\omega_0 R_0)^2}{a_0^2}$ we take a period of 30 days for the sun's rotation, so that $\omega_0 = \dfrac{2\pi}{30\ \mathrm{days}} = 2.4 \times 10^{-6}$. Hence,

$$\gamma = \frac{(2.4 \times 10^{-6} \times 6.96 \times 10^{10})^2}{2.7 \times 10^{14}} = 1.0 \times 10^{-4}$$

Hence the solar values:

$$\alpha = 7.1 \qquad \beta = 0.25 \qquad \gamma = 1.0\times10^{-4}$$

Clearly the value of γ shows that the effect of centrifugal force in the sun is negligible.

To solve for the cusp point, we put $x = \frac{R_0}{R}$ and solve the following equation for x:

$$f(\alpha, \beta, \gamma, x) = x^6 - \beta\exp[-\alpha(1-x)] \times \exp\left[\frac{1}{2}\gamma(x^{-2}-x)\right]$$

This can be easily programmed in MATLAB or MAPLE, for example and the solution for the sun is

$$x = 0.382 \Rightarrow \frac{R}{R_0} = 2.62$$

The corresponding colatitude corresponding to this field line is determined from

$$\frac{R}{R_0} = \frac{\sin^2\theta}{\sin^2\theta_0} = \frac{1}{\sin^2\theta_0}$$

$$\Rightarrow \sin\theta_0 = \left(\frac{R_0}{R}\right)^{1/2} = x^{1/2}$$

$$\Rightarrow \theta_0 = \sin^{-1}0.382 = 22.5°$$

These calculated parameters represent a first order accurate estimation of the parameters of the dead zone in the solar corona. However, the assumption of a dipole field is too simplistic and the structure of the solar corona is more complicated. See the picture on p 298 of M.S. Longair *High Energy Astrophysics*. Nevertheless, one clearly sees the cusp-like structure of the solar corona in the dead zones.

6.5 The dead zone as a function of stellar rotation

The centrifugal parameter γ depends upon the rotation of the star. Moreover, it is also generally believed that the magnetic field in stars depends upon the stellar rotation rate via dynamo processes, which are not well understood at the present time; this affects the parameter β.

Mestel & Spruit adopt the following relationship for β and γ.

$$\beta = \beta_{solar} \times \left(\frac{\omega}{\omega_0}\right)^{-1} \qquad \gamma = \gamma_{solar} \times \left(\frac{\omega}{\omega_{solar}}\right)^{2}$$

