

# Derivation of the Magnetohydrodynamic Equations from Kinetic Theory

## Course references

### Astrophysics references

S. Chandrasekhar: *Hydrodynamic and Hydromagnetic Stability*

T.G. Cowling: *Magnetohydrodynamics*

E. Parker: *Magnetic Fields*

B. Rossi and S. Olbert: *Introduction to the Physics of Space*

S. Shore: *An Introduction to Astrophysical Hydrodynamics*

F. Shu: *The Physics of Astrophysics, Vol. 2, Gas Dynamics*

If you purchase any one of these, the book by Shu is recommended.

### General physical references

J.D. Jackson: *Classical Electrodynamics*

L.D. Landau & E.M. Lifshitz: *The Electrodynamics of Continuous*

*Media*

E.M. Lifshitz & L.P. Pitaevskii: *Physical Kinetics*

K. Huang: *Statistical Mechanics*

## 1Basis of kinetic theory

### 1.1 Equations of motion

The aim of kinetic theory is to derive continuum equations involving macroscopic variables (density, bulk velocity, pressure etc.) given fundamental equations of motion for the particles constituting the continuum. In the case of MHD, the fundamental equations of motion are:

$$m \frac{d\mathbf{v}}{dt} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) - m \nabla \phi$$

Lorentz force |  
Gravitational force

where

$\mathbf{v}$  = Velocity

$m$  = mass

$t$  = time

$q$  = electric charge

$\mathbf{E}$  = Electric field

$c$  = speed of light

$\mathbf{B}$  = Magnetic field

$\phi$  = Gravitational potential

The electric and magnetic fields are described by Maxwell's equations given here in Gaussian units:

**Dyadic notation:**

Note the “e” for electric

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho_e & \nabla \times \mathbf{B} &= \frac{4\pi}{c} \mathbf{j}_e + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0 \end{aligned}$$

**Tensor notation:**

Particle equations of motion:

$$m \frac{dv_i}{dt} = q \left( E_i + \epsilon_{ijk} \frac{v_j}{c} B_k \right) - m \frac{\partial \phi}{\partial x_i}$$

Maxwell’s equations:

Gauss’s law of electrostatics

$$\frac{\partial E_i}{\partial x_i} = 4\pi\rho_e$$

$$\frac{\partial B_i}{\partial x_i} = 0$$

No magnetic monopoles

Ampere’s law with Maxwell displacement current

$$\epsilon_{ijk} \frac{\partial B_k}{\partial x_j} = \frac{4\pi}{c} j_{e,i} + \frac{1}{c} \frac{\partial E_i}{\partial t}$$

$$\epsilon_{ijk} \frac{\partial E_k}{\partial x_j} + \frac{1}{c} \frac{\partial B_i}{\partial t} = 0$$

Faraday’s law of induction

**1.2 Energy density, Poynting flux and Maxwell stress**

## tensor

### Definitions

$$\epsilon_{EM} = \frac{E^2 + B^2}{8\pi} = \text{Electromagnetic energy density}$$

$$S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j B_k = \text{Electromagnetic Energy Flux density}$$

$$\Pi_i^{EM} = \frac{S_i}{c^2} = \text{Electromagnetic momentum density}$$

$$M_{ij}^B = \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right) = \begin{array}{l} \text{Magnetic component of Maxwell} \\ \text{stress tensor} \end{array}$$

$$M_{ij}^E = \frac{1}{4\pi} \left( E_i E_j - \frac{1}{2} E^2 \delta_{ij} \right) = \begin{array}{l} \text{Electric component of Maxwell} \\ \text{stress tensor} \end{array}$$

### Relationships

$$\frac{\partial \epsilon_{EM}}{\partial t} + \frac{\partial S_i}{\partial x_i} = -j_{e,i} E_i$$

$$\frac{\partial \Pi_i}{\partial t} - \frac{\partial M_{ij}}{\partial x_j} = - \left( \rho_e E_i + \epsilon_{ijk} \frac{j_{e,j}}{c} B_k \right)$$

**Gaussian and SI units compared (following Jackson)**

Physical quantity	Symbol	SI	1 SI unit =? Gaussian	Gaussian/cgs
Length	$l$	meter (m)	$10^2$	centimetre (cm)
Mass	$m$	kilogram (kg)	$10^3$	gram (gm)
Time	$t$	second (s)	1	second (s)
Frequency	$\nu$	Hertz (Hz)	1	Hertz (Hz)
Force	$F$	Newton (N)	$10^5$	dyne
Work	$W$	Joule (J)	$10^7$	erg
Energy	$E$	Joule (J)	$10^7$	erg
Power	$P$	Watt (W)	$10^7$	ergs s <sup>-1</sup>
Pressure	$pP$	N m <sup>-2</sup>	10	dyne cm <sup>-2</sup>
Charge	$q$	Coulomb (C)	$3 \times 10^9$	statcoulomb
Charge density	$\rho_e$	C m <sup>-3</sup>	$3 \times 10^3$	statcoulomb cm <sup>-3</sup>
Current	$I$	ampere (amp)	$3 \times 10^9$	statampere
Current density	$j$	amp m <sup>-2</sup>	$3 \times 10^5$	statamp cm <sup>-2</sup>

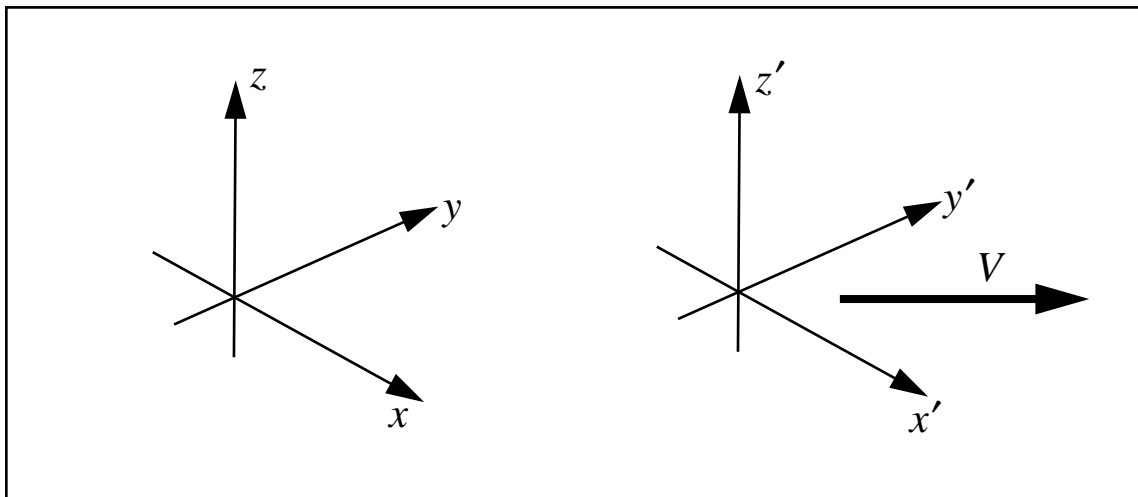
*Magnetic Virial Theorem*

Physical quantity	Symbol	SI	1 SI unit =? Gaussian	Gaussian/cgs
Electric field	$E$	V m <sup>-1</sup>	$\frac{1}{3} \times 10^{-4}$	statvolt cm <sup>-1</sup>
Potential	$\Phi, V$	Volt (V)	$\frac{1}{300}$	statvolt
Polarization	$P$	C m <sup>-2</sup>	$3 \times 10^5$	dipole moment cm <sup>-3</sup>
Displacement	$D$	C m <sup>-2</sup>	$12\pi \times 10^5$	statvolt cm <sup>-1</sup>
Conductivity	$\sigma$	mho m <sup>-1</sup>	$9 \times 10^9$	sec <sup>-1</sup>
Resistance	$R$	ohm	$\frac{1}{9} \times 10^{-11}$	sec cm <sup>-1</sup>
Capacitance	$C$	farad	$9 \times 10^{11}$	cm
Magnetic flux	$\Phi, F$	weber	$10^8$	gauss cm <sup>2</sup> or maxwells
Magnetic induction	$B$	tesla	$10^4$	gauss
Magnetic field	$H$	amp turn m <sup>-1</sup>	$4\pi \times 10^{-3}$	oersted

Physical quantity	Symbol	SI	1 SI unit =? Gaussian	Gaussian/cgs
Magnetization	$M$	amp m <sup>-1</sup>	10 <sup>-3</sup>	magnetic moment cm <sup>-3</sup>
Inductance	$L$	henry	$\frac{1}{9} \times 10^{-11}$	

### 1.3 Transformation properties under Galilean transformations

It is important to know the way in which variables change under the effect of a Galilean transformation



$$x' = x - Vt \Leftrightarrow x = x' + Vt$$

For particle velocities:

$$v' = v - V \Leftrightarrow v = v' + V$$

For particle momenta:

$$\mathbf{p}' = \mathbf{p} - m\mathbf{V} \Leftrightarrow \mathbf{p} = \mathbf{p}' + m\mathbf{V}$$

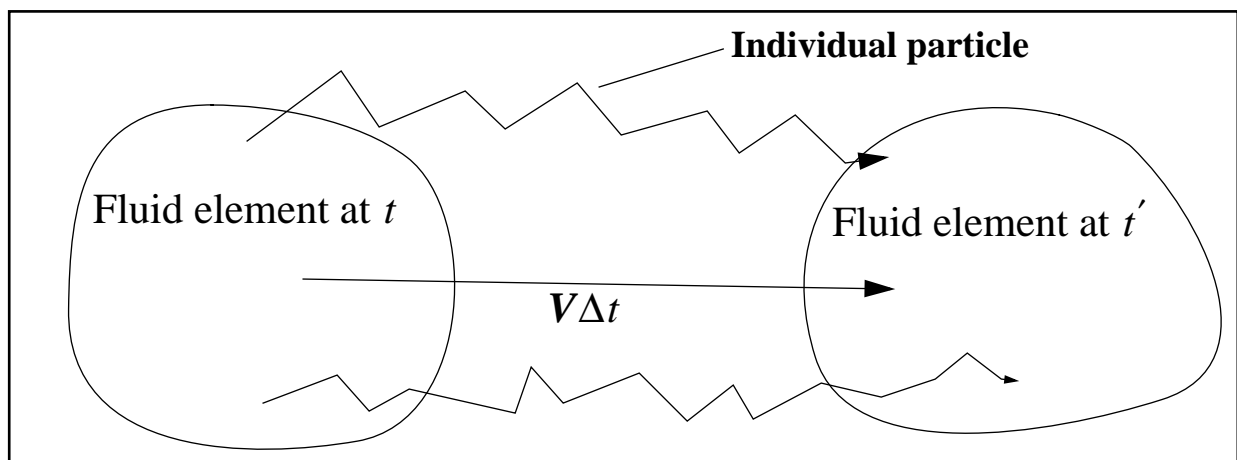
For the electromagnetic field

$$\mathbf{B}' = \mathbf{B}$$

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B}$$

Note that the transformation properties of the electromagnetic field are different under a Lorentz transformation. The above equations for  $\mathbf{E}$  and  $\mathbf{B}$  are for the case where  $V^2/c^2 \ll 1$

## 2Basis for fluid dynamics



If the mean free path of a particle is much less than the length scale of the system under consideration, then a fluid approximation is valid. There is some diffusion of the particles wrt the mean flow and this results in viscosity.

$l$  = Mean free path

$L$  = Length scale of flow

$l \ll L \Rightarrow$  Fluid

## **Velocities of particles:**

$$\mathbf{v} = \mathbf{V} + \mathbf{v}'$$

$\mathbf{V}$  = Mean velocity

$\mathbf{v}'$  = Fluctuating component

## **Mean free path:**

Can be due to:

- Collisions between ions and/or neutrals with neutrals
- Coulomb collisions between charged particles
- Collisions between particles and waves
- Mean free path perpendicular to the magnetic field also determined by gyroradius of charged particle.

$$\text{Mean free path} = l_{\text{mfp}} = \frac{1}{n\sigma}$$

$n$  = no density of collision targets

$\sigma$  = Cross-section

## **Cross sections:**

Neutral atom  $\sigma \approx 10^{-19} \text{ m}^2$

Earth's atmosphere:

$$n \sim 10^{25} \text{ m}^{-3} \Rightarrow l_{\text{mfp}} \approx \frac{1}{10^{25} \times 10^{-19}} \approx 10^{-6} \text{ m} = 1 \mu$$

Hence the atmosphere can be treated as a fluid down to these scales. NB This is not true in the rarefied regions of the upper atmosphere.

