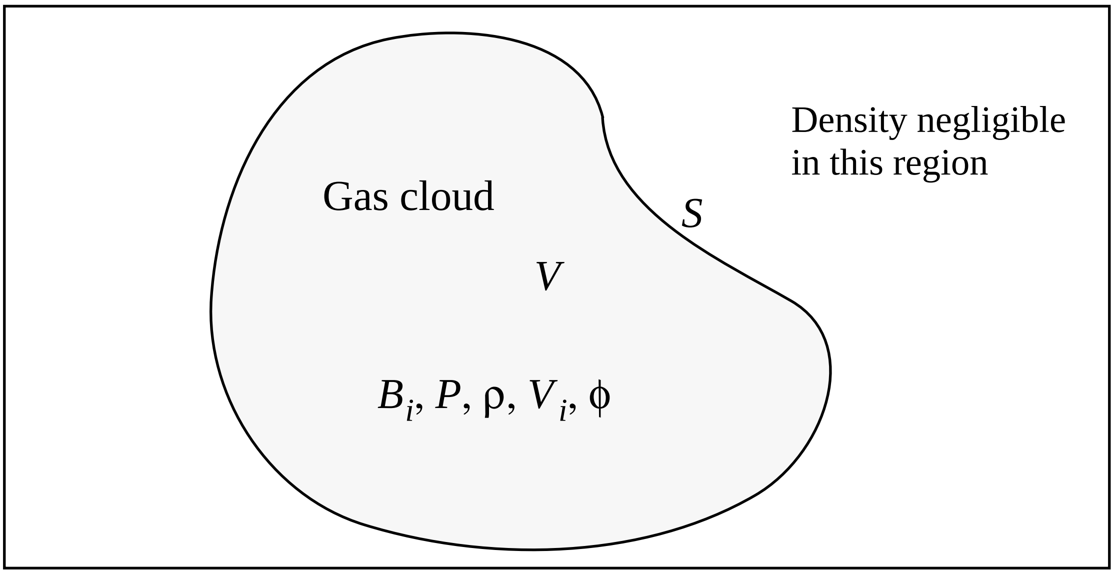
The Magnetic Virial Theorem

1 Introduction

The magnetic virial theorem is an example of how we can use integral constraints on a dynamical system to extract useful information. We do this without a detailed knowledge of the partial differential equations which are involved in a more detailed analysis.

2 Derivation of the virial theorem



2.1 Aside: The gravitational field

In the following we use some results related to the solutions of Poisson's equation for the gravitational potential. This leads to the gradient of the gravitational potential in closed form.

Poisson's equation is:

$$\nabla^2\phi = 4\pi G\rho$$

and the solution for a finite volume V is

$$\phi = -G\int_V \frac{\rho(x_j')}{|x_j - x_{j'}|} d^3x'$$

You can think of this solution as integrating the infinitesimal potentials $-\frac{G\delta m}{r_{12}}$ from mass elements δm, where r_{12} is the distance from δm to the source point. The negative gradient of ϕ is

$$-\frac{\partial\phi}{\partial x_i} = -G\int_V \frac{\rho(x_j')}{|x_j - x_{j'}|^3}(x_i - x_i')d^3x'$$

(Note that we have differentiated wrt x_i within the integral sign.)

Gravitational energy

The self gravitational energy is calculated by integrating the contributions $-G\frac{\delta m_1 \delta m_2}{r_{12}}$ from the entire body, i.e.

$$W = -\frac{1}{2}G\int_V\int_V \frac{\rho(x_i)\rho(x_j')}{|x_j - x_{j'}|} d^3x d^3x'$$

where the factor of $1/2$ is necessary because the integration counts each contribution twice.

2.2 The virial theorem

We begin with the momentum equation:

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j) = -\frac{\partial P}{\partial x_i} - \rho\frac{\partial\phi}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi}\delta_{ij}\right],$$

multiply by x_i and integrate over an isolated volume of gas. This gives us a number of terms.

First term:

$$T_1 = \int_V x_i \frac{\partial}{\partial t}(\rho V_i) d^3x = \int_V \frac{\partial}{\partial x_i}\left(\frac{1}{2}x_j x_j\right)\frac{\partial}{\partial t}(\rho V_i) d^3x$$

$$= \int_V \left[\frac{\partial}{\partial x_i}\left(\frac{1}{2}x_j x_j \frac{\partial}{\partial t}\rho V_i\right) - \frac{1}{2}x_j x_j \frac{\partial^2}{\partial x_i \partial t}(\rho V_i)\right] d^3x$$

