# Characterisation of Magnetic Forces

## 1 Introduction

The momentum equation

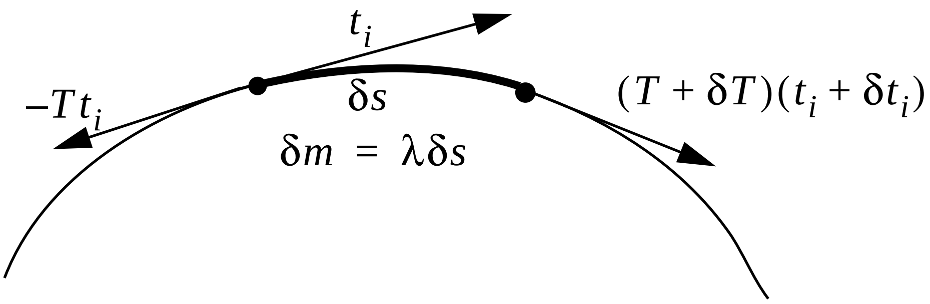
$$\rho \frac{dV_i}{dt} = -\rho \frac{\partial \phi}{\partial x_i} - \frac{\partial P}{\partial x_i} + \frac{\partial M_{ij}}{\partial x_j}$$

$$M_{ij} = \left[ \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right]$$

contains pressure gradient terms and gravitational force terms that we are familiar with together with the divergence of the term $M_{ij}$ that we have referred to as “magnetic stresses”. The purpose of the following is to come to a better physical understanding of what this term represents physically and what effect it can have on magnetised gas.

## 2 Aside: the forces on a stretched string

Before going further it is helpful to consider the forces acting on a stretched string. This analogy is useful for one part of the magnetic force.



Forces on an element of a stretched string

Take the tension in a stretched string to be $T$. This is the force exerted over a cross-section of the string by the rest of the string.

Take $t_i$ to be the unit tangent to the string, $s$ to be the arc-length along the string, the mass per unit length to be $\lambda$ so that the mass of the element is $\delta m = \lambda\delta s$ The force on an element of the string as shown in the diagram is

$$\begin{aligned} \delta F_i &= (T(s) + \delta T)(t_i + \delta t_i) - T(s)t_i \\ &= T(s)t_i + \delta T t_i + T(s)\delta t_i - T(s)t_i \\ &= \delta T t_i + T(s)\delta t_i \\ &= \left[\frac{dT}{ds}t_i + T(s)\frac{dt_i}{ds}\right]\delta s \end{aligned}$$

Now the Frenet-Serret relations for a curve tell us that

$$\frac{dt_i}{ds} = \kappa n_i$$

where $\kappa$ is the curvature and $n_i$ is the unit normal. Hence the equation of motion of the mass element is

$$\lambda\frac{dV_i}{dt} = \frac{dT}{ds}t_i + T\kappa n_i$$

i.e. there is a force along the string equal to the rate of the change of the tension with arc-length and there is a force in the direction of curvature proportional to the curvature times the tension.

# 3 Decomposition of the magnetic forces

We can write

$$\frac{\partial M_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{B_iB_j}{4\pi}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)$$

