Derivation of the Fluid Equations

References:

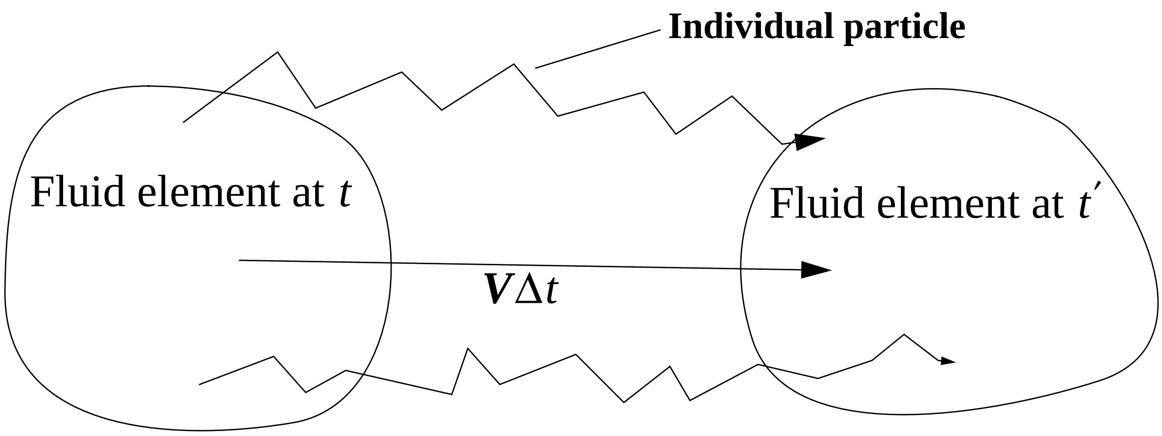
B. Rossi and S. Olbert: ***Introduction to the Physics of Space***

F. Shu: ***The Physics of Astrophysics, Vol. 2, Gas Dynamics***

E.M. Lifshitz & L.P. Pitaevskii: ***Physical Kinetics***

K. Huang: ***Statistical Mechanics***

1 Basis for fluid dynamics



If the mean free path of a particle is much less than the length scale of the system under consideration, then a fluid approximation is valid. There is some diffusion of the particles wrt the mean flow and this results in viscosity.

$$l = \text{Mean free path}$$
$$L = \text{Length scale of flow}$$
$$l \ll L \Rightarrow \text{Fluid}$$

Velocities of particles:

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{v}'$$
$$\boldsymbol{V} = \text{Mean velocity}$$
$$\boldsymbol{v}' = \text{Fluctuating component}$$

Mean free path:

Can be due to:

- Collisions between ions and/or neutrals with neutrals
- Coulomb collisions between charged particles
- Collisions between particles and waves
- Mean free path perpendicular to the magnetic field also determined by gyroradius of charged particle.

$$\text{Mean free path} = l_{\text{mfp}} = \frac{1}{n\sigma}$$
$$n = \text{no density of collision targets}$$
$$\sigma = \text{Cross-section}$$

Cross sections:

Neutral atom $\sigma \approx 10^{-19}\ \text{m}^2$

Earth's atmosphere:

$$n \sim 10^{25}\ \text{m}^{-3} \Rightarrow l_{\text{mfp}} \approx \frac{1}{10^{25} \times 10^{-19}} \approx 10^{-6}\ \text{m} \quad = 1\mu$$

Hence the atmosphere can be treated as a fluid down to these scales. NB This is not true in the rarefied regions of the upper atmosphere.

Neutral Hydrogen Gas Cloud:

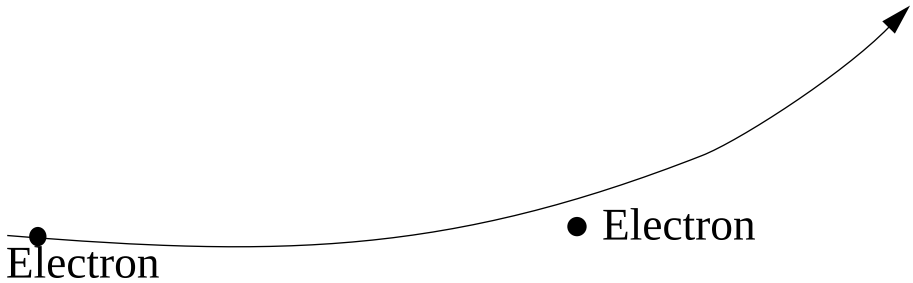
$$n \sim 10^7 \text{ m}^{-3} \Rightarrow l_{\text{mfp}} \approx \frac{1}{10^7 \times 10^{-19}} \approx 10^{12} \text{ m}$$

But

$$L \sim \text{few pc} \sim 10^{17} \text{ m} \Rightarrow \frac{l}{L} \sim 10^{-5} \ll 1 \ .$$

(Note $1 \text{ pc} \approx 3.1 \times 10^{16} \text{ m}$.)