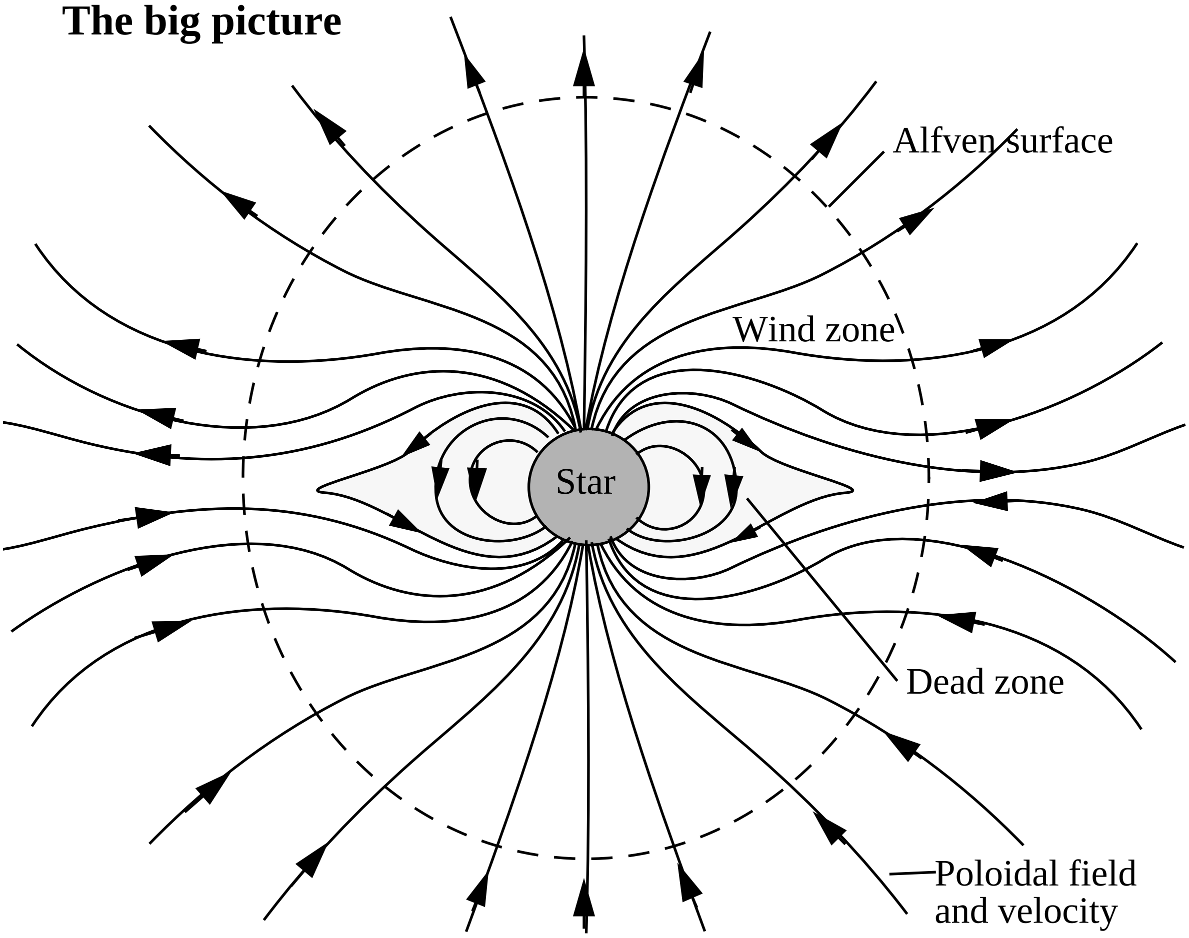
This leads to the following dependence upon rotation rate for a star of solar mass and radius:

$\frac{\omega}{\omega_{solar}}$	$\frac{R}{R_0}$	θ_0
1	2.62	22°
2	3.19	18°
10	4.59	13°
20	4.93	12°
40	4.48	13°
80	3.36	17°

You can see that the size of the dead zone increases as the magnetic field increases (reflecting the increased of the coronal plasma by the magnetic field) but that eventually the dead zone decreases in size as a result of the effect of the centrifugal force. At the same time the colatitude defining the boundary between the dead and the wind zones initially decreases (indicating a more restricted wind zone over the surface of the star) and then increases again showing the effect of the centrifugal force in opening up the wind zone.

7 Magnetic braking

The big picture



7.1 Expression for the rate of angular momentum loss

Remember that in an axisymmetric flow, the flux of angular momentum per unit mass flux is given by

$$L = rV_\phi - \frac{rB_\phi}{4\pi\alpha} = r_A^2\Omega$$

the last condition coming from the regularity of the flow at the Alfven point. The rate of loss of angular momentum by a star with a magnetised wind can therefore be evaluated by integrating over a sphere of radius R,

$$\begin{aligned} -\dot{J} &= \int_0^{2\pi}\int_0^{\pi} L\rho(\boldsymbol{V}_p \cdot \hat{\boldsymbol{R}})R^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} [r_A^2\Omega]\rho V_R R^2 \sin\theta d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} [R_A^2 \sin^2\theta\Omega]\rho V_R R^2 \sin\theta d\theta d\phi \\ &= 2\pi \times 2\int_0^{\pi/2} \rho V_R \Omega R_A^2 \sin^2\theta R^2 \sin\theta d\theta \end{aligned}$$

With the approximations we have been making, $\Omega \approx \omega_0$, the angular velocity at the surface of the star. Note that R here, is arbitrary and we take it to be the radius corresponding to that of the cusp, i.e. $R = R_c$ where $\frac{R_c}{R_0}$ is as evaluated in the previous section. Hence,

$$\begin{aligned} -\dot{J} &= 4\pi\int_0^{\pi/2} \rho V_R \Omega R_A^2 R^2 \sin^3\theta d\theta \\ &= 4\pi R_0^4 \omega_0 \int_0^{\pi/2} \rho V_R \left(\frac{R_A}{R_0}\right)^2 \left(\frac{R_c}{R_0}\right)^2 \sin^3\theta d\theta \end{aligned}$$