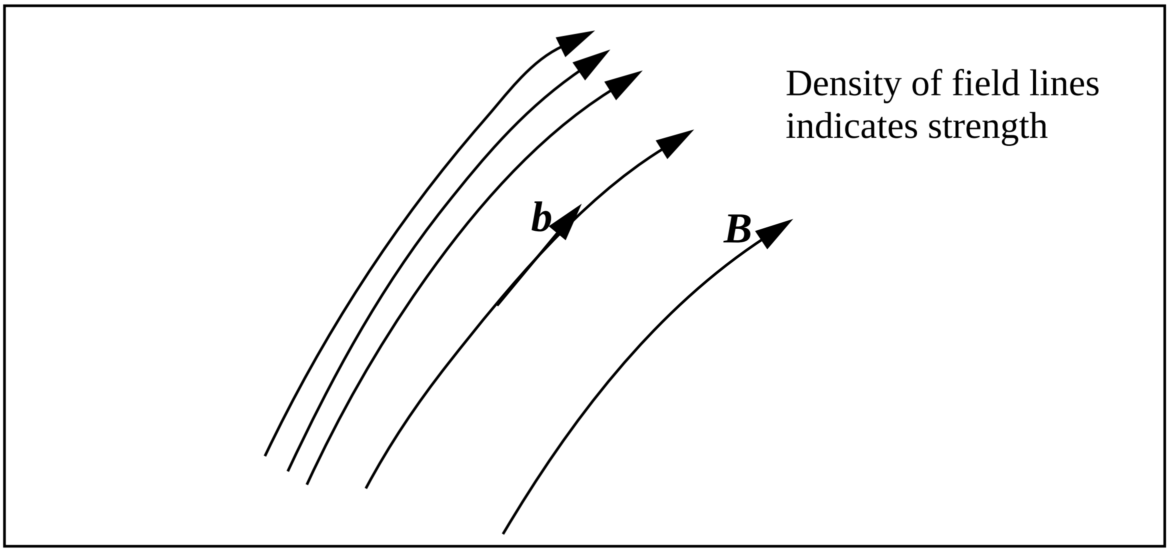
$$= \left(\frac{B_j}{4\pi}\frac{\partial B_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)$$

We now write

$$B_i = Bb_i$$

where $b_i$ is a unit vector in the direction of the magnetic field and is therefore tangent to the magnetic field lines. If $x_i = x_i(s)$ are the coordinates of a field line with arclength $s$, then

$$\frac{dx_i}{ds} = b_i$$



We can therefore write the magnetic force terms as:

$$\begin{aligned}\frac{\partial M_{ij}}{\partial x_j} &= \frac{1}{4\pi}Bb_j\frac{\partial}{\partial x_j}(Bb_i) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) \\ &= \frac{Bb_jb_i}{4\pi}\frac{\partial B}{\partial x_j} + \frac{B^2}{4\pi}b_j\frac{\partial b_i}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right) \\ &= b_ib_j\frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi}\left(b_j\frac{\partial b_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{B^2}{8\pi}\right)\end{aligned}$$

The first and thrid terms can be combined in the form:

$$-P_{ij}\frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right)$$

where the projection operator

$$P_{ij} = \delta_{ij} - b_ib_j$$

projects vectors into the space normal to the magnetic field. That is, suppose we have a vector $U_i$, then $P_{ij}U_j$

is normal to the magnetic field, since,

$$b_i P_{ij} U_j = b_i(\delta_{ij} - b_i b_j) U_j = (b_j - b_j) U_j = 0$$

The operator $P_{ij}$ therefore projects the gradient operator $\frac{\partial}{\partial x_j}$ perpendicular to the magnetic field, ie. the operator $P_{ij}\frac{\partial}{\partial x_j}$ is the component of the gradient perpendicular to $\boldsymbol{B}$.

Hence, we express the divergence of the stress tensor in the form:

$$\frac{\partial M_{ij}}{\partial x_j} = -P_{ij}\frac{\partial}{\partial x_j}\left(\frac{B^2}{8\pi}\right) + \left(\frac{B^2}{4\pi}\right)\kappa_B n_i$$

ie, the sum of a gradient perpendicular to the magnetic field plus a term proportional to the curvature of the magnetic field lines. It is the last term, in particular that distinguishes magnetic forces from pure hydrostatic forces. Note also that the component of magnetic force along the field lines is zero:

$$B_i \frac{\partial M_{ij}}{\partial x_j} = 0$$

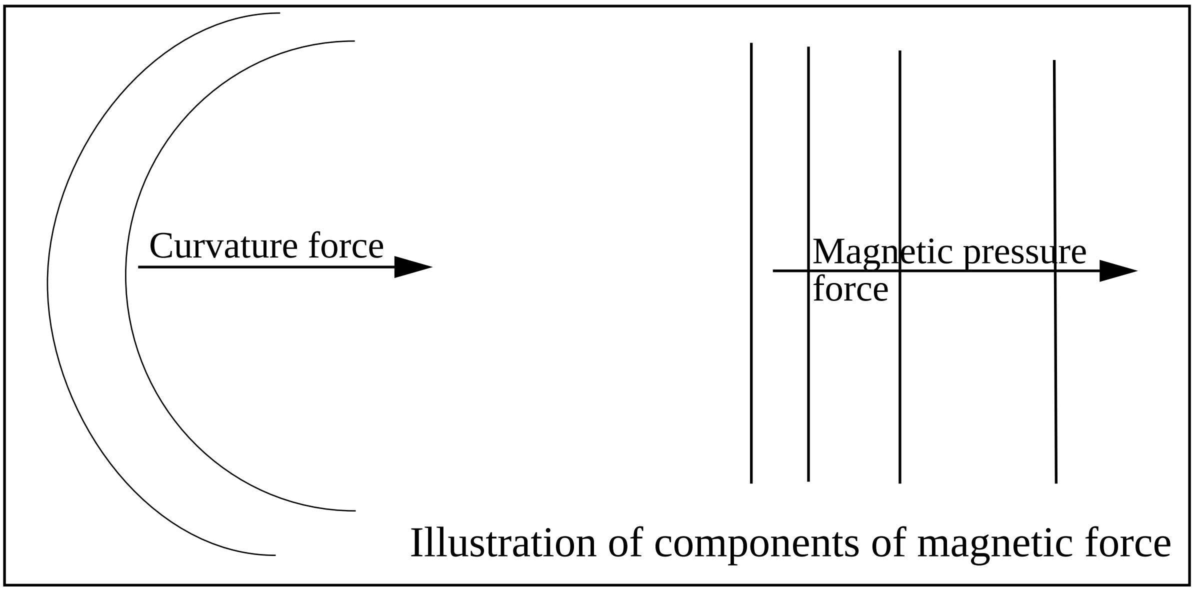
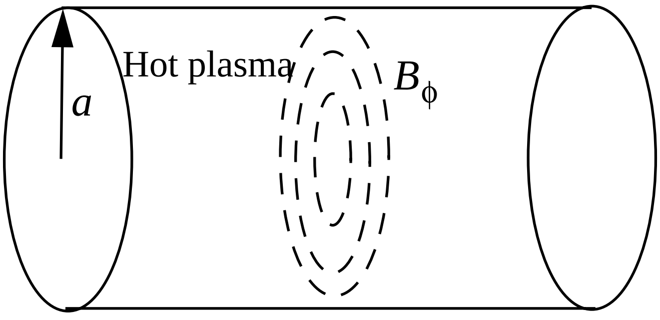


Illustration of components of magnetic force

# 4 The magnetic pinch

The confinement of a plasma by a toroidal magnetic field is an example of the different forces provided by a magnetic field. We can also analyse the stability of this configuration using the physical concepts derived above.



Confinement of a plasma column by a magnetic field

## 4.1 Magnetostatic equilibrium

From the above diagram, once can see that it is feasible that the “curvature force" associated with the magnetic "tension" can plausibly confine a hot plasma. To see if this is possible, we analyse the magnetostatic configuration using the momentum equations:

$$\rho\frac{d\boldsymbol{V}}{dt} = -\nabla P + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \boldsymbol{0}$$

We analyse this situation in cylindrical polars, and take

$$\boldsymbol{B} = (B_r, B_\phi, B_z) = (0, B_\phi, 0)$$

so that

$$\nabla\times\boldsymbol{B} = \left[\frac{1}{r}\frac{\partial B_z}{\partial\phi} - \frac{\partial B_\phi}{\partial z}\right]\hat{\boldsymbol{r}} + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial B_r}{\partial\phi}\right]\hat{\boldsymbol{z}}$$

$$= -\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{z}}$$

and

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{B_\phi}{r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{r}} - B_\phi\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{z}}$$

We now take $B_\phi$ to be independent of $z$ and

$$\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi)\hat{\boldsymbol{r}}$$

and the force on the plasma is in the inward radial direction if $rB_\phi$ increases outwards.