Collisions between charged particles



Long range force implies that Total cross-section (averaged over all impact parameters) is infinite. Ignore this problem for the time being.

Order of magnitude estimate:

Collision of two thermal electrons: Define effective radius of collisional cross-section by:

$$\frac{e^2}{r_{\text{eff}}} \sim m_e w^2 \sim kT$$

Electrostatic PE ~ Relative KE ~ Thermal Energy

Cross-section:

$$r_{\text{eff}} = \frac{e^2}{kT}$$

$$\sigma \approx \pi r_{\text{eff}}^2$$

$$l \approx \frac{1}{n\pi r_{\text{eff}}^2} \sim \frac{(kT)^2}{n\pi e^4} = \frac{m_e v_{\text{th}}^4}{n_e e^4}$$

$$\text{where } v_{\text{th}} = \left(\frac{kT}{m_e}\right)^{1/2} \sim \text{Thermal Speed}$$

e.g. Hot ISM

$$n_e \sim 10^{-2} \qquad T \sim 10^6 \qquad k = 1.38\times10^{-16} \text{ ergs/K}$$

$$l \sim 4\times10^{19} \text{ cm} \sim 10 \text{ pc}$$

cf. Size of Galaxy: Distance of Sun to Galactic Centre = 8.5 kpc

$$\Rightarrow \frac{l}{L} \sim 10^{-3}$$

2 Liouville and Boltzmann equations

The basis for the derivation of the equations of magnetohydrodynamics is a statistical mechanics approach based upon the Boltzmann equation.

2.1 Distribution function

Phase space consists of coordinates and momenta of particles, i.e.

$$\text{Phase Space} = \{x_1, x_2, x_3, p_1, p_2, p_3\}$$

Each component of the fluid denoted by superscript a.

$$f^a(x_i, p_i)d^3xd^3p = \begin{array}{l}\text{No of particles of species a} \\ \text{occupying volume element} \\ d^3xd^3p \text{ of phase space.}\end{array}$$

$$\text{Number Density of component} = n^a(x_i) = \int f^a(x_i, p_i, t)d^3p$$

2.1.1 Transformation properties

A Galilean transformation is defined by:

$$x_i' = x_i - V_i t$$

where V_i is a constant. The transformation of velocities implies the Galilean transformation is:

$$v_i' = v_i' - V_i$$

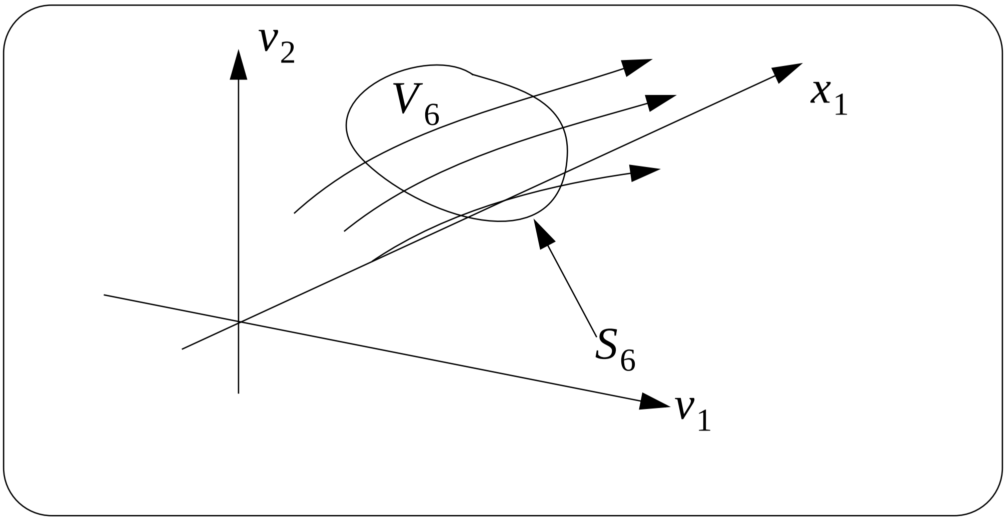
Let us now consider the transformation of the distribution function under such a transformation.