We eliminate ρV_p using the equation of continuity in the form:

$$\rho \boldsymbol{V}_p = \alpha \boldsymbol{B}_p \Rightarrow \frac{\rho V_R}{B_R} = \text{streamline constant}$$

Therefore, we can estimate ρV_R at $R = R_c$ in terms of the velocity and magnetic field at the surface of the star by:

$$\frac{\rho V_R}{B_R} = \frac{\rho_0 V_0}{B(R_0, \theta_0)} \Rightarrow \rho V_R = \left(\frac{B_R}{B(R_0, \theta_0)}\right)\rho_0 V_0$$

where ρ_0 is the density at the coronal base and $B(R_0, \theta_0)$ is the magnetic field at the base of the field line.

Substituting into the above expression for $-\dot{J}$, we obtain

$$-\dot{J} = 4\pi\rho_0 V_0 R_0^4 \omega_0 \int_0^{\pi/2} \left(\frac{B_R}{B(R_0, \theta_0)}\right)\left(\frac{R_A}{R_0}\right)^2\left(\frac{R_c}{R_0}\right)^2 \sin^3\theta d\theta$$

Note that we have not integrated wrt θ yet, since the variables in side the integral, in principle, can depend upon θ.

7.2 The location of the Alfven surface

One thing we require in order to evaluate the above integral is the location of the Alfven surface, i.e. the value of R_A. Previously we determined R_A for the case of a rapid rotator. We can elaborate on this for the present case of a dipole field with an R^{-3} radial dependence, making the transition to a wind zone with an R^{-2} dependence. To do this, first note that the density at the Alfven point is given by

$$\rho_A = 4\pi\alpha^2 \qquad \text{where} \qquad \alpha = \frac{\rho V_p}{B_p}$$

Hence,

$$\frac{\rho_A}{\alpha} = \left.\frac{B_p}{V_p}\right|_{R = R_A}$$

We now make the approximation that the field is approximately dipolar out to $R \approx R_c$ and then goes as R^{-2} beyond. With $B = B_0$ at $R = R_0$ and $\theta = \pi/2$, the strength of the dipole field is

$$B = B_0 \left(\frac{R_0}{R}\right)^3 (1 + 3\cos^2\theta)^{1/2}$$

so that at $R = R_c$, the magnitude of the dipole field is given by

$$B = B_0 \left(\frac{R_0}{R_c}\right)^3 (1 + 3\cos^2\theta)^{1/2}$$

and near $\theta = \pi/2$,

$$B \approx B_0 \left(\frac{R_0}{R_c}\right)^3$$

There is no point in adopting a more detailed angular dependence because of the distortion of the dipole field by the wind and also because, the angular momentum loss is dominated by values of θ near $\pi/2$. (cf. the factor of $\sin^3\theta$ in the integral for $-\dot{J}$.)

Hence,

$$\frac{\rho_A}{\alpha} = \left.\frac{B_p}{V_p}\right|_{R = R_A}$$

becomes

$$\frac{4\pi\rho_0 V_0}{B_0} = \frac{B_0\left(\frac{R_0}{R_c}\right)^3\left(\frac{R_c}{R_A}\right)^2}{\frac{2\sqrt{6}}{9}R_A\sin\theta\Omega} = \frac{B_0\left(\frac{R_0}{R_c}\right)^3\left(\frac{R_c}{R_0}\right)^2\left(\frac{R_0}{R_A}\right)^2}{\frac{2\sqrt{6}}{9}(R_0\Omega)\left(\frac{R_A}{R_0}\right)\sin\theta}$$

$$= \frac{9}{2\sqrt{6}}B_0\left(\frac{R_0}{R_c}\right)\left(\frac{R_0}{R_A}\right)^3(R_0\Omega)^{-1}(\sin\theta)^{-1}$$