The first integral can be converted to a surface integral over the surface S. For the second part we exchange the order of integration and use the continuity equation, so that

$$\frac{\partial^2}{\partial x_i \partial t}(\rho V_i) = \frac{\partial^2}{\partial t \partial x_i}(\rho V_i) = -\frac{\partial^2 \rho}{\partial t^2}$$

Hence,

$$\int_V \left[\frac{\partial}{\partial x_i}\left(\frac{1}{2}x_j x_j \frac{\partial}{\partial t}\rho V_i\right) - \frac{1}{2}x_j x_j \frac{\partial^2}{\partial x_i \partial t}(\rho V_i)\right] d^3x = \int_S \frac{1}{2}x_j x_j \frac{\partial}{\partial t}\rho V_i n_i dS$$

$$+ \frac{1}{2}\int_V \frac{\partial^2 \rho}{\partial t^2} r^2 d^3x$$

We define the inertia tensor:

$$I_{ij} = \int_V \rho x_i x_j d^3 x$$

$$\Rightarrow I_{ii} = \int_V \rho r^2 d^3 x$$

Therefore,

$$T_1 = \int_S \frac{1}{2} r^2 \frac{\partial}{\partial t} \rho V_i n_i dS + \frac{1}{2} \frac{d^2 I_{ii}}{dt^2}$$

For our “isolated” volume in which the velocity is zero outside, the surface integral vanishes and

$$T_1 = \frac{1}{2} \frac{d^2 I_{ii}}{dt^2}$$

Second term:

$$
\begin{aligned}
{}_2 &= \int_V x_i \frac{\partial}{\partial x_j} (\rho V_i V_j) d^3 x \\
&= \int_V \left[\frac{\partial}{\partial x_j} (x_i \rho V_i V_j) - \delta_{ij} \rho V_i V_j \right] d^3 x \\
&= \int_S x_i \rho V_i V_j n_i dS - \int_V \rho V^2 d^3 x
\end{aligned}
$$

Again the surface integral vanishes and

$$T_2 = -\int_V \rho V^2 d^3 x = -2K$$

where the kinetic energy

$$K = \frac{1}{2} \int_V \rho V^2 d^3 x$$

Third term:

$$T_3 = -\int_V x_i \frac{\partial P}{\partial x_i} d^3x = -\int_V \frac{\partial}{\partial x_i}(Px_i) d^3x + \int_V 3P d^3x$$

$$= -\int_S P x_i n_i dS + 2U$$

where, the total internal energy,

$$U = \int_V \varepsilon d^3x$$

Fourth term:

$$T_4 = -\int_V x_i \rho \frac{\partial \phi}{\partial x_i}$$

$$= -G\int_V x_i \rho(x_i) d^3x \int_V \frac{\rho(x_j')(x_i - x_i')}{|x_j - x_j'|^3} d^3x'$$

$$= -G\int_V \int_V \rho(x_i)\rho(x_j') \frac{x_i(x_i - x_i')}{|x_j - x_j'|^3} d^3x d^3x'$$

Now consider:

$$x_i(x_i - x_i') = x_i x_i - x_i x_i'$$

and

$$(x_i - x_i')(x_i - x_i') = x_i x_i - 2x_i x_i' + x_i' x_i'$$

$$= (x_i x_i - x_i x_i') + (x_i' x_i' - x_i' x_i)$$

Since the double integral over V involves both x_i and x_i' symmetrically, then we can write

$$\begin{aligned} T_4 &= -\frac{1}{2}G\int_V\int_V \rho(x_i)\rho(x_j{}')\frac{|x_i - x_i|^2}{|x_j - x_j{}'|^3}d^3x d^3x' \\ &= -\frac{1}{2}G\int_V\int_V \frac{\rho(x_i)\rho(x_j{}')}{|x_j - x_j{}'|}d^3x d^3x' \\ &= W \end{aligned}$$

Fifth term:

$$\begin{aligned} T_5 &= \int_V x_i\frac{\partial M_{ik}}{\partial x_k}d^3x = \int_V \left[\frac{\partial}{\partial x_k}(x_i M_{ik}) - \delta_{ik}M_{ik}\right]d^3x \\ &= \int_S x_i M_{ik} n_k dS - \int_V M_{ii} d^3x \\ &= \int_S x_i M_{ik} n_k dS + \int_V \frac{B^2}{8\pi}d^3x \\ &= \int_S x_i M_{ik} n_k dS + U_B \end{aligned}$$

where, the magnetic energy,

$$U_B = \int_V \frac{B^2}{8\pi}d^3x$$

Collecting terms,

$$T_1 + T_2 = T_3 + T_4 + T_5$$

$$\frac{1}{2}\frac{d^2 I_{ii}}{dt^2} - 2K = -\int_S P x_i n_i dS + 2U + W + \int_S x_i M_{ik} n_k dS + U_B$$

$$\frac{1}{2}\frac{d^2 I_{ii}}{dt^2} = 2K + 2U + W + U_B - \int_S P x_i n_i dS + \int_S x_i M_{ik} n_k dS$$

The last equation is the magnetic virial theorem.