The number of particles, $f^a d^3xd^3p$, within the phase-space volume d^3xd^3p is obviously invariant under a Galilean transformation since we are just counting particles. Also, the Jacobean of the transformation:

$$x_i' = x_i - V_i t$$
$$p_i' = p_i - m^a V_i t$$

is unity so that $d^3xd^3p \to d^3x'd^3p'$. Hence $f^a(x_i', p_i') = f^a(x_i, p_i)$, i.e. the distribution function is invariant under a Galilean transformation.

2.2 Flow of particles in phase space



Suppose particles are moving in such a way that there are no sudden changes in velocity due to collisions; then the flow of particles in phase space is a smooth one.

Suppose particles moving in phase space on smooth trajectories. The the rate of change of the number of particles within a six dimensional volume in phase space is given by the negative flux of particles through the 5 dimensional boundary of the volume. We define:

$$\text{Six-dimensional Velocity} = v_\alpha = \left(\frac{dx_i}{dt}, \frac{dp_i}{dt}\right)$$

$$\text{Six-dimensional Divergence operator} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial p_i}\right)$$

and the conservation of particle number tells us that:

$$\frac{\partial}{\partial t}\int_{V_6} f^a dV_6 + \int_{S_5} f^a v_\alpha n_\alpha dS_5 = 0$$

The surface integral over S_5 can be converted to an in tegral throughout the volume V_6 so that

$$\frac{\partial}{\partial t}\int_{V_6} f^a dV_6 + \int_{V_6} \frac{\partial}{\partial x_\alpha}(f^a v_\alpha) dV_6 = 0$$

and since the volume is arbitrary:

$$\frac{\partial}{\partial t} f^a + \frac{\partial}{\partial x_\alpha}(f^a v_\alpha) = 0$$

Liouville's equation

Splitting the above 6-dimensional divergence into 2 three dimensional parts related to space and momentum gives:

$$\frac{\partial}{\partial t} f^a(x_i, p_i, t) + \frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt} f^a(x_i, p_i, t)\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt} f^a(x_i, p_i, t)\right) = 0$$

where

$$\frac{dx_i}{dt} = v_i$$

$$\frac{dp_i}{dt} = F_i = eZ^a(E_i + \varepsilon_{ijk} v_j B_k) + m^a g_i$$

where F_i is the force on the particle.

Notation:

$$
\begin{aligned}
e &= \text{Electronic Charge} \\
Z^a &= \text{Atomic No. of component} \\
E_i &= \text{Electric Field} \\
B_k &= \text{Magnetic Field} \\
v_i^a &= \text{Velocity} \\
g_i &= \text{Gravitational Field}
\end{aligned}
$$

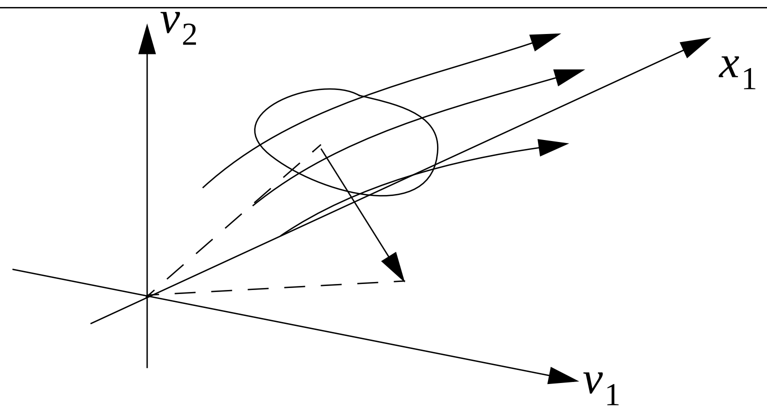
In considering Liouville's equation it is important to keep in mind the three levels of approximation in astrophysical plasmas:

- ***Very dilute plasma.*** The force components are unaffected by the plasma and collisions between particles are unimportant. e.g. Cosmic Rays in Earth's magnetic Field. Here the force components are given functions of x_i and Liouville's equation provides a satisfactory basis for theoretical work.
- ***Collisionless plasma***. Collisions are unimportant but substantial charge and current densities due to the ions themselves. In this case E_i and B_i depend upon f^a and the situation is highly non-linear. Liouville's equations is used.
- ***Fluid***. Collisions are important. Here Liouville's equation is unsatisfactory since one is required to take into account the collisions between elements of the fluid. Boltzmann's equation forms the basis for the derivation of the equations of gas dynamics.