Neutral Hydrogen Gas Cloud:

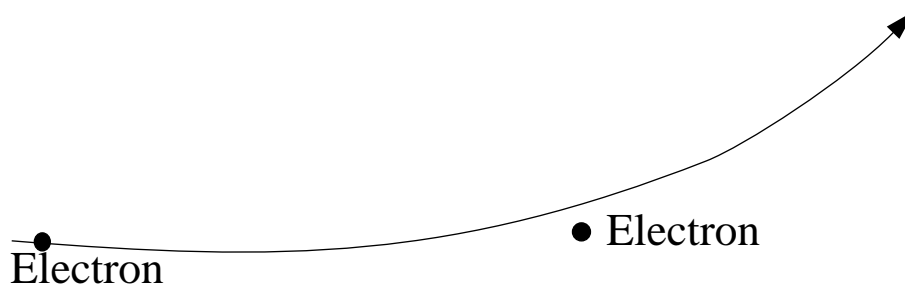
$$n \sim 10^7 \text{ m}^{-3} \Rightarrow l_{\text{mfp}} \approx \frac{1}{10^7 \times 10^{-19}} \approx 10^{12} \text{ m}$$

But

$$L \sim \text{few pc} \sim 10^{17} \text{ m} \Rightarrow \frac{l}{L} \sim 10^{-5} \ll 1 .$$

(Note  $1 \text{ pc} \approx 3.1 \times 10^{16} \text{ m}$ .)

## **Collisions between charged particles**



Long range force implies that Total cross-section (averaged over all impact parameters) is infinite. Ignore this problem for the time being.

## Order of magnitude estimate:

Collision of two thermal electrons: Define effective radius of collisional cross-section by:

$$\frac{e^2}{r_{\text{eff}}} \sim m_e v^2 \sim kT$$

Electrostatic PE  $\sim$  Relative KE  $\sim$  Thermal Energy

Cross-section:

$$r_{\text{eff}} = \frac{e^2}{kT}$$

$$\sigma \approx \pi r_{\text{eff}}^2$$

$$l \approx \frac{1}{n\pi r_{\text{eff}}^2} \sim \frac{(kT)^2}{n\pi e^4} = \frac{m_e v_{\text{th}}^4}{n_e e^4}$$

where  $v_{\text{th}} = \left(\frac{kT}{m_e}\right)^{1/2} \sim$  Thermal Speed

e.g. Hot ISM

$$n_e \sim 10^{-2} \quad T \sim 10^6 \quad k = 1.38 \times 10^{-16} \text{ ergs/K}$$

$$l \sim 4 \times 10^{19} \text{ cm} \sim 10 \text{ pc}$$

cf. Size of Galaxy: Distance of Sun to Galactic Centre = 8.5 kpc

$$\Rightarrow \frac{l}{L} \sim 10^{-3}$$

## 3Liouville and Boltzmann equations

The basis for the derivation of the equations of magnetohydrodynamics is a statistical mechanics approach based upon the Boltzmann equation.

### 3.1 Distribution function

Phase space consists of coordinates and momenta of particles, i.e.

$$\text{Phase Space} = \{x_1, x_2, x_3, p_1, p_2, p_3\}$$

Each component of the fluid denoted by superscript  $a$ .

$$f^a(x_i, p_i) d^3x d^3p = \begin{array}{l} \text{No of particles of species } a \\ \text{occupying volume element} \\ d^3x d^3p \text{ of phase space.} \end{array}$$

$$\text{Number Density of component} = n^a(x_i) = \int f^a(x_i, p_i, t) d^3p$$

### 3.2 Transformation properties

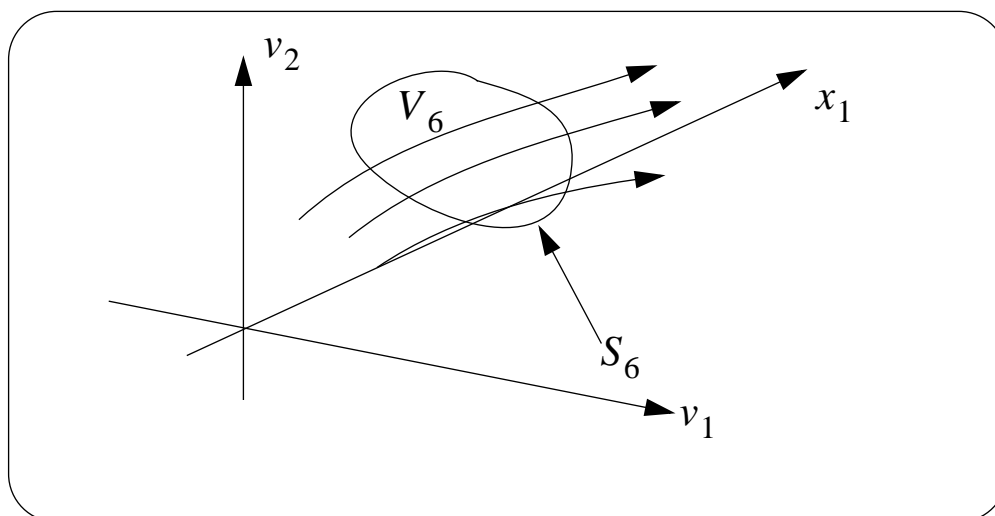
Let us consider the transformation of the distribution function under a Galilean transformation.

The number of particles,  $f^a d^3x d^3p$ , within the phase-space volume  $d^3x d^3p$  is obviously invariant under a Galilean transformation since we are just counting particles. Also, the Jacobean of the transformation:

$$\begin{aligned} x'_i &= x_i - V_i t \\ p'_i &= p_i - m^a V_i \end{aligned}$$

is unity so that  $d^3x d^3p \rightarrow d^3x' d^3p'$ . Hence  $f^a(x'_i, p'_i) = f^a(x_i, p_i)$ , i.e. the distribution function is invariant under a Galilean transformation.

### 3.3 Flow of particles in phase space



Suppose particles are moving in such a way that there are no sudden changes in velocity due to collisions; then the flow of particles in phase space is a smooth one.

Suppose particles moving in phase space on smooth trajectories. The rate of change of the number of particles within a six dimensional volume in phase space is given by the negative flux of particles through the 5 dimensional boundary of the volume. We define:

$$\text{Six-dimensional Velocity} = v_\alpha = \left( \frac{dx_i}{dt}, \frac{dp_i}{dt} \right)$$

$$\text{Six-dimensional Divergence operator} = \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial p_i} \right)$$

and the conservation of particle number tells us that:

$$\frac{\partial}{\partial t} \int_{V_6} f^a dV_6 + \int_{S_5} f^a v_\alpha n_\alpha dS_5 = 0$$

The surface integral over  $S_5$  can be converted to an integral throughout the volume  $V_6$  so that

$$\frac{\partial}{\partial t} \int_{V_6} f^a dV_6 + \int_{V_6} \frac{\partial}{\partial x_\alpha} (f^a v_\alpha) dV_6 = 0$$

and since the volume is arbitrary:

$$\frac{\partial}{\partial t} f^a + \frac{\partial}{\partial x_\alpha} (f^a v_\alpha) = 0$$

## Liouville's equation

Splitting the above 6-dimensional divergence into 2 three dimensional parts related to space and momentum gives:

$$\frac{\partial}{\partial t} f^a(x_i, p_i, t) + \frac{\partial}{\partial x_i} \left( \frac{dx_i}{dt} f^a(x_i, p_i, t) \right) + \frac{\partial}{\partial p_i} \left( \frac{dp_i}{dt} f^a(x_i, p_i, t) \right) = 0$$

where

$$\frac{dx_i}{dt} = v_i$$

$$\frac{dp_i}{dt} = F_i = eZ^a \left( E_i + \varepsilon_{ijk} \frac{v_j}{c} B_k \right) + m^a g_i$$

where  $F_i$  is the force on the particle.

## Notation:

$$e = \text{Electronic Charge} = 4.8 \times 10^{-10} \text{ esu}$$

$$Z^a = \text{Atomic No. of component}$$

$$E_i = \text{Electric Field}$$

$$B_k = \text{Magnetic Field}$$

$$v_i^a = \text{Velocity}$$

$$g_i = \text{Gravitational Field}$$

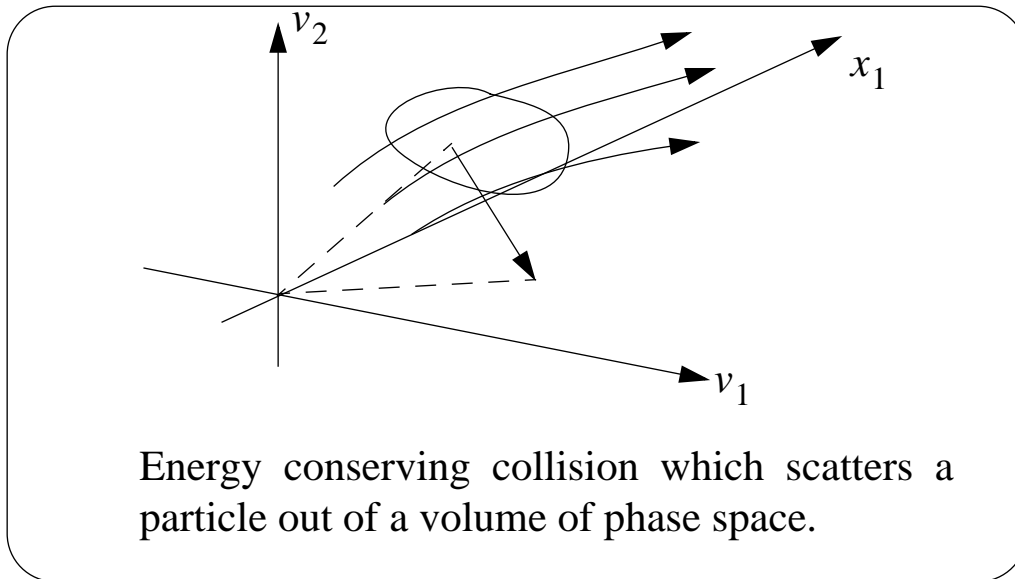
In considering Liouville's equation it is important to keep in mind the three levels of approximation in astrophysical plasmas:

- ***Very dilute plasma.*** The force components are unaffected by the plasma and collisions between particles are unimportant. e.g. Cosmic Rays in Earth's magnetic Field. Here the force components are given functions of  $x_i$  and Liouville's equation provides a satisfactory basis for theoretical work.
- ***Collisionless plasma.*** Collisions are unimportant but substantial charge and current densities due to the ions themselves. In this case  $E_i$  and  $B_i$  depend upon  $f^a$  and the situation is highly non-linear. Liouville's equation is used.
- ***Fluid.*** Collisions are important. Here Liouville's equation is unsatisfactory since one is required to take into account the collisions between elements of the fluid. Boltzmann's equation forms the basis for the derivation of the equations of gas dynamics.

## **Other applications of Liouville's equation**

Liouville's equation also provides the basis for the treatment of the stellar dynamics of collisionless systems such as galaxies.

## 4 The effect of collisions.



In contrast to the smooth flow in momentum space implied by Liouville's equation, collisions scatter particles in and out of a given volume of phase space.

## Boltzmann equation

$$\frac{\partial}{\partial t} f^a(x_i, p_i, t) + \frac{\partial}{\partial x_i} \left( \frac{dx_i}{dt} f^a(x_i, p_i, t) \right) + \frac{\partial}{\partial p_i} \left( \frac{dp_i}{dt} f^a(x_i, p_i, t) \right) = \left( \frac{\delta f^a}{\delta t} \right)_{\text{coll}}$$

where the right hand side formally represents the effect of collisions. Integrals over momentum space of the Boltzmann equation provide the basis for the equations of fluid dynamics.

## 5 Moments of the distribution function

Before moving on to the implications of the Boltzmann equation it is useful to consider the various moments of the distribution function.