In many circumstances, we can simplify the first surface integral by assuming that the pressure on the surface of our isolated cloud is constant. Also,

$$\int_S x_i n_i dS = \int_V \frac{\partial x_i}{\partial x_i} d^3x = 3V$$

Therefore the magnetic virial theorem becomes:

$$\frac{1}{2}\frac{d^2 I_{ii}}{dt^2} = 2K + 2U + W + U_B - 3P_{\text{ext}}V + \int_S x_i M_{ik} n_k dS$$

3 The virial theorem in a static configuration

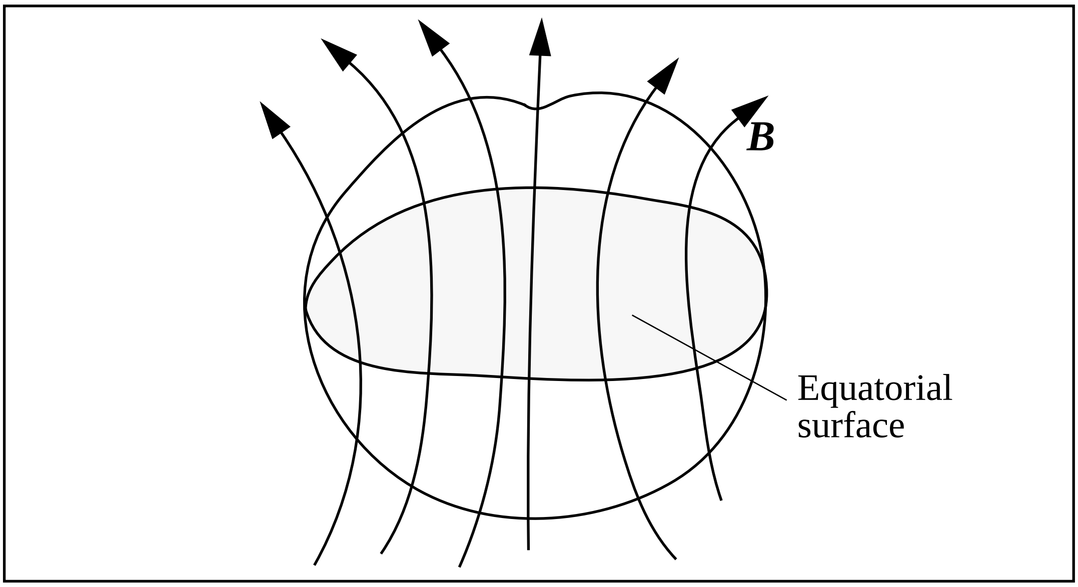
When the gas cloud is in a static equilibrium, then

$$\frac{d^2 I_{ii}}{dt^2} = K = 0$$

and

$$2U + W + U_B + \int_S x_i M_{ik} n_k dS = 3P_{\text{ext}}V$$

4 Application: gravitational collapse of a magnetised cloud



4.1 Derivation of equilibrium equation

In making use of the virial theorem, we utilise various “factor of order unity” estimates for the various terms.

Internal energy:

We assume that the cloud is isothermal. Therefore,

$$2U = 3\int_V P d^3x = 3\int_V a^2 \rho d^3x = 3a^2 M$$

where

$$a^2 = \frac{kT}{\mu m_p}$$

is the square of the isothermal sound speed.

The gravitational energy

$$W = -\frac{1}{2}G\int_V\int_V \frac{\rho(x_i)\rho(x_j')}{|x_j - x_{j'}|} d^3x d^3x'$$

We can see that dimensionally

$$W \sim -\frac{GM^2}{R}$$

where R is a mean radius for the cloud defined by

$$\frac{4\pi}{3}R^3 = V$$

We put

$$W = -\alpha\frac{GM^2}{R}$$

The magnetic energy

$$U_B = \int_V \frac{B^2}{8\pi} d^3x \sim \frac{B^2}{8\pi}\frac{4\pi}{3}R^3 = \frac{1}{6\pi^2}\frac{(\pi BR^2)^2}{R} = \frac{1}{6\pi^2}\frac{\Phi^2}{R}$$

where Φ is the flux through the central region of the cloud. The reason we concentrate on the flux Φ is that we know this is conserved from the flux-freezing theorem.