Solving for $\frac{R_A}{R_0}$,

$$\begin{aligned}
\left(\frac{R_A}{R_0}\right)^3 &= \frac{9}{2\sqrt{6}}\frac{B_0^2}{4\pi\rho_0 V_0(R_0\Omega)}\left(\frac{R_0}{R_c}\right)(\sin\theta)^{-1} \\
&= \frac{9}{2\sqrt{6}}\frac{V_{A,0}^2}{V_0(R_0\Omega)}\left(\frac{R_0}{R_c}\right)(\sin\theta)^{-1} \\
\Rightarrow \frac{R_A}{R_0} &= \left(\frac{9}{2\sqrt{6}}\right)^{1/3}\left(\frac{V_{A,0}^2}{V_0(R_0\Omega)}\right)^{1/3}\left(\frac{R_0}{R_c}\right)^{1/3}(\sin\theta)^{-1/3} \\
&= \left(\frac{9}{2\sqrt{6}}\right)^{1/3}\left(\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}\right)^{1/3}\left(\frac{R_0}{R_c}\right)^{1/3}(\sin\theta)^{-1/3} \\
&\approx 1.2\left(\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}\right)^{1/3}\left(\frac{R_0}{R_c}\right)^{1/3}(\sin\theta)^{-1/3}
\end{aligned}$$

It is interesting to compare this expression with that for the “rapid rotator” which we considered earlier, viz,

$$\left(\frac{R_A}{R_0}\right)^3 = \frac{9}{2\sqrt{6}}\frac{V_{A,0}^2}{V_0(R_0\Omega)} = \frac{9}{2\sqrt{6}}\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}$$

The difference in the expressions for $\frac{R_A}{R_0}$ is the factor $\left(\frac{R_0}{R_c}\right)^{1/3}$ which reflects the effect of the dead zone in determining the Alfven radius and ultimately the loss of angular momentum.

7.2.1 Alfven radius for the sun

Earlier we evaluated the position of the cusp point for the sun at

$$\frac{R_c}{R_0} \approx 2.6$$

The parameter $\frac{V_{A,0}^2}{V_0 V_{\text{rot}}}$ may be estimated in the following way: We write

$$\frac{V_{A,0}^2}{V_0 V_{\text{rot}}} = \frac{B_0^2}{4\pi\rho_0 V_0 V_{\text{rot}}}$$
$$= \frac{B_0^2 R_0^2}{4\pi\rho_0 V_0 R_0^2 V_{\text{rot}}}$$

The parameter $\rho_0 V_0 R_0^2$ may be related to the total mass flux $\dot{M}$ by

$$\dot{M} = 2 \times \int_0^{2\pi} \int_0^{\theta_0} \rho_0 V_0 R_0^2 \sin\theta d\theta d\phi = 4\pi\rho_0 V_0 R_0^2 (1 - \cos\theta_0)$$
$$\Rightarrow 4\pi\rho_0 V_0 R_0^2 = \frac{\dot{M}}{(1 - \cos\theta_0)}$$

Hence,

$$\frac{V_{A,0}^2}{V_0 V_{\text{rot}}} = \frac{B_0^2 R_0^2 (1 - \cos\theta_0)}{\dot{M} V_{\text{rot}}}$$

For the sun, we take

$$B_0 = 1\ \text{G} \qquad R_0 = 6.96\times10^{10}\ \text{cm} \qquad \theta_0 = 22.5°$$
$$\dot{M} = 2.5\times10^{-14}\ \text{solar masses / yr} = 1.5\times10^{12}\ \text{gm/s}$$
$$V_{\text{rot}} = 6.96\times10^{10}\ \text{cm} \times 2.4\times10^{-6}\ \text{s}^{-1} = 1.7\times10^{5}\text{cm/s}$$

and

$$\frac{V_{A,0}^2}{V_0 V_{\text{rot}}} = 1400$$

Also we take the same value for the parameter

$$\beta = \frac{8\pi\rho_0 a_0^2}{B^2}$$

as before, i.e. $\beta \approx 0.25$ so that $\frac{R_c}{R_0} \approx 2.6$ and $\theta_0 \approx 22.5°$

Therefore,

$$\frac{R_A}{R_0} \approx 1.2 \times 1400^{1/3} \times 2.6^{-1/3}(\sin\theta)^{-1/3} \approx 9.7(\sin\theta)^{-1/3}$$