## 5.1 Definitions of continuum variables

### Number density

$$n^a(x_i, t) = \int f^a(x_i, p_i, t) d^3 p$$

### Particle flux density

$$\chi_i^a = \int v_i^a f^a(x_i, p_i, t) d^3 p$$

### Mean velocity

$$V_i^a = \frac{\chi_i^a}{n^a} \Rightarrow n^a V_i^a = \chi_i^a$$

### Mass density

$$\rho^a = n^a m^a \quad \rho = \sum_a \rho^a$$

### Electric charge density

$$\rho_e^a = e Z^a n^a \quad \rho_e = \sum_a \rho_e^a$$

### Electric current density

$$j_{e,i}^a = e Z^a \chi_i^a \quad j_{e,i} = \sum_a j_{e,i}^a$$

## Total energy density

$$E^a = \int \frac{1}{2} m^a v^{a2} f^a(x_i, p_i, t) d^3 p \quad E = \sum_a E^a$$

## Momentum density

$$\Pi_i^a = \int p_i f^a(x_i, p_i, t) d^3 p \quad \Pi_i = \sum_a \Pi_i^a$$

## Kinetic tensor

$$\begin{aligned} \Pi_{ij}^a &= \int v_i^a p_j f^a(x_i, p_i, t) d^3 p = \int v_i^a p_i f^a(x_i, p_i, t) d^3 p \\ &= \text{Flux of } j^{\text{th}} \text{ component of momentum in the } i\text{-direction} \\ &= \Pi_{ji}^a \end{aligned}$$

## 5.2 Centre of mass frame for a component

This is defined as the frame in which the total momentum of the fluid is zero. Define the velocity of the Galilean transformation to this frame as follows:

$$\text{Momentum density} = \Pi_i^a = \int p_i f^a d^3 p$$

The mean velocity was defined earlier by

$$V_i^a = \frac{\int v_i^a f^a d^3 p}{n^a} = \frac{\int p_i f^a d^3 p}{n^a m^a} = \frac{\Pi_i^a}{\rho^a}$$

Momentum density in a frame moving with velocity  $V_i^a$  is

$$\begin{aligned}\Pi_i^{a'} &= \int (p_i - m^a V_i^a) f^a d^3 p = \Pi_i - n^a m^a V_i^a \\ &= \Pi_i^a - \rho^a V_i^a = 0\end{aligned}$$

so that  $V_i^a$  defines the centre of mass frame.

### 5.3 Kinetic tensor in the centre of mass frame

The velocity of a component in the centre of mass frame is

$$v_i^{a'} = v_i^a - V_i^a \Rightarrow v_i^a = V_i^a + v_i^{a'}$$

Therefore the kinetic tensor of a component is:

$$\begin{aligned}\Pi_{ij}^a &= \int m^a (V_i^a + v_i^{a'}) (V_j^a + v_j^{a'}) f^a d^3 p \\ &= \int m^a (V_i^a V_j^a + v_i^{a'} V_j^a + V_i^a v_j^{a'} + v_i^{a'} v_j^{a'}) f^a d^3 p \\ &= m^a V_i^a V_j^a \int f^a d^3 p + m^a V_j^a \int v_i^{a'} f^a d^3 p \\ &\quad + m^a V_i^a \int v_j^{a'} f^a d^3 p + m^a \int v_i^{a'} v_j^{a'} f^a d^3 p\end{aligned}$$

The 2nd and 3rd terms vanish by virtue of the definition of the mean velocity so that

$$\Pi_{ij}^a = \rho^a V_i^a V_j^a + P_{ij}^a$$

which introduces the partial pressure tensor

$$P_{ij}^a = m^a \int v_i^{a'} v_j^{a'} f^a d^3 p$$

## 5.4 Isotropic distribution function

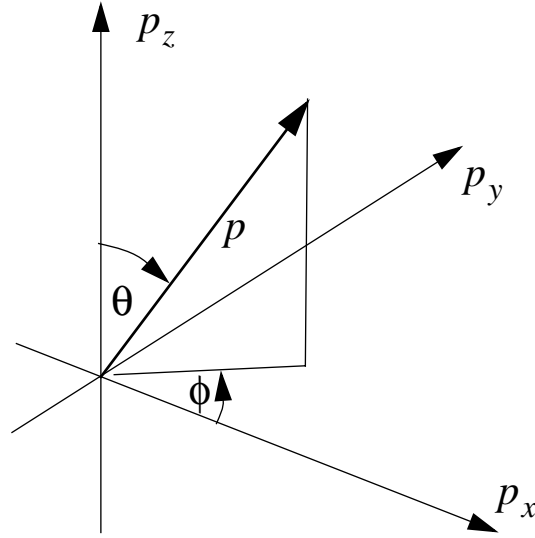
If the distribution function is isotropic in the centre of mass (rest) frame, i.e.

$$f^a(x_i, p_i) = f^a(x_i, p)$$

then

$$P_{ij}^a = m_a^{-1} \int p_i p_j f^a(x_i, p) p^2 \sin\theta dp d\theta d\phi$$

where  $(p, \theta, \phi)$  constitute spherical polars in momentum space.



Polar coordinates in momentum space

Since  $f^a$  is isotropic, then it is easily shown that the integral is zero for  $i \neq j$  and that when  $i = j$

$$\begin{aligned} P_{ij}^a &= m_a^{-1} \int p_z^2 f^a(x_i, p) p^2 \sin\theta dp d\theta d\phi \\ &= m_a^{-1} \int p^4 f^a(x_i, p) \cos^2\theta \sin\theta dp d\theta d\phi \\ &= \frac{4\pi}{3m_a} \int p^4 f^a(x_i, p) dp \end{aligned}$$

We have taken  $i = j = z$  in the above, since the  $xx$ ,  $yy$  and  $zz$  com-

ponents of the pressure tensor are identical, i.e. there is no preferred axis.

Hence

$$P_{ij}^a = P^a \delta_{ij}$$

where

$$P^a = \frac{4\pi}{3m^a} \int p^4 f^a(x_i, p) dp$$

Thus in the case of an isotropic distribution function, the pressure tensor is also isotropic. We shall see that  $P$  is what we normally associate with the pressure when we think of a fluid in terms of a continuum.

When the particle distribution is anisotropic we obtain terms relating to *bulk and shear viscosity* in the pressure tensor.

## 5.5 Relationship of pressure to thermal velocity and temperature

When the pressure tensor is isotropic

$$\begin{aligned} P_{ii}^a = 3P^a &\Rightarrow P^a = \frac{1}{3} P_{ii}^a = \frac{1}{3} \int v_i p_i f^a d^3 p \\ &= \frac{1}{3} \int m^a v^2 f^a d^3 p \\ &= \frac{1}{3} n^a m^a \overline{v^2} \end{aligned}$$

where  $\overline{v^2}$  is the rms velocity of the ions.

For monatomic ions in thermal equilibrium at temperature  $T$  there is  $(kT)/2$  energy for each of the three degrees of freedom so that

$$\frac{1}{3}n^a m^a \overline{v^2} = \frac{1}{3} \times n^a \times 2 \times \frac{1}{2} m^a \overline{v^2} = \frac{2}{3} n^a \times \frac{3}{2} kT = n^a kT$$

Hence

$$P^a = n^a kT$$

## 5.6 Derivation of pressure from the Maxwell-Boltzmann distribution

For a uniform gas in thermodynamic equilibrium the Maxwell-Boltzmann distribution function is given (in terms of velocity) by:

$$f_{MB}(\mathbf{v}) d^3\mathbf{v} = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) d^3\mathbf{v}$$

By calculating the above integral for the pressure, viz,

$$P^a = \frac{4\pi}{3m^a} \int p^4 f^a(x_i, p) dp$$

one can also regain  $P^a = n^a kT$ .

## 5.7 Centre of mass frame for the entire fluid

This is defined as the frame in which the momentum density of the entire fluid vanishes. That is, we determine a Galilean transformation for which

$$\Pi'_i = \sum_a \Pi_i^{a'} = 0$$

We proceed similarly to the case for the CoM frame for a single component, viz, for transformation velocity  $V_i$ , the transformed momentum density is

$$\begin{aligned}\Pi'_i &= \sum_a \Pi_i^a - \rho^a V_i = \sum_a \Pi_i^a - V_i \sum_a \rho^a \\ &= \Pi_i - \rho V_i\end{aligned}$$

and this is zero for

$$V_i = \frac{\Pi_i}{\rho}$$

If the centre of mass frames for all components are equal, then we can use the *Single Fluid Approximation* in which all components have the same centre of mass velocity  $V_i$ . In this case the Kinetic Tensor

$$\begin{aligned}\Pi_{ij} &= \sum_a \Pi_{ij}^a = \sum_a \rho^a V_i V_j + \sum_a P_{ij}^a \\ &= V_i V_j \sum_a \rho^a + \sum_a P_{ij}^a \\ &= \rho V_i V_j + \sum_a P_{ij}^a\end{aligned}$$

and when the distribution is isotropic,

$$\Pi_{ij} = \rho V_i V_j + P \delta_{ij}$$

where

$$\text{Total Pressure} = P = \sum_a P^a$$

i.e the total pressure is the sum of the partial pressures.

In many cases a single fluid approximation is appropriate e.g. stellar winds, hydrostatic atmospheres in elliptical galaxies, supernova blast waves, geophysical fluid dynamics, aerodynamics.

An example of where a single fluid approximation is inadequate is in the study of partially ionised gases where the ions and neutrals may

move differently. This leads to *ambipolar diffusion* which is discussed later.

## 6 Conservation of particle number, current and mass

### 6.1 Conservation of particle number

Integrate the Boltzmann equation wrt momentum

$$\int \frac{\partial f^a}{\partial t} d^3 p + \int v_i^a \frac{\partial f^a}{\partial x_i} d^3 p + \int \frac{\partial(F_i^a f^a)}{\partial p_i} d^3 p = \int \frac{\delta f^a}{\delta t} d^3 p$$

The integral of the momentum space divergence can be transformed over an arbitrary surface enclosing all of the momenta. This can be made arbitrarily large so that the distribution function on the surface is zero. Hence this term disappears.

The partial derivative wrt  $x_i$  can be taken outside the integral since the integral is over momentum and  $v_i^a = (m^a)^{-1} p_i$  is not a function of  $x_i$ .

The right hand side represents the rate of scattering of particles out of all momentum space. Although particles are scattered from one value of momentum to another there is a balance, i.e. the total number of particles is not destroyed by the collisions which redistribute momenta. Hence the right hand side equals zero and

$$\frac{\partial}{\partial t} \int f^a d^3 p + \frac{\partial}{\partial x_i} \int v_i^a f^a d^3 p = 0$$

Using the moments defined earlier:

$$\frac{\partial n^a}{\partial t} + \frac{\partial(n^a V_i^a)}{\partial x_i} = \frac{\partial n^a}{\partial t} + \frac{\partial(\chi_i^a)}{\partial x_i} = 0$$

This represents the conservation of particles of component  $a$ .

## 6.2 Electric current

Multiply above equation by  $eZ^a$

$$\Rightarrow \frac{\partial \rho_e^a}{\partial t} + \frac{\partial j_{e,i}^a}{\partial x_i} = 0$$

i.e. the conservation law for electric current associated with component  $a$ .

Summing over components gives the conservation of current for the fluid as a whole:

$$\frac{\partial \rho_e}{\partial t} + \frac{\partial j_{e,i}}{\partial x_i} = 0$$

## 6.3 Mass

Multiply number conservation equation by  $m^a$ .

$$\Rightarrow \frac{\partial \rho^a}{\partial t} + \frac{\partial (\rho^a V_i^a)}{\partial x_i} = 0$$

i.e. conservation of mass of component  $a$ .