Other applications of Liouville's equation

Liouville's equation also provides the basis for the treatment of the stellar dynamics of collisionless systems such as galaxies.

3 The effect of collisions.



Energy conserving collision which scatters a particle out of a volume of phase space.

In contrast to the smooth flow in momentum space implied by Liouville's equation, collisions scatter particles in and out of a given volume of phase space.

Boltzmann equation

$$\frac{\partial}{\partial t} f^a(x_i, p_i, t) + \frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt} f^a(x_i, p_i, t)\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt} f^a(x_i, p_i, t)\right)$$

$$= \left(\frac{\delta f^a}{\delta t}\right)_{\mathrm{coll}}$$

where the right hand side formally represents the effect of collisions. Integrals over momentum space of the Boltzmann equation provide the basis for the equations of fluid dynamics.

4 Moments of the distribution function

Before moving on to the implications of the Boltzmann equation it is useful to consider the various moments of the distribution function.

Number density

$$n^a(x_i, t) = \int f^a(x_i, p_i, t) d^3 p$$

Flux density

$$\chi_i^a = \int v_i^a f^a(x_i, p_i, t) d^3 p$$

Mean velocity

$$V_i^a = \frac{\chi^a{}_i}{n^a} \Rightarrow n^a V^a{}_i = \chi^a{}_i$$

Mass density

$$\rho^a = n^a m^a$$

$$\rho = \sum_a \rho^a$$

Electric charge density

$$\rho_e^a = eZ^a n^a$$

$$\rho_e = \sum_a \rho_e$$

Electric current density

$$J_i^a = eZ_a \chi_i^a$$

$$J_i = \sum_a J_i^a$$

Energy density

$$E^a = \int \frac{1}{2} m^a v^{a2} f^a(x_i, p_i, t) d^3 p$$

$$E = \sum_a E^a$$

Momentum density

$$\Pi_i^a = \int p_i f^a(x_i, p_i, t) d^3 p$$

$$\Pi_i = \sum_a \Pi_i^a$$

Kinetic tensor

$$\Pi_{ij}^a = \int v_i^a p_j f^a(x_i, p_i, t) d^3p = \int v_j^a p_i f^a(x_i, p_i, t) d^3p$$

$$= \text{Flux of } j^{\text{th}} \text{ component of momentum in the } i\text{–direction}$$

$$= \Pi_{ji}^a$$

4.1 Centre of mass frame for a component

This is defined as the frame in which the total momentum of the fluid is zero. Define the velocity of the Galilean transformation to this frame as follows:

$$\text{Momentum density} = \Pi_i^a = \int p_i f^a d^3p$$

The mean velocity was defined earlier by

$$V_i^a = \frac{\int v_i^a f^a d^3p}{n^a} = \frac{\int p_i f^a d^3p}{n^a m^a} = \frac{\Pi_i^a}{\rho^a}$$

Momentum density in a frame moving with velocity V_i^a is

$$\Pi_i^{a\prime} = \int (p_i - m^a V_i^a) f^a d^3p = \Pi_i - n^a m^a V_i^a$$

$$= \Pi_i^a - \rho^a V_i^a = 0$$

so that V_i^a defines the centre of mass frame.

4.2 Kinetic tensor in the centre of mass frame

The velocity of a component in the centre of mass frame is

$$v^{a'}_i = v^a_i - V^a_i \Rightarrow v^a_i = V^a_i + v^{a'}_i$$

Therefore the kinetic tensor of a component is:

$$\begin{aligned}
\Pi^a_{ij} &= \int m^a (V^a_i + v^{a'}_i)(V^a_j + v^{a'}_j) f^a d^3 p \\
&= \int m^a (V^a_i V^a_j + v^{a'}_i V^a_j + V^a_i v^{a'}_j + v^{a'}_i v^{a'}_j) f^a d^3 p \\
&= m^a V^a_i V^a_j \int f^a d^3 p + m^a V^a_j \int v^{a'}_i f^a d^3 p \\
&\quad + m^a V^a_i \int v^{a'}_j f^a d^3 p + m^a \int v^{a'}_i v^{a'}_j f^a d^3 p
\end{aligned}$$