7.3 Final expression for the rate of magnetic braking

Previously we showed,

$$-\dot{J} = 4\pi\rho_0 V_0 R_0^4 \omega_0 \int_0^{\pi/2} \left(\frac{B_R(R_c, \theta)}{B(R_0, \theta_0)}\right)\left(\frac{R_A}{R_0}\right)^2\left(\frac{R_c}{R_0}\right)^2 \sin^3\theta d\theta$$

In the spirit of the previous approximations, we take

$$\frac{B_R(R_c, \theta)}{B(R_0, \theta_0)} \approx \left(\frac{R_c}{R_0}\right)^{-3}$$

and from the solution for $\frac{R_A}{R_0}$

$$\left(\frac{R_A}{R_0}\right)^2 \approx \left(\frac{9}{2\sqrt{6}}\right)^{2/3}\left(\frac{V_{A,0}^2}{V_0 V_{\rm rot}}\right)^{2/3}\left(\frac{R_0}{R_c}\right)^{2/3}(\sin\theta)^{-2/3}$$

so that

$$-\dot{J} \approx 4\pi\left(\frac{9}{2\sqrt{6}}\right)^{2/3}\omega_0\rho_0 V_0 R_0^4\left(\frac{V_{A,0}^2}{V_0 V_{\rm rot}}\right)^{2/3}\left(\frac{R_c}{R_0}\right)^{-5/3}\int_0^{\frac{\pi}{2}} \frac{\sin^3\theta}{\sin^{2/3}\theta} d\theta$$

The value of the integral is approximately 0.739. Combining with the other numerical values:

$$4\pi\left(\frac{9}{2\sqrt{6}}\right)^{2/3} \times 0.739 \approx 13.93$$

and

$$\begin{aligned} -\dot{J} &\approx 13.9\omega_0\rho_0 V_0 R_0^4\left(\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}\right)^{2/3}\left(\frac{R_c}{R_0}\right)^{-5/3} \\ &= \frac{1.1}{(1-\cos\theta_0)}\dot{M}R_0^2\left(\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}\right)^{2/3}\left(\frac{R_c}{R_0}\right)^{-5/3} \end{aligned}$$

where we have again eliminated $\rho_0 V_0 R_0^2$ in favour of the mass flux in the wind.

The total angular momentum is:

$$J = I\omega_0$$

where I is the moment of inertia. Hence the inverse time scale for angular momentum loss is:

$$\tau_J^{-1} = -\frac{\dot{J}}{J} = \frac{1.1}{(1-\cos\theta_0)}\frac{\dot{M}R_0^2}{I}\left(\frac{V_{A,0}^2}{V_0 V_{\mathrm{rot}}}\right)^{2/3}\left(\frac{R_c}{R_0}\right)^{-5/3}$$

To obtain an approximate estimate we take

$$I \approx \frac{2}{5}MR_0^2,$$

the moment of inertia of a sphere of uniform density. The real value of the moment of inertia is less since the density of the sun is centrally concentrated. Using this value therefore underestimates τ_J^{-1} and overestimates τ_J. However, using this value gives

$$\tau_J^{-1} \approx \frac{2.8}{(1-\cos\theta_0)} \frac{\dot{M}}{M} \left(\frac{V_{A,0}^2}{V_0 V_{\text{rot}}}\right)^{2/3} \left(\frac{R_c}{R_0}\right)^{-5/3}$$

For the sun

$$\theta_0 \approx 22.5° \qquad \frac{V_{A,0}^2}{V_0 V_{\text{rot}}} \approx 1400$$

$$\frac{R_c}{R_0} \approx 2.6 \qquad M \approx 2 \times 10^{33} \text{ gm}$$

$$\dot{M} \approx 1.5 \times 10^{12} \text{ gm/s}$$

These values give

$$\tau_J^{-1} \approx 7.0 \times 10^{-19} \text{ s}^{-1}$$

$$\Rightarrow \tau_J \approx 1.4 \times 10^{18} \text{ s} \approx 4.5 \times 10^{10} \text{ yrs}$$

Despite the approximations, this estimate should be accurate to order of magnitude and shows us that the sun is slowing down as a result of magnetic braking on a timescale of 10^{10} years. As you can see from the expression for τ_J^{-1} the rate of magnetic braking increases as the parameter $\frac{V_{A,0}^2}{V_0 V_{\text{rot}}}$ increases, that is the importance of the magnetic field increases.