## 7 Conservation of Momentum

### 7.1 Derivation of momentum equation from the Boltzmann equation

Now multiply the Boltzmann equation by momentum and integrate.

$$\begin{aligned} \int p_i \frac{\partial f^a}{\partial t} d^3 p + \int p_i v_j^a \frac{\partial f^a}{\partial x_j} d^3 p + \int p_i \frac{\partial (F_j^a f^a)}{\partial p_j} d^3 p \\ = \int p_i \left( \frac{\delta f^a}{\delta t} \right)_{\text{coll}} f^a d^3 p \end{aligned}$$

Again, with the second integral, the partial derivative can be taken outside. For the third integral, note that

$$p_i \frac{\partial (F_j^a f^a)}{\partial p_j} = \frac{\partial}{\partial p_j} (p_i F_j^a f^a) - \delta_{ij} F_j^a f^a = \frac{\partial}{\partial p_j} (p_i F_j^a f^a) - F_i^a f^a$$

The divergence integrates to zero as before leaving the integral of  $-F_i^a f^a$ . This gives:

$$\begin{aligned} \frac{\partial}{\partial t} \int p_i \frac{\partial f^a}{\partial t} d^3 p + \frac{\partial}{\partial x_j} \int p_i v_j^a f^a d^3 p - \int F_i^a f^a d^3 p \\ = \int p_i \left( \frac{\delta f^a}{\delta t} \right)_{\text{coll}} f^a d^3 p \end{aligned}$$

Now

$$\begin{aligned} \int F_i^a f^a d^3 p &= \int e Z^a E_i f^a d^3 p + \int e Z^a \epsilon_{ijk} \frac{v_j^a}{c} B_k f^a d^3 p + \int m^a g_i f^a d^3 p \\ &= e Z^a E_i \int f^a d^3 p + e Z^a \epsilon_{ijk} B_k \int \frac{v_j^a}{c} f^a d^3 p + m^a g_i \int f^a d^3 p \\ &= e Z^a n^a E_i + \frac{e Z^a}{c} \epsilon_{ijk} \chi_i^a B_k + \rho^a g_i \\ &= \rho_e^a E_i + \epsilon_{ijk} \frac{j_{e,j}^a}{c} B_k + \rho^a g_i \end{aligned}$$

i.e. integrating the force term in the Boltzmann equation over momentum space gives the Lorentz force on the component.

Rearranging terms and recognizing the kinetic tensor in the above:

$$\frac{\partial \Pi_i^a}{\partial t} + \frac{\partial \Pi_{ij}^a}{\partial x_j} = \rho_e^a E_i + \varepsilon_{ijk} \frac{j_{e,j}^a}{c} B_k + \rho^a g_i + \int p_i \left( \frac{\delta f^a}{\delta t} \right)_{\text{coll}} f^a d^3 p$$

The first and second terms on the right represent the Lorentz force on component  $a$ .

The third term represents the gravitational force.

The last term represents the rate of gain of momentum of component  $a$  as a result of collisions. Now, contrary to the case of particle conservation, we cannot set the collision term on the right hand side to zero. However, the nett momentum gain to the entire fluid as a result of collisions is zero.

Using the expressions for the momentum density and the kinetic tensor derived above:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho^a V_i^a) + \frac{\partial}{\partial x_j}(\rho^a V_i^a V_j^a + P_{ij}^a) &= \rho_e^a E_i + \varepsilon_{ijk} \frac{j_{e,j}^a}{c} B_k + \rho^a g_i \\ &+ \left( \frac{\delta \Pi_i^a}{\delta t} \right) \end{aligned}$$

where the last term represents the nett rate of gain of momentum to component  $a$  from collisions.

## 7.2 Advective term and pressure forces

The terms

$$\begin{aligned} \frac{\partial}{\partial t}(\rho^a V_i^a) + \frac{\partial}{\partial x_j}(\rho^a V_i^a V_j^a) &= V_i^a \left( \frac{\partial \rho}{\partial t} + \frac{\partial(\rho^a V_j^a)}{\partial x_j} \right) \\ &+ \rho^a \left( \frac{\partial V_i^a}{\partial t} + V_j^a \frac{\partial V_j^a}{\partial x_j} \right) \end{aligned}$$

and the first group of terms on the RHS is zero as a result of the continuity equation derived earlier.

Hence the  $a$ -component momentum equation becomes:

$$\rho^a \left( \frac{\partial V_i^a}{\partial t} + V_j^a \frac{\partial V_j^a}{\partial x_j} \right) = - \frac{\partial P_{ij}^a}{\partial x_j} + \rho_e^a E_i + \epsilon_{ijk} \frac{j_{e,j}^a}{c} B_k + \rho^a g_i + \left( \frac{\delta \Pi_i^a}{\delta t} \right)$$

or

$$\rho^a \frac{DV_i^a}{Dt} = - \frac{\partial P_{ij}^a}{\partial x_j} + \rho_e^a E_i + \epsilon_{ijk} \frac{j_{e,j}^a}{c} B_k + \rho^a g_i + \left( \frac{\delta \Pi_i^a}{\delta t} \right)$$

where the operator  $\frac{D}{Dt}$  represents *differentiation following the motion* defined by the streamlines with velocity  $V_i^a$ . In the two above equations, the negative gradient of the pressure tensor plays the role of a force.

### 7.3 Single fluid approximation

Summing the integrated momentum equation over components gives:

$$\frac{\partial \Pi_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = \rho_e E_i + \epsilon_{ijk} \frac{j_{e,j}}{c} B_k + \rho g_i + \sum_a \int p_i \left( \frac{\delta f^a}{\delta t} \right)_{\text{coll}} f^a d^3 p$$

In this case, the momentum collision term can be set to zero since there is no nett gain to the fluid as a whole as a result of collisions. Thus,

$$\frac{\partial \Pi_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = \frac{\partial(\rho V_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho V_i V_j + P_{ij}) = \rho_e E_i + \varepsilon_{ijk} \frac{j_{e,j}}{c} B_k + \rho g_i$$

or

$$\rho \left( \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right) = -\frac{\partial P_{ij}}{\partial x_j} + \rho_e E_i + \varepsilon_{ijk} \frac{j_{e,j}}{c} B_k + \rho g_i$$

This follows from the continuity equation, as before.

## 7.4 MHD approximation

We make two important approximations in deriving the momentum equation for the MHD approximation:

1. We assume that the fluid is charge neutral (i.e.  $\rho_e = 0$ ). This is generally an excellent approximation that can easily be justified *a posteriori*
2. We neglect the displacement current,  $\frac{1}{c} \frac{\partial E_i}{\partial t}$ , in Maxwell's equations. This assumption is equivalent to saying that the fields are slowly varying. Again this can be justified *a posteriori*. The consequence of this is that the current which appears in the Lorentz force term can be eliminated from the momentum equation in favour of the magnetic field, via

$$\frac{j_{e,i}}{c} = \frac{1}{4\pi} \varepsilon_{ijk} B_{k,j}$$

so that

$$\begin{aligned}
 \varepsilon_{ijk} \frac{j_{e,j}}{c} B_k &= \frac{1}{4\pi} \varepsilon_{ijk} \varepsilon_{jlm} B_{m,l} B_k \\
 &= \frac{1}{4\pi} [\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}] (B_{m,l} B_k) \\
 &= \frac{1}{4\pi} (-B_{k,i} B_k + B_{i,k} B_k) \\
 &= \frac{1}{4\pi} \frac{\partial}{\partial x_j} \left( B_i B_j - \frac{1}{2} B^2 \delta_{ij} \right) \\
 &= \frac{\partial}{\partial x_j} M_{ij}^B
 \end{aligned}$$

where

$$M_{ij}^B = \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij}$$

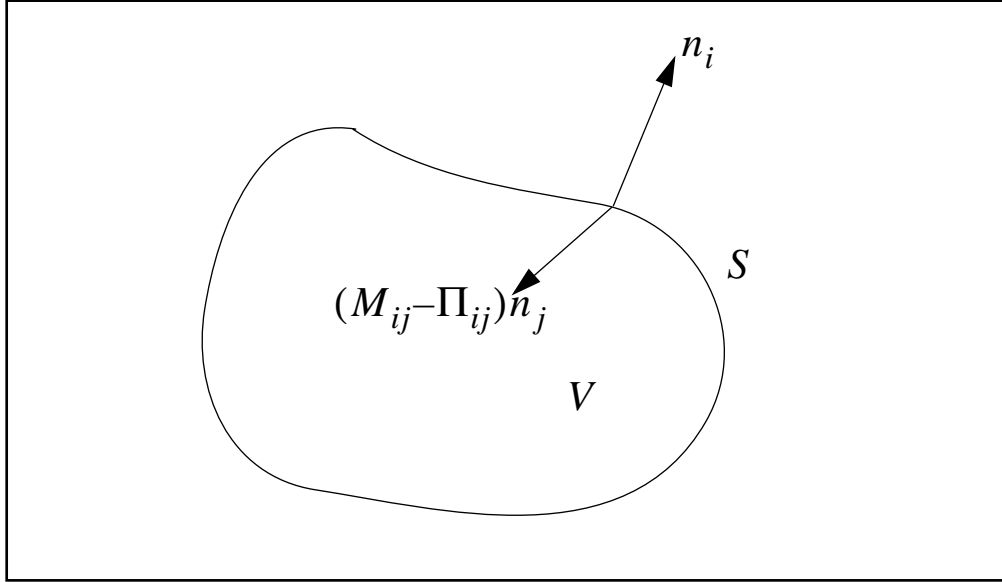
is the magnetic component of the Maxwell stress tensor.

We therefore have the final form of the momentum equations:

$$\begin{aligned}
 \rho \left( \frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right) &= \frac{\partial}{\partial t} (\rho V_i) + \frac{\partial}{\partial x_j} (\rho V_i V_j) = -\frac{\partial P_{ij}}{\partial x_j} + \frac{\partial M_{ij}}{\partial x_j} + \rho g_i \\
 &= -\frac{\partial P_{ij}}{\partial x_j} + \frac{\partial M_{ij}}{\partial x_j} - \frac{\partial \phi}{\partial x_i}
 \end{aligned}$$

## 7.5 Continuum interpretation

### 7.5.1 Integral conservation law for momentum



Using Green's theorem we can write the momentum equation as

$$\frac{\partial}{\partial t} \int_V \Pi_i d^3v = \int_S (M_{ij} - \Pi_{ij}) n_j dS + \int_V \rho g_i d^3v$$

i.e. the rate of change of momentum within  $V$  is equal to minus the flux of momentum across  $S$ , plus the stress on the surface  $S$  due to the magnetic field, as well as the integrated body force due to gravity.

Expressing the kinetic tensor in terms of bulk momentum and pressure components, this equation can be written:

$$\begin{aligned} \frac{\partial}{\partial t} \int_V \rho V_i d^3x &= - \int_S \rho V_i V_j n_j dS - \int_S P_{ij} n_j dS + \int_S M_{ij} n_j dS \\ &\quad + \int_V \rho g_i d^3x \end{aligned}$$

The left hand side of this equation represents the time rate of change of

the momentum within the volume  $V$ ; the first term on the right represents the flow of momentum associated with the bulk velocity; the second term represents the flow of momentum associated with the pressure and the last term represents the gravitational force per unit volume integrated throughout  $V$ .

This provides another way of thinking of the pressure tensor is that the term  $P_{ij}n_j$  represents the force per unit area on the surface  $S$  due to the flux of momentum or in other words  $P_{ij}n_j dS$  is the force exerted on the fluid inside  $S$  by the fluid outside  $S$ . When the pressure tensor is isotropic

$$P_{ij}n_j = P\delta_{ij}n_j = Pn_i$$

and the pressure force is normal to the surface. This coincides with the way in which we think of pressure in a perfect fluid.

Similarly, the term  $M_{ij}n_j$  represents the force on the surface resulting from the magnetic field.  $M_{ij}$  is thought of as a magnetic stress.

## 7.5.2 Stress tensor for the gravitational field

Consider the tensor

$$\Gamma_{ij} = -\frac{1}{4\pi G} \left( g_i g_j - \frac{1}{2} g^2 \delta_{ij} \right)$$

where  $G = 6.67 \times 10^{-8} \text{ N kg}^{-2} \text{ m}^2 = 6.67 \times 10^{-11} \text{ dyn gm}^{-2} \text{ cm}^2$  is the constant of gravitation. The divergence of this tensor

$$\Gamma_{ij,j} = -\frac{1}{4\pi G} (g_{i,j} g_j + g_i g_{j,j} - g_j g_{j,i})$$

Now the gravitational field is expressed as the gradient of a scalar potential  $\phi$

$$g_i = -\frac{\partial\phi}{\partial x_i} \quad \text{with } \nabla^2\phi = 4\pi G\rho$$

Therefore,

$$\Gamma_{ij,j} = -\frac{1}{4\pi G}(g_j(\phi_{,ij} - \phi_{,ji}) + g_i \times -4\pi G\rho) = \rho g_i$$

and

$$\frac{\partial}{\partial t} \int_V \Pi_i d^3x = \int_S (M_{ij} - \Pi_{ij} + \Gamma_{ij}) n_j dS$$

This shows that one can formally identify a flux of momentum per unit area,  $-\Gamma_{ij}n_j$ , associated with the gravitational field. However, this formal result is rarely exploited.

## 8 Evolution of the Magnetic Field

### 8.1 Development of evolution equation

So far we have only considered the effect of the magnetic field on the fluid. We also need to know how the magnetic field evolves with time as a result of the motion of the fluid. In principle this is given by Maxwell's equations. However, there are some simplifications in the MHD approximation which have some interesting consequences.