## Radial magnetostatic equilibrium

Because of the limitations we have imposed, we only have to consider the radial force balance which is expressed by the equation:

$$-\frac{\partial P}{\partial r}-\frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi) = 0$$

There is a wide variety of magnetostatic equilibria that we could envisage. For the sake of simplicity, we consider one in which the current density in the plasma is uniform. Ampere's law becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = \frac{4\pi}{c}j_z = \text{constant}$$

The solution of this is

$$\begin{aligned} rB_\phi &= \frac{2\pi}{c}r^2 j_z + C \\ \Rightarrow B_\phi &= \frac{2\pi}{c}r j_z + \frac{C}{r} \\ &= \frac{2\pi}{c}r j_z \end{aligned}$$

The constant $C$ is put to zero so that the magnetic field is finite at $r = 0$.

The magnetostatic equilibrium equation becomes an equation for the pressure:

$$\begin{aligned} -\frac{\partial P}{\partial r} &= \frac{B_\phi}{4\pi r}\frac{\partial}{\partial r}(rB_\phi) \\ &= \frac{2\pi}{c^2}r j_z^2 \end{aligned}$$

Hence

$$P = A-\frac{\pi}{c^2}j_z^2 r^2$$

where $A$ is a constant which is determined by the condition that the plasma be confined to $r < a$, i.e.

$$P = 0 \qquad \text{at } r = a \qquad \Rightarrow P = \frac{\pi j_z^2}{c^2}(a^2 - r^2)$$

Note that, with this solution,

$$P + \frac{B_\phi^2}{4\pi} = \text{constant}$$

## Magnetic field outside $r = a$

The region outside $r = a$ is envisaged as a vacuum, so that Ampere's law in this region becomes:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = 0$$

$$\Rightarrow rB_\phi = \text{constant}$$

$$B_\phi = \frac{C}{r}$$

This constant of integration is determined by continuity at $r = a$. Hence,

$$B_\phi = \frac{C}{a} = \frac{2\pi}{c}a j_z$$

$$\Rightarrow C = \frac{2\pi}{c}a^2 j_z$$

$$\Rightarrow B_\phi = \frac{2\pi}{c}\frac{a^2}{r}j_z$$

We can also easily derive this form of the solution from the integral form of Ampere's law, viz,

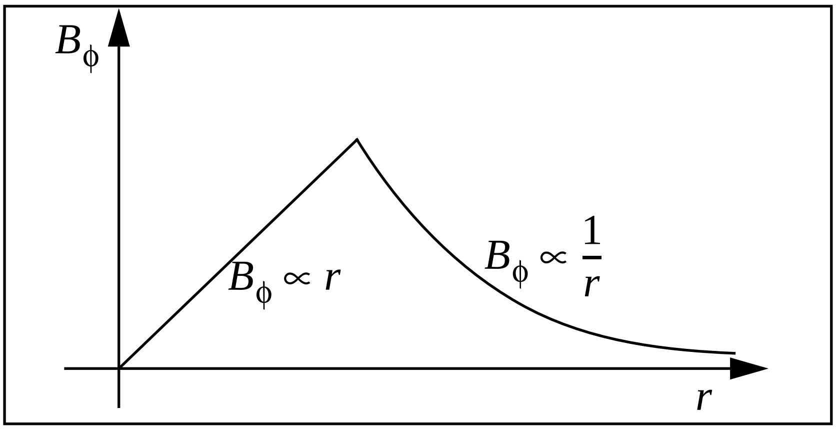
$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \frac{4\pi}{c}\int_A (j \cdot n)ds$$

where *C* encloses the area *A*. Here we just take C to be a circle of radius $2\pi r$ outside the plasma column, so that the above integral formulation reads:

$$B \times 2\pi r = \frac{4\pi}{c} j_z \times \pi a^2 \Rightarrow B_\phi = \frac{2\pi}{c}\frac{a^2}{r} j_z$$

as before.