The term

$$\int_S x_i M_{ik} n_k dS \sim \frac{B^2}{8\pi} \times 4\pi R^3$$

so that it also similar to $\frac{\Phi^2}{R}$. If the field is perfectly uniform, then the two magnetic contributions cancel exactly. However, if the field “bows out” as shown in the diagram, then its value on the surface is less than its value in the centre of the cloud and we put

$$U_B + \int_S x_i M_{ik} n_k dS = \beta \frac{\Phi^2}{R}$$

The external pressure term

$$3P_{ext}V \approx 4\pi P_{ext} R^3$$

Our virial equation, therefore is

$$P_{ext} = \frac{1}{4\pi}\left[-\frac{\alpha G M^2}{R^4} + \beta\frac{\Phi^2}{R^4} + \frac{3a^2 M}{R^3}\right]$$

4.2 Special cases

(i) $\Phi = G = 0$ i.e. a cloud confined by pressure alone.

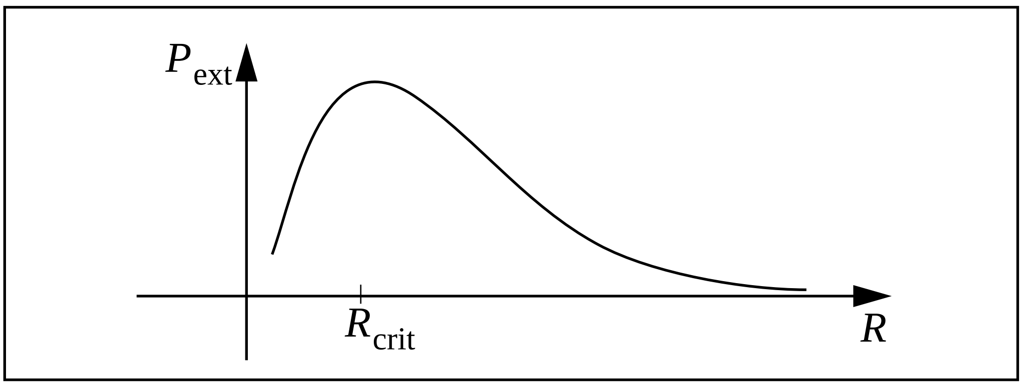
In this case the equilibrium equation becomes:

$$P_{ext} = \frac{3a^2 M}{4\pi R^3} = a^2 \rho$$

i.e. the pressure $a^2\rho$ of the cloud is balanced by the external pressure.

(ii) $\Phi = 0$ i.e. gravity but no magnetic field.

$$P_{ext} = -\frac{\alpha G M^2}{4\pi R^4} + \frac{3a^2 M}{4\pi R^3}$$



There are 2 terms of opposite sign. As R decreases the negative term (the self gravity term) starts to win out and the curve of P_{ext} as a function of R has a maximum at

$$R_{crit} = \frac{4\alpha GM}{9a^2}$$

For $R < R_{crit}$ the external pressure required to keep the cloud in equilibrium decreases with decreasing R, so that for a given P_{ext} there is nothing to stop the cloud collapsing.

(iii) $\Phi \neq 0$ Magnetic field included.

$$P_{ext} = \frac{1}{4\pi R^4}[-\alpha GM^2 + \beta\Phi^2] + \frac{3}{4\pi R^3}a^2M$$
$$= \frac{1}{4\pi}\left\{\frac{\alpha G}{R^4}(M_\Phi^2 - M^2) + \frac{3a^2M}{R^3}\right\}$$

where the mass, M_Φ, associated with the magnetic flux, is given by

$$M_\Phi = \left(\frac{\beta}{\alpha}\right)^{1/2} G^{-1/2}\Phi \approx 0.13 G^{-1/2}\Phi$$

The value of $\left(\frac{\beta}{\alpha}\right)^{1/2} \approx 0.13$ is the only part of this calculation which is derived from an equilibrium calculation. It is actually close to the value of $\left(\frac{1}{6\pi^2}\right)^{1/2}$ estimated from the magnetic energy density alone.

$M < M_\Phi$

If this is true, then P_{ext} always increases with decreasing radius. Equivalently, if we "squeeze" the cloud then it finds that it would need a larger pressure to confine it at the smaller radius and it therefore springs back to its original radius.

Hence a cloud whose mass

$$M < 0.13G^{-1/2}\pi BR^2 \approx 10^3 M_O \left(\frac{B}{30\mu G}\right)\left(\frac{R}{2\text{ pc}}\right)^2$$

will always be stable to gravitational collapse.

$M > M_\Phi$

When this is true, then the curve of P_{ext} as a function of R has the same form as it has without a magnetic field and gravitational collapse can occur for radii less than a critical one.

5 Concluding comments

The functional form of equilibrium provided by the virial theorem gives us a way of understanding the stability of equilibrium, without a detailed knowledge of the entire system.