Let us first examine the consequences of the normally high conductivity of magnetised gases. Let us assume an Ohm's type law for the conduction current (this is justified later):

$$j_{c,i} = \sigma E'_i$$

where  $\sigma$  is the conductivity and the prime refers to the rest frame of the

gas. Remembering the behaviour of electric fields under a Galilean transformation, we have

$$E_i' = E_i + \epsilon_{ijk} \frac{V_j}{c} B_k$$

where  $V_j$  is the velocity of the gas in the lab frame. Hence,

$$j_{c,i} = \sigma \left( E_i + \epsilon_{ijk} \frac{V_j}{c} B_k \right)$$

We now use the following two of Maxwell's equations, neglecting the displacement current in the first,

$$\begin{aligned} \epsilon_{ijk} \frac{\partial B_k}{\partial x_j} &= \frac{4\pi}{c} j_{e,i} + \frac{1}{c} \frac{\partial E_i}{\partial t} = \frac{4\pi}{c} j_{e,i} \\ \epsilon_{ijk} \frac{\partial E_k}{\partial x_j} + \frac{1}{c} \frac{\partial B_i}{\partial t} &= 0 \end{aligned}$$

Using the expression for the current to solve for the magnetic field, gives

$$E_i = \frac{1}{\sigma} j_{c,i} - \epsilon_{ijk} \frac{V_j}{c} B_k$$

Also we assume that the gas is charge neutral so that the conduction current and the total current are equal ( $j_{e,i} = j_{c,i} + \rho_e V_i$ ) and we can substitute

$\frac{c}{4\pi} \nabla \times \mathbf{B}$  for the current to obtain

$$\begin{aligned} E_i &= \frac{c}{4\pi\sigma} \epsilon_{ijk} B_{k,j} - \epsilon_{ijk} \frac{V_j}{c} B_k \\ \mathbf{E} &= \frac{c}{4\pi\sigma} \nabla \times \mathbf{B} - \frac{\mathbf{V}}{c} \times \mathbf{B} \end{aligned}$$

and we substitute this into Faraday's law to obtain

$$\epsilon_{ijk} \left[ \frac{c}{4\pi\sigma} \epsilon_{jlm} B_{m,l} - \epsilon_{jlm} \frac{V_l}{c} B_m \right] = -\frac{1}{c} \frac{\partial B_i}{\partial t}$$

$$\nabla \times \left[ \frac{c}{4\pi\sigma} \nabla \times \mathbf{B} - \frac{\mathbf{V}}{c} \times \mathbf{B} \right] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

Solving for  $\frac{\partial \mathbf{B}}{\partial t}$ ,

$$\frac{\partial B_i}{\partial t} + \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} B_l V_m) = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \left( \frac{c^2}{4\pi\sigma} \epsilon_{klm} B_{m,l} \right)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}) = -\nabla \times \left( \frac{c^2}{4\pi\sigma} \nabla \times \mathbf{B} \right)$$

We denote the electrical resistivity by

$$\eta = \frac{c^2}{4\pi\sigma}$$

The “curl curl” term on the right can be simplified as follows:

$$\epsilon_{ijk} \epsilon_{klm} \frac{\partial B_{m,l}}{\partial x_j} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial B_{m,l}}{\partial x_j} = \frac{\partial^2 B_j}{\partial x_j \partial x_i} - \frac{\partial^2 B_i}{\partial x_j \partial x_j}$$

$$= \frac{\partial^2 B_i}{\partial x_j \partial x_j}$$

that is,

$$\nabla \times \nabla \times \mathbf{B} = -\nabla^2 \mathbf{B}$$

since  $\nabla \cdot \mathbf{B} = 0$ .

Hence for  $\eta = \text{constant}$ ,

$$\frac{\partial B_i}{\partial t} + \epsilon_{ijk}(\epsilon_{klm} V_l B_m) = \eta \frac{\partial^2 B_i}{\partial x_j \partial x_j}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{V} \times \mathbf{B}) = \eta \nabla^2 \mathbf{B}$$

## 8.2 Diffusion time scale

Obviously, if  $\mathbf{V} = 0$ , then we have a diffusion equation for  $\mathbf{B}$ , viz,

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$$

The diffusion time scale,  $t_D$ , associated with this is determined by order of magnitude estimate of each side of this equation. Let the length scale of the magnetic field be  $L$ , then

$$\frac{B}{t_D} \sim \frac{\eta B}{L^2} \Rightarrow t_D \sim \frac{L^2}{\eta}$$

Normally, for astrophysical plasmas, the length scale is so long and the conductivity is so high that this time scale is very long. For many phenomena we are interested in, the timescales are much less than the characteristic time,  $t_D$ . (Estimates of times in various regions are given in the exercises.)

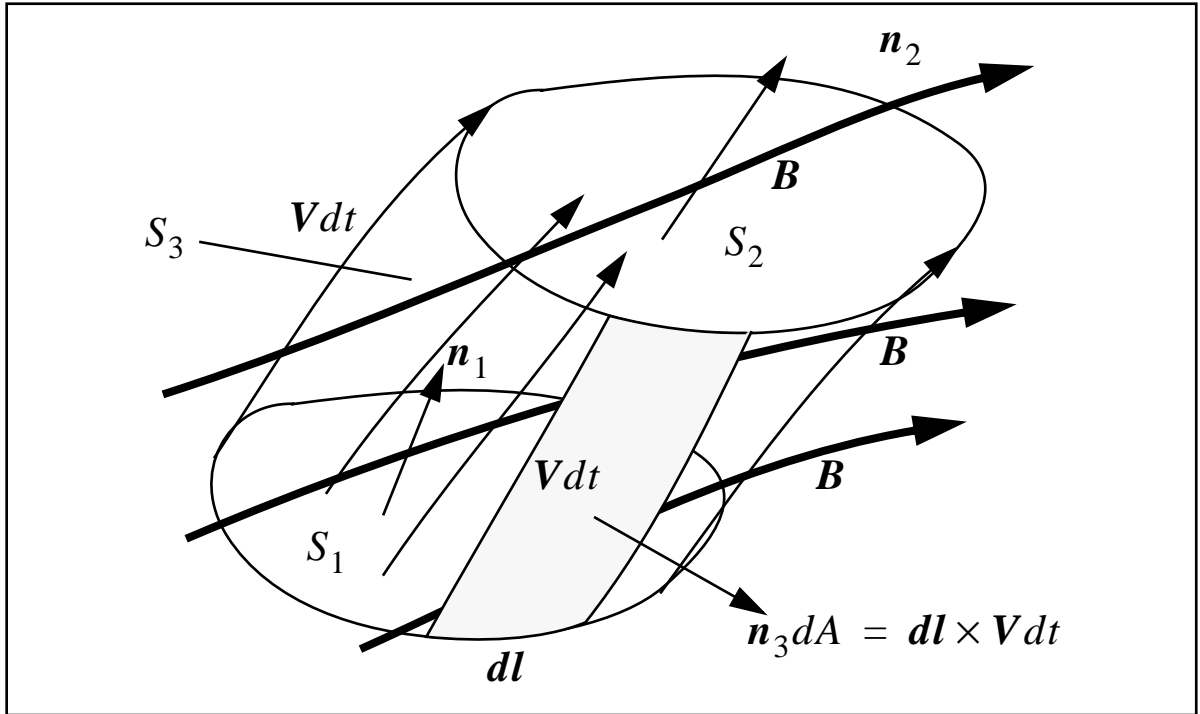
## 8.3 Alfven's flux-freezing theorem

When we can ignore the effects of conductivity, the equation for the evolution of the magnetic field becomes:

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} B_l V_m) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}) = 0$$

The implications of this are extremely interesting: The flux through a comoving loop is conserved, as we now show.



In the above diagram, the surface  $S_1$  evolves to the surface  $S_2$  in the element of time  $dt$  due to the motion of the fluid. The magnetic field is transported by the fluid according to the transport equation above. In the figure the magnetic field is shown at the time  $t + dt$ . The flux through the moving surface is given by

$$\Phi(t) = \int_{S_1} \mathbf{B}(r, t) \cdot \mathbf{n} dA$$

$$\Phi(t + dt) = \int_{S_2} \mathbf{B}(r, t + dt) \cdot \mathbf{n} dA$$

The directed area formed by the sides of the tube generated by the motion of the fluid is  $\mathbf{n}_3 dA = \mathbf{V} dt \times d\mathbf{l}$ . The flux through the sides of the volume generated by the moving surface is given to first order in  $dt$  by

$$\Phi_{\text{sides}} = \int_{S_3} \mathbf{B}(t) \cdot [\mathbf{V} dt \times d\mathbf{l}] = -dt \int_{S_3} [\mathbf{B}(t) \times \mathbf{V}] \cdot d\mathbf{l}$$

Now, since  $\text{div} \mathbf{B} = \mathbf{0}$ , the total flux through the correctly oriented surfaces  $S_1$ ,  $S_2$  and  $S_3$  at a fixed time, is zero, since these surfaces enclose a fixed volume. Hence,

$$\begin{aligned} \int_{S_2} \mathbf{B}(r, t + dt) \cdot \mathbf{n} dA - \int_{S_1} \mathbf{B}(r, t + dt) \cdot \mathbf{n} dA - dt \int_{S_3} [\mathbf{B}(t) \times \mathbf{V}] \cdot d\mathbf{l} \\ = 0 \end{aligned}$$

The integral through  $S_1$  can be expanded to first order in  $dt$  to

$$- \int_{S_1} \mathbf{B}(r, t) \cdot \mathbf{n} dA - dt \int_{S_1} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} dA$$

and, using Green's theorem

$$\int_{S_3} [\mathbf{B}(t) \times \mathbf{V}] \cdot d\mathbf{l} = \int_{S_1} \nabla \times (\mathbf{B} \times \mathbf{V}) \cdot \mathbf{n} dA$$

so that we end up with

$$\Phi(t + dt) - \Phi(t) - dt \int_{S_1} \left[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}) \right] \cdot \mathbf{n} dA = 0$$

However, the integral over  $S_1$  is zero because of the transport equation, so that

$$\Phi(t + dt) - \Phi(t) = 0$$

i.e.

$$\frac{d\Phi}{dt} = 0$$

where the time derivative refers to the time derivative following the motion of the loop. This elegant result is known as Alfvén's flux-freezing theorem.

### 8.3.1 Motion of the field lines

There is another way to characterize the motion of the field lines when diffusion is negligible. We expand

$$\frac{\partial B_i}{\partial t} + \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{klm} B_l V_m) = 0$$

to

$$\begin{aligned} \frac{\partial B_i}{\partial t} + (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} (B_l V_m) &= 0 \\ \Rightarrow \frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j} (B_i V_j) - \frac{\partial}{\partial x_j} (B_j V_i) &= 0 \\ \frac{\partial B_i}{\partial t} + V_j \frac{\partial B_i}{\partial x_j} + B_i \frac{\partial V_j}{\partial x_j} - V_i \frac{\partial B_j}{\partial x_j} - B_j \frac{\partial V_i}{\partial x_j} &= 0 \end{aligned}$$

Using

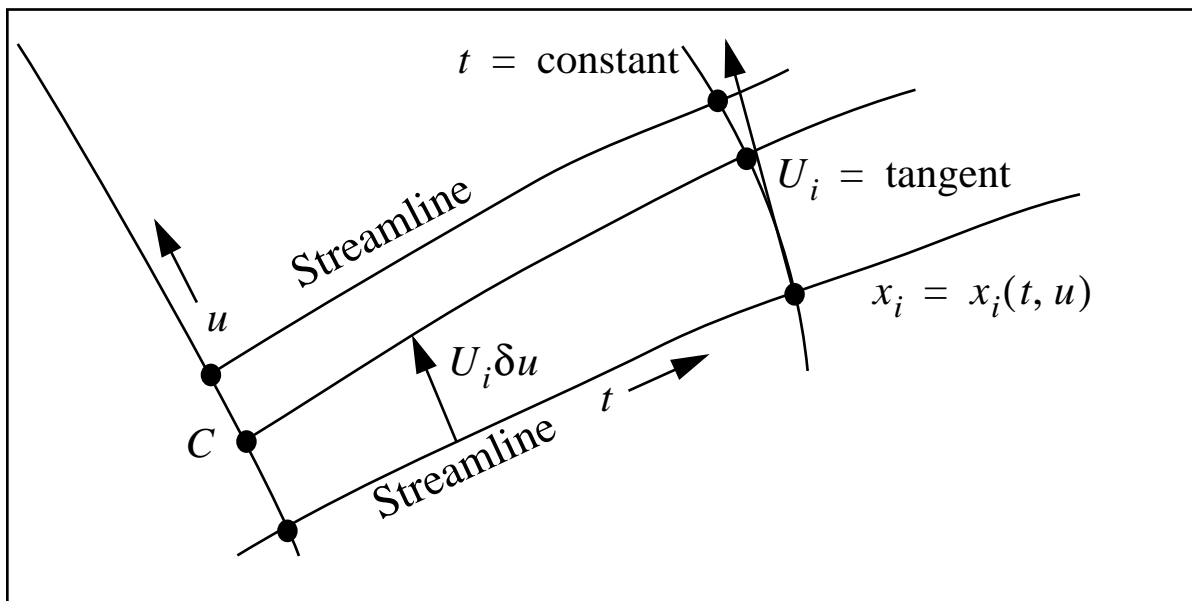
$$\frac{\partial B_j}{\partial x_j} = 0 \quad \text{and} \quad \frac{\partial V_j}{\partial x_j} = -\frac{1}{\rho} \frac{D\rho}{Dt}$$

gives

$$\frac{DB_i}{Dt} - \frac{B_i D\rho}{\rho Dt} = B_j \frac{\partial V_i}{\partial x_j}$$

$$\Rightarrow \frac{D}{Dt} \left( \frac{B_i}{\rho} \right) = \frac{B_j}{\rho} \frac{\partial V_i}{\partial x_j}$$

To consider the implications of this equation, we need to make a slight diversion into the theory of two-dimensional congruences of curves.