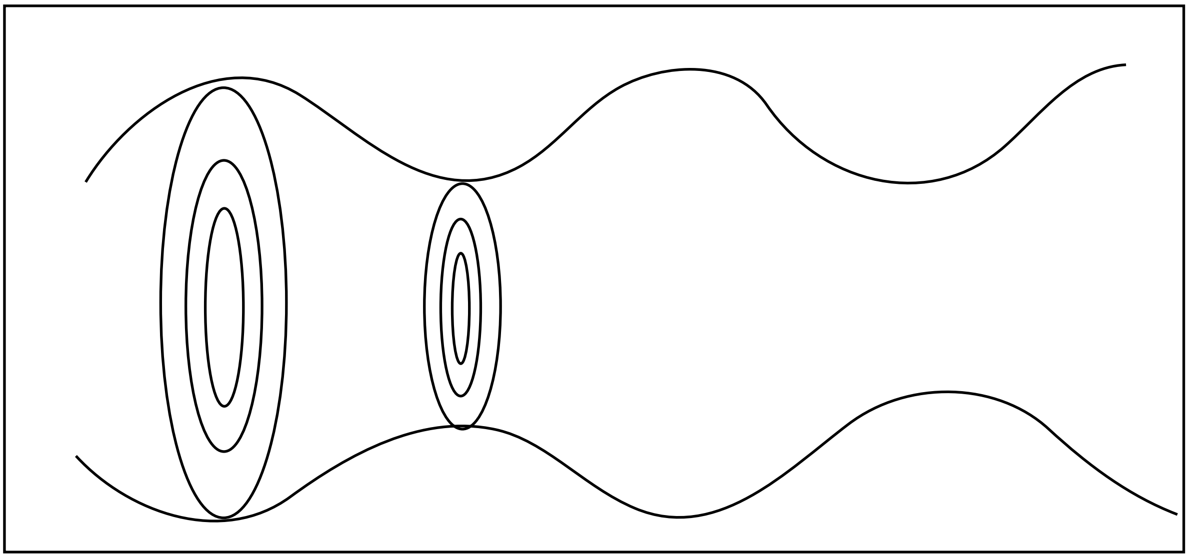
The radial profile of the toroidal field therefore looks like the following diagram:



## 4.2 Stability of the magnetic pinch

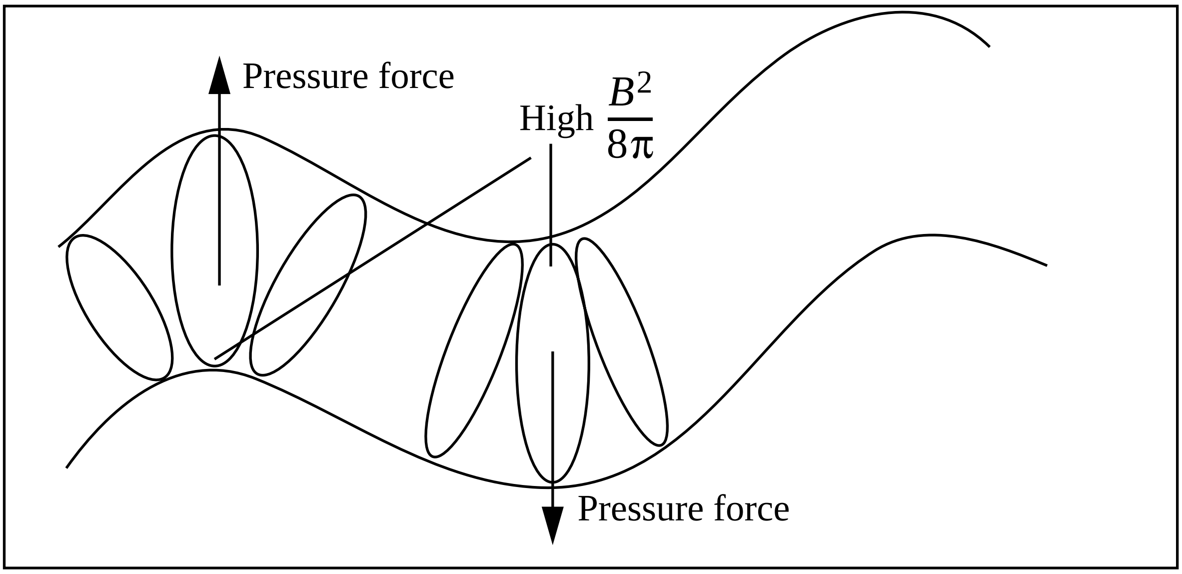
The magnetic pinch is subject to two well-known instabilities – the “sausage” or “pinch” instability and the “firehose” instability. With our knowledge of the nature of magnetic forces, we can analyse these instabilities as follows.