We consider streamlines, originating from a curve,  $C$ , so that the *congruence* of streamlines defines a two dimensional space,

$$x_i = x_i(t, u)$$

with the velocity along each streamline being defined by

$$V_i = \frac{\partial}{\partial t} x_i(t, u)$$

We define the *separation vector*

$$U_i = \frac{\partial}{\partial u} x_i(t, u)$$

which is a tangent vector to the lines formed by  $t = \text{constant}$ .

Sometimes it is intuitively easier to think in terms of the infinitesimal separation between two neighbouring streamlines. This is  $U_i \delta u$  and motivates the use of the term separation vector for  $U_i$ .

Now consider the *rate of change* of the separation vector with respect to time, as we move along a trajectory. This is

$$\frac{\partial U_i}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial u} x_i(t, u) \right] = \frac{\partial}{\partial u} \left[ \frac{\partial}{\partial t} x_i(t, u) \right] = \left. \frac{\partial V_i}{\partial u} \right|_t$$

Now the rate at which the components of the velocity,  $V_i$ , change at a fixed time, is given by their spatial derivatives, i.e.

$$\frac{\partial V_i}{\partial u} = \frac{\partial V_i}{\partial x_j} \frac{\partial}{\partial u} x_j(t, u) = \frac{\partial V_i}{\partial x_j} U_j$$

Hence,

$$\frac{\partial U_i}{\partial t} = \frac{\partial V_i}{\partial x_j} U_j$$

The operator  $\frac{\partial}{\partial t}$  is differentiation following the motion, so that we have

$$\frac{DU_i}{Dt} = V_{i,j} U_j$$

In terms of the infinitesimal separation vector  $\delta x_i = U_i \delta u$

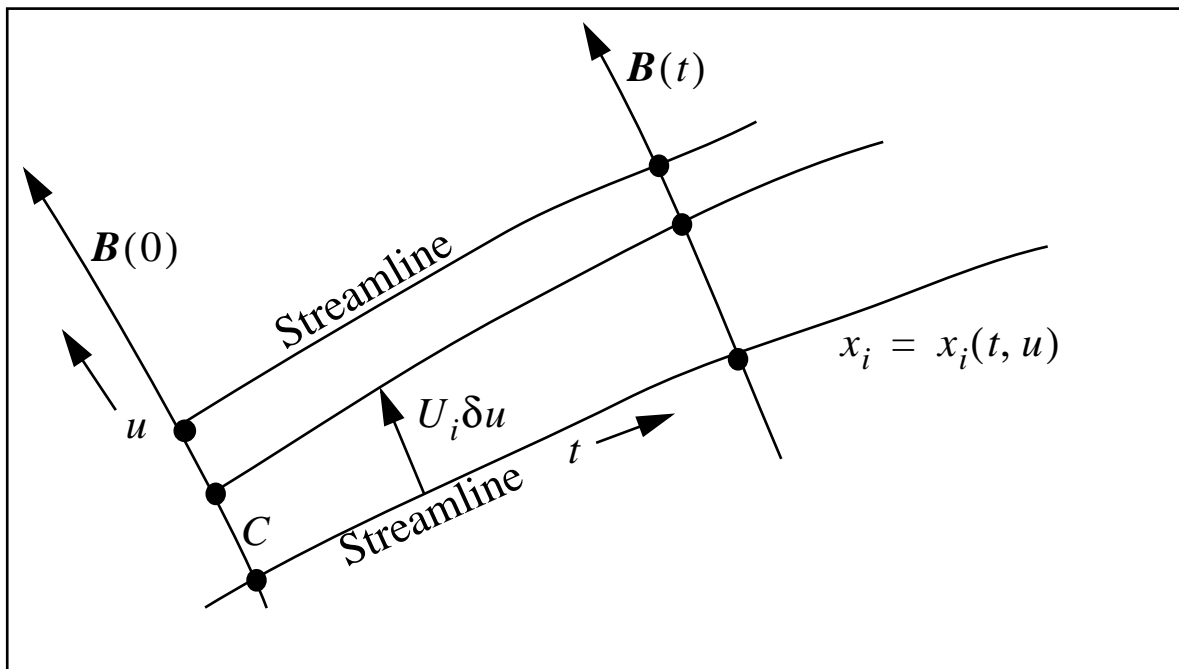
$$\frac{\partial}{\partial t} \delta x_i = V_{i,j} \delta x_j$$

since  $\delta u$  is constant between neighbouring streamlines.

These equations show us that the way in which points on neighbouring streamlines separate is determined by the gradient of the velocity.

Note that the separation vector satisfies the same equation as the magnetic field divided by the density and this leads to the following interpretation. Consider the vector  $U_i - \frac{B_i}{\rho}$ . This satisfies the equation,

$$\frac{D}{Dt} \left( U_i - \frac{B_i}{\rho} \right) = V_{i,j} \left( U_j - \frac{B_j}{\rho} \right)$$



If we now take our initial curve  $C$  such that it is tangent to a magnetic field line and choose the parametrization of that curve so that  $U_i = \frac{B_i}{\rho}$ ,

then from the above equation we can see that  $U_i - \frac{B_i}{\rho} = 0$  always.

Therefore the line  $t = \text{constant}$  formed by the evolution of fluid elements along the magnetic field will remain parallel to  $\mathbf{B}$ . In other words, the magnetic field remains parallel to the curve defined by the new positions of the fluid elements. Thus, the magnetic field behaves as though it is carried along by the fluid.

## 8.4 A posteriori justification of MHD assumptions

In deriving the MHD equations we have assumed that

1. The fluid is charge neutral
2. Maxwell's displacement current can be neglected.

To justify these, consider the relationship between magnetic and electric fields in a fluid of high conductivity. We have

$$\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} \approx \mathbf{0}$$

so that

$$\mathbf{E} \sim \frac{V\mathbf{B}}{c}$$

where  $T$  is the characteristic timescale.

Therefore the displacement current

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \sim \frac{V\mathbf{B}}{c^2 T}$$

This is to be compared with

$$\nabla \times \mathbf{B} \sim \frac{B}{L}$$

and we can see that the displacement current can be neglected if

$$\frac{VL}{c^2 T} \sim \frac{V^2}{c^2} \ll 1$$

and this is consistent with our restriction to sub-relativistic velocities.

Now, consider the current in a fluid. We take  $V_i^{\text{ions}}$  to be the velocity of the ions,  $V_i^e$  as the electron velocity,  $n_i$  as the ion density, with average charge  $Ze$  and  $n_e$  as the electron density. The current can be expressed as

$$j_{e,i} = Zen_i V_i^{\text{ions}} - en_e V_i^e = (Zen_i - en_e) V_i^{\text{ions}} - en_e (V_i^e - V_i^{\text{ions}})$$

Now when the fluid is charge neutral,  $Zen_i - en_e = 0$  and

$$j_{e,i} = -en_e (V_i^e - V_i^{\text{ions}}) = -en_e V_i^{\text{drift}}$$

where

$$V_i^{\text{drift}} = V_i^e - V_i^{\text{ions}}$$

is the drift velocity of the electrons with respect to the ions. We can estimate the drift velocity from

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}_e \Rightarrow j \sim \frac{Bc}{4\pi L} \Rightarrow V^{\text{drift}} \sim \frac{Bc}{4\pi en_e L}$$

In most cases of astrophysical interest, the drift velocity is negligible compared to the thermal velocity (see exercises) and is certainly much smaller than the velocity of light.

We can estimate the charge density from

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \Rightarrow \rho_e \sim \frac{E}{4\pi L} \sim \frac{VB}{4\pi cL}$$

from our estimate for the magnetic field. From Ampere's law,

$$\frac{B}{L} \sim \frac{4\pi}{c} en_e V^{\text{drift}}$$

and,

$$\frac{\rho_e}{en_e} \sim \frac{V V^{\text{drift}}}{c^2}$$

i.e. the total electric charge density is very small compared to the total electronic charge which must therefore be balanced by the ions.

## 8.5 General comments on the electric field

We have, in many cases, that

$$\mathbf{E} = -\frac{\mathbf{V}}{c} \times \mathbf{B}$$

so that, in MHD, the electric field is usually replaced by the equivalent expression involving the magnetic field wherever possible. The electromagnetic energy density

$$\epsilon_{\text{EM}} = \frac{E^2 + B^2}{8\pi} \approx \frac{B^2}{8\pi}$$

since  $E^2 \sim \frac{V^2}{c^2} B^2$ .

## 8.6 The physical basis for conductivity

As we have discussed above it is the relative motion between ions and electrons which provides the current which generates the magnetic field. Also, as we have seen, the current required is usually very small compared to what could potentially be produced. We have also used a

type of Ohm's law,  $\mathbf{j}_e = \sigma \mathbf{E}$  and we now discuss the physical basis for this ansatz.

We perform our calculations in the frame of reference provided by the ions. This is very close to the centre of mass frame of reference since the electrons are not very massive. In this (primed) frame, we write the equation of motion of an electron in the form

$$m_e \frac{d\mathbf{v}_e'}{dt} = -e \left( \mathbf{E}' + \frac{\mathbf{v}_e'}{c} \times \mathbf{B}' \right) - m_e \nu_c \mathbf{v}_e'$$

Lorentz force      Effect of collisions

The last term is a phenomenological one which accounts for the loss of momentum of electrons in collisions with ion. The parameter  $\nu_c$  is the collision frequency.

We assume that the drag force on the electrons results in a terminal electron velocity, satisfying

$$-m_e \nu_c \mathbf{v}_e' - e \left( \mathbf{E}' + \frac{\mathbf{v}_e'}{c} \times \mathbf{B}' \right) = 0$$

Take  $n_e$  to be the electron density, then the current in the ion frame is

$$\mathbf{j}_e' = -en_e \mathbf{v}_e'$$

so that on multiplying by  $en_e$ , the equation for the electron terminal velocity can be written

$$m_e \nu_c \mathbf{j}_e' - e^2 n_e \mathbf{E}' + \frac{e}{c} \mathbf{j}_e' \times \mathbf{B}' = 0$$

and we can solve for the electric field in this frame:

$$\begin{aligned} \mathbf{E}' &= \frac{m_e v_c}{e^2 n_e} \mathbf{j}_e' + \frac{1}{e n_e c v_c} \mathbf{j}_e \times \mathbf{B}' \\ &= \frac{1}{\sigma} \left[ \mathbf{j}_e + (\omega v_c^{-1}) \frac{\mathbf{j}_e \times \mathbf{B}'}{B} \right] \end{aligned}$$

where, the conductivity,  $\sigma$ , and gyrofrequency,  $\omega_B$ , are given by

$$\sigma = \frac{e^2 n_e}{m_e v_c} \quad \omega_B = \frac{eB}{m_e c}$$

When the gyrofrequency is much less than the collision frequency, ( $\omega_B v_c \ll 1$ ), this relation reduces to what we have used earlier, viz.

$$\mathbf{E}' = \frac{1}{\sigma} \mathbf{j}_e$$

and we can take  $\mathbf{E}' \approx 0$

However, even when  $\omega_B v_c \gg 1$ , and the second term in the equation for the electric field dominates, the drift velocity is so small and the conductivity is so large that the approximation  $\mathbf{E}' \approx 0$  is still valid (see exercises).

## 9 The energy equation

It is valuable to know how the energy is transported in both magnetised and unmagnetised fluids.

An equation for the total energy follows from multiplying the Boltzmann equation by  $K^a = \frac{1}{2} m^a v^2$  and integrating over momentum space.

In the following we do this, incorporating electromagnetic effects without approximations. We then introduce the MHD approximations. Before carrying out the integration, however, we define a few more terms.

## 9.1 Bulk kinetic energy and internal energy

Since  $v_i^a = V_i^a + v_i^{a'}$  then the total energy

$$\begin{aligned}
 E &= \int \frac{1}{2} m^a v^2 f^a d^3 p \\
 &= \int \left( \frac{1}{2} m^a V^a{}^2 + m^a V_i^a v_i^{a'} + \frac{1}{2} m^a v^{a'2} \right) f^a d^3 p \\
 &= \frac{1}{2} \rho^a V^a{}^2 + \int \frac{1}{2} m^a v^{a'2} f^a d^3 p \\
 &= \frac{1}{2} \rho^a V^a{}^2 + \epsilon^a
 \end{aligned}$$

This breaks up the total energy into the kinetic energy associated with the centre of mass velocity and the internal energy density  $\epsilon^a$ .

The internal energy density is the kinetic energy of the plasma in the centre of mass frame.

## 9.2 Heat flux

This is defined for a single component by:

$$q_i^a = \int \frac{1}{2} m^a v'^2 v_i^{a'} f^a d^3 p$$

The heat flux is the flux of kinetic energy in the centre of mass frame.

The heat flux for the entire fluid:

$$q_i = \sum_a q_i^a$$

## 9.3 Derivation of the energy equation

Energy times Boltzmann equation =>

$$\int K^a \frac{\partial f^a}{\partial t} d^3 p + \int K^a v_i^a \frac{\partial f^a}{\partial x_i} d^3 p + \int K^a \frac{\partial (F_i f^a)}{\partial p_i} d^3 p = \int K^a \frac{\delta f^a}{\delta t} d^3 p$$

$K$  does not depend on either  $x_i$  or  $t$  so that the space and time derivatives can be taken outside the integral. We rearrange the terms within the third integral to give a divergence. This gives us:

$$\begin{aligned} \frac{\partial}{\partial t} \int K^a f^a d^3 p + \frac{\partial}{\partial x_i} \int K^a v_i^a f^a d^3 p + \int \left[ \frac{\partial}{\partial p_i} (K^a F_i f^a) - \left( \frac{\partial K^a}{\partial p_i} \right) F_i f^a \right] d^3 p \\ = \int K^a \frac{\delta f^a}{\delta t} d^3 p \end{aligned}$$

The integral over the momentum space divergence vanishes for the same reasons as before. The term

$$\frac{\partial K^a}{\partial p_i} = \frac{1}{2m^a} \frac{\partial}{\partial p_i} (p_j p_j) = \frac{1}{m^a} p_i = v_i^a$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial t} \int \frac{1}{2} m_a v^{a2} f^a d^3 p + \frac{\partial}{\partial x_i} \int \frac{1}{2} m_a v^{a2} v_i^a f^a d^3 p - \int v_i^a F_i f^a d^3 p \\ = \int K^a \frac{\delta f^a}{\delta t} d^3 p \end{aligned}$$

Let us take each of these terms separately. As we had above, the first term is:

$$\int \frac{1}{2} m_a v^{a2} f^a d^3 p = \frac{1}{2} \rho^a V^2 + \epsilon^a$$

For the second term (leaving out the  $a$  superscript):

$$\begin{aligned}
 -v^2 v_i &= \left( \frac{1}{2} V^2 + V_j v'_j + \frac{1}{2} v'_j v'_j \right) (V_i + v'_i) \\
 &= \frac{1}{2} V^2 V_i + V_j V_i v'_j + \frac{1}{2} v'_j v'_j V_i + \frac{1}{2} V^2 v'_i + V_j v'_i v'_j + \frac{1}{2} v'_j v'_j v'_i
 \end{aligned}$$

When multiplied by  $f^a$  and integrated over momentum space these terms give:

$$\int \frac{1}{2} m^a v^a{}^2 v_i^a f^a d^3 p = \frac{1}{2} \rho^a V^a{}^2 V_i + 0 + \varepsilon^a V_i^a + 0 + P_{ij}^a V_j^a + q_i^a$$

We also have:

$$\begin{aligned}
 \int v_i^a F_i d^3 p &= e Z^a \int v_i E_i f^a d^3 p + \int p_j g_j f^a d^3 p \\
 &= j_{e,i}^a E_i + \rho^a V_i g_i
 \end{aligned}$$

Putting all of this together:

$$\begin{aligned}
 \frac{\partial}{\partial t} \left( \frac{1}{2} \rho^a V^a{}^2 + \varepsilon^a \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho^a V^a{}^2 V_i^a + \varepsilon^a V_i + P_{ij}^a V_j + q_i \right) \\
 = j_{e,i}^a E_i + \rho^a V_i^a g_i + \int K^a \left( \frac{\delta f^a}{\delta t} \right) d^3 p
 \end{aligned}$$

As with other expressions we have derived, the collision term is not zero for an individual component. However, if all components are moving with the same mean motion, then we can sum over components to obtain:

where the collision term sums to zero when we assume that energy is conserved in collisions.

The gravitational term  $\rho V_i g_i$  can be manipulated in the following way:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \varepsilon \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + q_i \right) \\ = j_{e,i} E_i + \rho V_i g_i \end{aligned}$$

$$\rho V_i g_i = -\rho V_i \frac{\partial \phi}{\partial x_i} = -\frac{\partial}{\partial x_i} (\rho \phi V_i) + \frac{\partial}{\partial x_i} (\rho V_i) \phi$$

and using the continuity equation for mass:

$$\rho V_i g_i = -\frac{\partial}{\partial x_i} (\rho \phi V_i) - \phi \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i} (\rho \phi V_i) - \frac{\partial}{\partial t} (\rho \phi) + \rho \frac{\partial \phi}{\partial t}$$

Hence, the energy equation becomes:

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \varepsilon + \rho \phi \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + \rho \phi V_i + q_i \right) \\ = \rho \frac{\partial \phi}{\partial t} + j_{e,i} E_i \end{aligned}$$

If  $\frac{\partial \phi}{\partial t} = 0$  then the only contribution to the right hand side is  $j_{e,i} E_i$ .

We break this term into a number of components referring to the rest frame and the bulk velocity of the fluid. Consider

$$\begin{aligned} j_{e,i} &= \sum_a Z^a e n^a v_i = \sum_a Z^a e n^a (V_i + v_i') \\ &= \left( \sum_a Z^a e n^a \right) V_i + \sum_a Z^a e n^a v_i^{a'} \\ &= \sum_a Z^a e n^a v_i^{a'} \end{aligned}$$

since we have already assumed that the fluid is charge neutral. This last term is the conduction current, to which we shall return later.

Poynting's theorem, summarised earlier, tells us that:

$$\frac{\partial \epsilon_{\text{EM}}}{\partial t} + \frac{\partial S_i}{\partial x_i} = -j_{e,i} E_i$$

where  $\epsilon_{\text{EM}} = \frac{E^2 + B^2}{8\pi}$  is the electromagnetic energy density and

$$S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j B_k$$

is the Poynting flux. Eliminating  $j_{e,i} E_i$  from the above energy equation,

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \epsilon + \epsilon_{\text{EM}} + \rho \phi \right) \\ & + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i + \epsilon V_i + P_{ij} V_j + S_i + \rho \phi V_i + q_i \right) \\ & = \rho \frac{\partial \phi}{\partial t} \end{aligned}$$

When the distribution function is isotropic,  $P_{ij} = P \delta_{ij}$  and  $q_i = 0$ .

## 9.4 Continuum interpretation

Integrating over a finite volume:

$$\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \rho V^2 + \varepsilon + \varepsilon_{EM} + \rho \phi \right) d^3x =$$

$$\int_S \left( \frac{1}{2} \rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + S_i + \rho \phi V_i + q_i \right) n_i dS + \int_V \frac{\partial \phi}{\partial t} d^3x$$

Generally we take the gravitational field to be time independent so that

$$\frac{\partial \phi}{\partial t} = 0.$$

The group of terms

$\frac{1}{2} \rho V^2 + \varepsilon + \varepsilon_{EM} + \rho \phi$  integrated over the volume  $V$ , represent the total energy (kinetic plus internal plus electromagnetic plus gravitational potential energy) within  $V$ . The integral over the bounding surface  $S$  represents the flux of energy through  $S$ . This gives the general form for the energy flux:

$$E_{,i} = \frac{1}{2} \rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + S_i + \rho \phi V_i + q_i$$

The various terms in this expression have the following meaning:

$$\frac{1}{2} \rho V^2 V_i n_i = \text{Flux of Kinetic Energy per unit area}$$

$$\varepsilon V_i n_i = \text{Flux of internal energy per unit area}$$

$$P_{ij} n_i V_j = \text{Rate of work per unit area by pressure forces on fluid}$$

$$S_i = \text{Poynting flux of electromagnetic energy}$$

$$\rho \phi V_i n_i = \text{Flux of gravitational potential energy}$$

$$q_i n_i = \text{Heat flux}$$

When the distribution function is isotropic, the pressure tensor is iso-

tropic and the heat flux is zero. Then

$$P_{ij}V_j = PV_i \quad \text{and} \quad q_i = 0$$

## **Enthalpy:**

The case of an isotropic distribution function (perfect fluid) introduces the enthalpy density defined by

$$H = \varepsilon + P$$

and the specific enthalpy (enthalpy per unit mass)

$$h = \frac{(\varepsilon + P)}{\rho}$$

For a monatomic gas

$$\varepsilon = \frac{3}{2}nkT \quad p = nkT \quad \text{and} \quad \rho = \mu nm_p$$

where  $\mu$  is the mean molecular weight ( $\mu \approx 0.62$  for a completely ionized gas) and  $m_p$  is the mass of a proton. Hence:

$$h = \frac{5}{2} \frac{kT}{\mu m_p}.$$

Using the specific enthalpy, the energy flux

$$F_E = \rho \left( \frac{1}{2} V^2 + h + \phi \right) V_i + S_i$$

It is, at first sight, surprising to see the specific enthalpy in this equation rather than the specific internal energy  $\frac{\varepsilon}{\rho}$ . However, the derivation above shows why this is the case. The source of the difference is the work done by the pressure in increasing the energy of a given volume

of fluid and this contributes an extra component to the energy flux. (This is a good way to impress non-experts!)

## 10 The internal energy equation

The above equation which is useful for indentifying the components of the energy flux applies to the total energy. A separate equation for the internal energy is also useful. This is derived by combining the total energy equation with the momentum equation.

We us the following form for the total energy equation for which no MHD approximations have been used:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \varepsilon \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + q_i \right) = j_{e,i} E_i + \rho V_i g_i$$

The point of the following is to eliminate the terms involving  $\rho V^2$ . To this end we look at

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i \right) &= \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V_j V_j \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V_j V_j V_i \right) \\ &= \frac{1}{2} V_j V_j \frac{\partial \rho}{\partial t} + \rho V_j \frac{\partial V_j}{\partial t} \\ &\quad + \frac{1}{2} V_j V_j \frac{\partial}{\partial x_i} (\rho V_i) + \rho V_i V_j \frac{\partial V_j}{\partial x_i} \end{aligned}$$

The first plus third terms on the RHS vanish by virtue of mass conser-  
vation.