## The “sausage” instability (pinch instability)



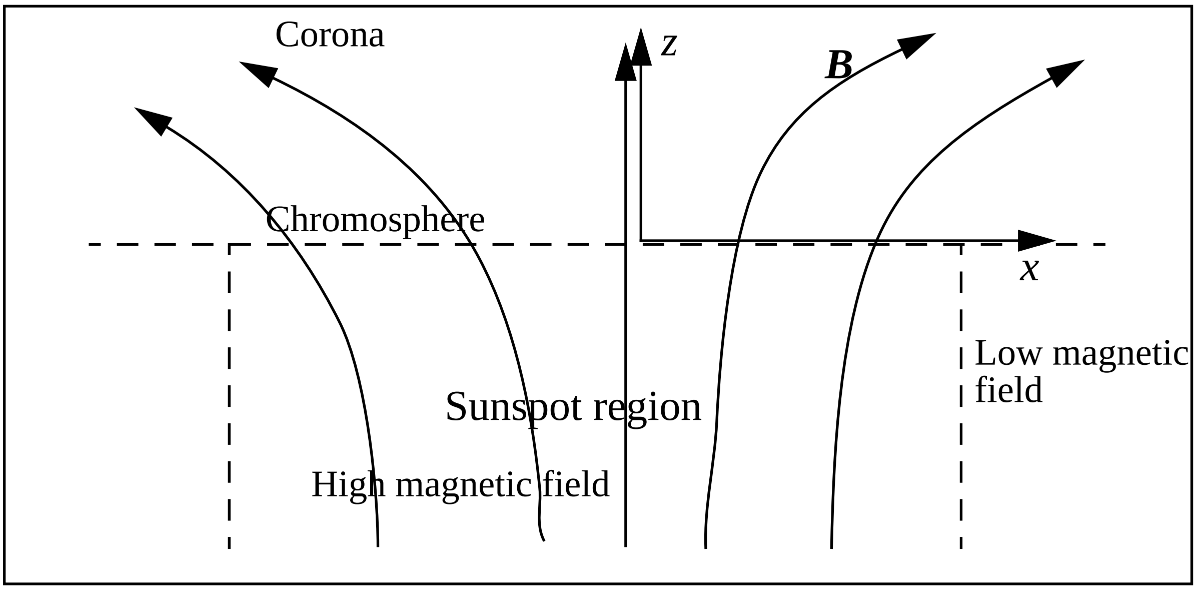
Consider an equilibrium plasma column which is perturbed by being “squeezed” as indicated. Since the field lines follow the motion they are squeezed as well. Hence the curvature force increases by virtue of the increased value of $\frac{B^2}{4\pi}$ and because of the higher curvature of the field lines. The pinching effect of the field is greater so that the toroidal magnetic field pinches the plasma even further. The end result is a sequence of blobs.