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 \right) + \frac{\partial}{\partial x_i} \left( \frac{1}{2} \rho V^2 V_i \right) &= V_i \left( \rho \frac{\partial V_i}{\partial t} + \rho V_j \frac{\partial V_i}{\partial x_j} \right) \\ &= -V_i \frac{\partial P_{ij}}{\partial x_j} + \rho_e V_i E_i + \epsilon_{ijk} V_i \frac{j_{e,j}}{c} B_k + \rho V_i g_i \end{aligned}$$

Substituting this into the total energy equation:

$$\begin{aligned} \frac{\partial \epsilon}{\partial t} + \frac{\partial}{\partial x_i} (\epsilon V_i + P_{ij} V_j + q_i) - V_i \frac{\partial P_{ij}}{\partial x_j} + \rho_e V_i E_i + \epsilon_{ijk} V_i \frac{j_{e,j}}{c} B_k + \rho V_i g_i \\ = j_{e,i} E_i + \rho V_i g_i \end{aligned}$$

A number of terms cancel and

$$\frac{\partial \epsilon}{\partial t} + V_i \frac{\partial \epsilon}{\partial x_i} + \epsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} = E_i (j_{e,i} - \rho_e V_i) - \epsilon_{ijk} V_i \frac{j_{e,j}}{c} B_k$$

Before proceeding further it is useful to distinguish between the conduction and advection currents. The  $a^{\text{th}}$  partial current is defined in terms of the distribution function by:

$$\begin{aligned} j_{e,i}^a &= eZ^a \int v_i^a f^a d^3p = eZ^a \int (V_i^a + v_i^{a'}) f^a d^3p \\ &= eZ^a n^a V_i^a + j_{c,i}^a = \rho_e^a V_i^a + j_{c,i}^a \end{aligned}$$

where  $j_{c,i}$  is the **conduction current** and the term  $\rho_e V_i^a$  is the current corresponding to the advection of bulk charge by the motion of the gas. Summing over components,

$$j_{e,i} = \rho_e V_i + j_{c,i}$$

and the energy equation becomes:

$$\begin{aligned} \frac{d\varepsilon}{dt} + \varepsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} &= E_i J_{c,i} - \varepsilon_{ijk} \frac{V_i}{c} j_{c,j} B_k \\ &= j_{c,i} \left( E_i + \varepsilon_{ijk} \frac{V_j}{c} B_k \right) \end{aligned}$$

The term  $E_i + \varepsilon_{ijk} \frac{V_j}{c} B_k$  is the electric field  $E'_i$  in the comoving frame of the gas, so that

$$\frac{d\varepsilon}{dt} + \varepsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} = j_{c,i} E'_i$$

(using differentiation following the motion). The term  $j_{c,i} E'_i$  describes the amount of Joule heating of the gas.

When the distribution function is isotropic:

$$\frac{d\varepsilon}{dt} + (\varepsilon + P) \frac{\partial V_i}{\partial x_i} = j_{c,i} E'_i$$

Writing this equation as

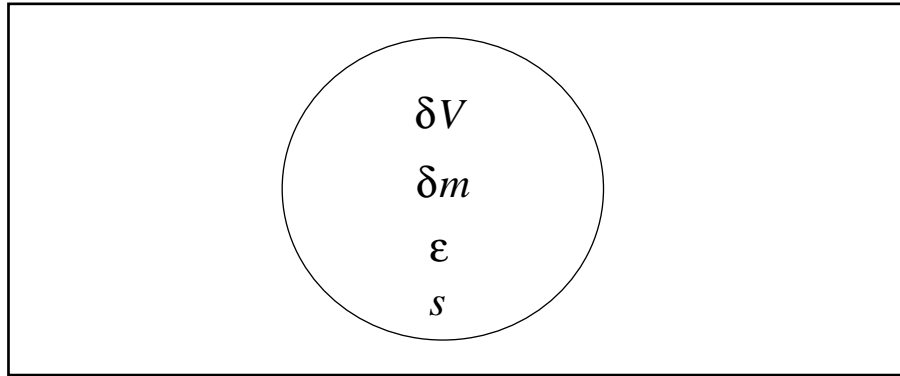
$$\frac{d\varepsilon}{dt} = -(\varepsilon + P) \frac{\partial V_i}{\partial x_i} + j_{c,i} E'_i$$

we see that the term  $(\varepsilon + P) \frac{\partial V_i}{\partial x_i}$  represents the effect of expansion (contraction) in cooling (heating) the gas.

Note that in this form of the energy equation we have not needed to make any assumptions about the neutrality of the fluid or the time variation of the electromagnetic field.

# 11 Relationship between the energy equation and thermodynamics

## 11.1 Entropy



Consider a comoving element of fluid with mass  $\delta m$ , volume  $\delta V = \frac{\delta m}{\rho}$ .

Let entropy per unit mass be  $s$ , then the entropy of the element is  $s\delta m$ , the internal energy is  $\frac{\epsilon\delta m}{\rho}$ , then the relationship between entropy, internal energy and volume is:

$$kTd(s\delta m) = d\left(\frac{\epsilon\delta m}{\rho}\right) + Pd\left(\frac{\delta m}{\rho}\right)$$

Since this is a comoving element, then  $\delta m = \text{constant}$  and

$$kTds = d\left(\frac{\epsilon}{\rho}\right) + Pd\left(\frac{1}{\rho}\right)$$

Expanding the differentials and multiply by  $\rho$ :

$$\rho kT ds = d\varepsilon - \frac{(\varepsilon + P)}{\rho} d\rho$$

Express in terms of derivatives along the trajectory of the element:

$$\rho kT \frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + P)}{\rho} \frac{d\rho}{dt}$$

The equation of continuity

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) &= 0 \\ \Rightarrow \frac{\partial \rho}{\partial t} + V_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial V_i}{\partial x_i} &= 0 \end{aligned}$$

can be expressed in the form

$$\frac{\partial V_i}{\partial x_i} = -\frac{1}{\rho} \frac{d\rho}{dt}$$

so that

$$\rho kT \frac{ds}{dt} = \frac{d\varepsilon}{dt} + (\varepsilon + P) \frac{\partial V_i}{\partial x_i}$$

and comparing this with the expression above:

$$\rho kT \frac{ds}{dt} = j_{c,i} E'_i$$

When there is no dissipation of energy by electromagnetic effects:

$$\rho kT \frac{ds}{dt} = 0$$

i.e. the flow is adiabatic.

## 11.2 Other forms of the entropy equation

Other forms of the relationship between entropy and other thermodynamic variables are also useful, e.g. in terms of the specific enthalpy, the above equation for entropy can be written:

$$\rho kT \frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho} \frac{dP}{dt}$$

## 11.3 Equation of state

In general for an ideal gas with internal degrees of freedom the pressure is still given by

$$p = nkT = \frac{\rho kT}{\mu m_p}$$

but the internal energy may be partitioned amongst extra degrees of freedom. e.g. rotational and vibrational. For the case of constant specific heats we have

$$p = (\gamma - 1)\varepsilon \Rightarrow \varepsilon = \frac{1}{\gamma - 1}p \quad \text{and} \quad h = \frac{\varepsilon + P}{\rho} = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

where  $\gamma = \frac{c_p}{c_v}$  the specific heat ratio. This gives rise to the well known

relation between pressure and density  $p \propto \rho^\gamma$  which is worthwhile re-deriving here in a slightly different form than is given in the usual Statistical Mechanics texts. Since

$$\rho kT \frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + P)d\rho}{\rho dt}$$

then

$$\mu m_p p \frac{ds}{dt} = \frac{1}{\gamma - 1} \left( \frac{dp}{dt} - \gamma \frac{p d\rho}{\rho dt} \right)$$

$$\Rightarrow \mu m_p \frac{ds}{dt} = \frac{1}{\gamma - 1} \left( \frac{1}{p} \frac{dp}{dt} - \frac{\gamma}{\rho} \frac{d\rho}{dt} \right)$$

Integrating:

$$\mu m_p s = \frac{1}{\gamma - 1} (\ln p - \gamma \ln \rho)$$

$$p = \exp[(\gamma - 1)(\mu m_p s)] \rho^\gamma$$

This is often written

$$p = K(s) \rho^\gamma$$

where  $K(s) = \exp[(\gamma - 1)(\mu m_p s)]$  is known as the *pseudo-entropy*.

## Some terms

*Adiabatic Flow:*  $s = \text{constant}$  along a streamline.

*Isentropic Flow:*  $s = \text{constant}$  everywhere (space and time).

## 12 Summary of single fluid equations for an isotropic distribution function and infinite conductivity

When the distribution function of a component is isotropic, i.e.

$$f(x_i, p_i) = f(x_i, p)$$

then the pressure tensor is isotropic and the heat flux is zero. If we further restrict ourselves to the case where all of the components of a fluid have the same mean velocity, then the fluid equations become:

**Continuity:**

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

**Momentum:**

$$\begin{aligned} \frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j) &= -\frac{\partial P}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \varepsilon_{ijk} \frac{j_{e,j}}{c} B_k \\ &= -\frac{\partial P}{\partial x_i} - \rho \frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j} \left[ \frac{B_i B_j}{4\pi} - \frac{B^2}{8\pi} \delta_{ij} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t}(\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) &= -\nabla P - \rho \nabla \phi + \frac{\mathbf{j}}{c} \times \mathbf{B} \\ &= -\nabla P - \rho \nabla \phi + \nabla \cdot \left[ \frac{\mathbf{B} \mathbf{B}}{4\pi} - \frac{B^2}{8\pi} \mathbf{I} \right] \end{aligned}$$

where the gravitational field

$$g_i = -\frac{\partial \phi}{\partial x_i} \quad \text{and} \quad \nabla^2 \phi = 4\pi G \rho$$

The advective terms in the momentum equation can also be expressed as:

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j) = \rho \frac{\partial V_i}{\partial t} + \rho V_j \frac{\partial V_i}{\partial x_j}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{V}) + \nabla \cdot (\rho \mathbf{V} \mathbf{V}) = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho \mathbf{V} \cdot \nabla \mathbf{V}$$

**Total energy:**

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \frac{1}{2} \rho V^2 + \varepsilon + \varepsilon_{EM} + \rho \phi \right] + \frac{\partial}{\partial x_i} \left[ \rho \left( \frac{1}{2} V^2 + h + \rho \phi \right) V_i + S_i \right] \\ = \rho \frac{\partial \phi}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \frac{1}{2} \rho V^2 + \varepsilon + \varepsilon_{EM} + \rho \phi \right] + \nabla \cdot \left[ \rho \left( \frac{1}{2} V^2 + h + \rho \phi \right) \mathbf{V} + \mathbf{S} \right] \\ = \rho \frac{\partial \phi}{\partial t} \end{aligned}$$

where the electromagnetic terms are:

$$\text{Poynting flux} = S_i = \frac{c}{4\pi} \varepsilon_{ijk} E_j B_k$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$$

$$\text{Energy density} = \varepsilon_{EM} = \frac{E^2 + B^2}{8\pi} \approx \frac{B^2}{8\pi}$$

## Internal energy equation

$$\frac{d\varepsilon}{dt} + (\varepsilon + P)\frac{\partial V_i}{\partial x_i} = j_{c,i}E'_i$$

$$\frac{d\varepsilon}{dt} + (\varepsilon + P)\nabla \cdot \mathbf{V} = j_c \cdot \mathbf{E}'$$

where the term on the right represents the Joule dissipation. In the limit of infinite conductivity,

$$\frac{d\varepsilon}{dt} + (\varepsilon + P)\frac{\partial V_i}{\partial x_i} = 0$$

$$\frac{d\varepsilon}{dt} + (\varepsilon + P)\nabla \cdot \mathbf{V} = 0$$

## Evolution of the magnetic field:

$$\frac{\partial B_i}{\partial t} + \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{klm}B_lV_m) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{B} \times \mathbf{V}) = \mathbf{0}$$

## The electric field:

$$E_i + \varepsilon_{ijk}\frac{V_j}{c}B_k = 0$$

$$\mathbf{E} + \frac{\mathbf{V}}{c} \times \mathbf{B} = \mathbf{0}$$

## Flux freezing:

The magnetic flux through a comoving element is conserved. Magnetic field lines move with the fluid.

## **Equation of state:**

For constant specific heats and  $\gamma = \frac{c_p}{c_v}$  the equation of state is

$$p = K(s)\rho^\gamma$$

where the function  $K(s)$  of the specific entropy  $s$  is called the pseudo-entropy.

## **Relationship between pressure and temperature:**

$$p = \frac{\rho kT}{\mu m_p}$$

where  $\mu \approx 0.62$  for a completely ionised gas and  $\mu \approx 1.4$  for a neutral gas.