## The “firehose” instability



Now consider a toroidally confined plasma column which is perturbed in an oscillatory fashion. Again because the field lines follow the motion of the plasma, the resulting perturbation to the field is as shown. The bunching up of field lines causing a magnetic pressure gradient as shown and the direction of this is to enhance the perturbation. The perturbation therefore grows in the manner of a hose with water flowing through it – hence the name *firehose instability*.

# 5 Sunspots



Sunspots are a classic example of the simple application of magnetostatics and provide us with an example of the importance of magnetic pressure. They are regions of the solar photosphere which are much cooler than average. ($T_{\text{sunspot}} \approx 3800$ K as opposed to $T \approx 5780$ K for the rest of the sun's photosphere.) Consider a model of a sunspot as indicated above. Equilibrium in the horizontal ($x$) direction implies for $\boldsymbol{B} = B_x\boldsymbol{i} + B_z\boldsymbol{k}$

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\frac{B_x^2}{4\pi}\right) + \frac{\partial}{\partial z}\left(\frac{B_x B_z}{4\pi}\right) - \frac{\partial}{\partial x}\left(\frac{B_x^2 + B_z^2}{8\pi}\right)$$

In this model we neglect $B_x$ in the region below the sunspot so that

$$P + \frac{B_z^2}{8\pi} = \text{constant}.$$

Therefore if we envisage the sunspot as having a high magnetic field inside and comparatively negligible field outside, then

$$\left.\left(P+\frac{B_z^2}{8\pi}\right)\right|_{\text{sunspot}} = P_{\text{photosphere}}$$

Since, in our model, we assume that the magnetic field is independent of height, then

$$\left.\frac{\partial P}{\partial z}\right|_{\text{sunspot}} = \left.\frac{\partial P}{\partial z}\right|_{\text{photopshere}}$$

Now consider the vertical equilibrium. This is expressed by the equation:

$$0 = -\frac{\partial P}{\partial z}+\rho g_z+\frac{\partial}{\partial x}\left(\frac{B_x B_z}{4\pi}\right)+\frac{\partial}{\partial z}\left(\frac{B_z^2}{4\pi}\right)-\frac{\partial}{\partial z}\left(\frac{B^2}{8\pi}\right)$$

where $g_z$ is the local acceleration due to gravity. Since $B_x \ll B_z$ below the photosphere boundary and there is no dependence of $B_z$ on height (why?), then

$$\frac{\partial P}{\partial z} = \rho g_z$$

in both sunspot and the surrounding photosphere. Since we have shown that the pressure gradient is the same in both, then the density must be the same in both regions. Hence the equation for horizontal equilibrium becomes

$$nkT_{\text{sunspot}}+\frac{B_{\text{sunspot}}^2}{8\pi} = nkT_{\text{photosphere}}$$

$$\Rightarrow \frac{B_{\text{sunspot}}^2}{8\pi} = (nk)(T_{\text{photosphere}}-T_{\text{sunspot}})$$

Typical parameters

$$\begin{aligned} n &\sim 10^{17}\ \ \mathrm{cm}^{-3} \\ T_{\mathrm{photosphere}} &\approx 5780\ \ \mathrm{K} \\ T_{\mathrm{sunspot}} &\approx 3800 \\ \Rightarrow B &\approx 830\ \mathrm{Gauss} \end{aligned}$$

This is typical of the magnetic field that is observed from Zeeman measurements of the sun's photosphere.

## 6 The $\beta$ parameter

Magnetic forces are important when the magnetic field is "large". What does large mean? We parameterize the relative importance of magnetic and thermal forces via the $\beta$ parameter, defined by:

$$\beta = \frac{P_{\mathrm{gas}}}{P_{\mathrm{magnetic}}} = \frac{nkT}{B^2/(8\pi)}$$

Thus "low $\beta$" means a stong magnetic field.