The 2nd and 3rd terms vanish by virtue of the definition of the mean velocity so that

$$\Pi^a_{ij} = \rho^a V^a_i V^a_j + P^a_{ij}$$

which introduces the partial pressure tensor

$$P^a_{ij} = m^a \int v^{a'}_i v^{a'}_j f^a d^3 p$$

4.3 Isotropic distribution function

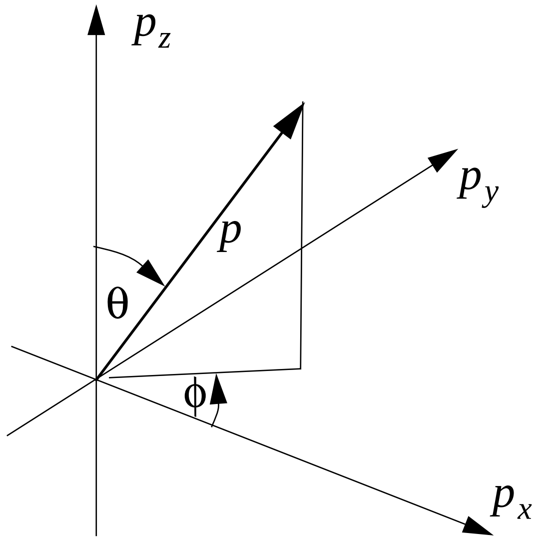
If the distribution function is isotropic in the centre of mass (rest) frame, i.e.

$$f^a(x_i, p_i) = f^a(x_i, p)$$

then

$${}^{a}_{ij} = m_a^{-1}\int p_i p_j f^a(x_i, p)p^2 \sin\theta\, dp\, d\theta\, d\phi$$

where (p, θ, ϕ) constitute spherical polars in momentum space.



Polar coordinates in momentum space

Since f^a is isotropic, then it is easily shown that the integral is zero for $i \neq j$ and that when $i = j$

$$P^a_{ij} = m_a^{-1}\int p_z^2 f^a(x_i, p)p^2 \sin\theta\, dp\, d\theta\, d\phi$$

$$= m_a^{-1}\int p^4 f^a(x_i, p)\cos^2\theta \sin\theta\, dp\, d\theta\, d\phi$$

$$= \frac{4\pi}{3m^a}\int p^4 f^a(x_i, p)dp$$

I have put $i = j = z$ in the above, since the xx, yy and zz components of the pressure tensor are identical, i.e. there is no preferred axis.

Hence

$$P^a_{ij} = P^a \delta_{ij}$$

where

$$ {}^{a} = \frac{4\pi}{3m^a}\int p^4 f^a(x_i, p)dp $$

Thus in the case of an isotropic distribution function, the pressure tensor is also isotropic. We shall see that P is what we normally associate with the pressure when we think of a fluid in terms of a continuum.

4.3.1 Relationship of pressure to thermal velocity and temperature

When the pressure tensor is isotropic

$$ \begin{aligned} P^a_{ii} = 3P^a \Rightarrow P^a &= \frac{1}{3}P^a_{ii} = \frac{1}{3}\int v_i p_i f^a d^3 p \\ &= \frac{1}{3}\int m^a v^2 f^a d^3 p \\ &= \frac{1}{3}n^a m^a \overline{v^2} \end{aligned} $$

where $\overline{v^2}$ is the rms velocity of the ions.

For monatomic ions in thermal equilibrium at temperature T there is $(kT)/2$ energy for each of the three degrees of freedom so that

$$ \frac{1}{3}n^a m^a \overline{v^2} = \frac{1}{3}\times n^a \times 2 \times \frac{1}{2}m^a \overline{v^2} = \frac{2}{3}n^a \times \frac{3}{2}kT = n^a kT $$

Hence

$$ P^a = n^a kT $$

4.3.2 Derivation of pressure from the Maxwell-Boltzmann distribution

For a uniform gas in thermodynamic equilibrium the Maxwell-Boltzmann distribution function is given (in terms of velocity) by:

$$f_{MB}(\mathbf{v})d^3\mathbf{v} = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp-\frac{mv^2}{2kT}d^3v$$

By calculating the above integral for the pressure, viz,

$$P^a = \frac{4\pi}{3m^a}\int p^4 f^a(x_i, p)dp$$

one can also regain $P^a = n^a kT$.

4.4 Centre of mass frame for the entire fluid

This is defined as the frame in which the momentum density of the entire fluid vanishes. That is, we determine a Galilean transformation for which

$$\Pi_i' = \sum_a \Pi_i^{a\prime} = 0$$

We proceed similarly to the case for the CoM frame for a single component, viz, for transformation velocity V_i, the transformed momentum density is

$$\Pi_i' = \sum_a \Pi_i^a - \rho^a V_i = \sum_a \Pi_i^a - V_i \sum_a \rho^a$$
$$= \Pi_i - \rho V_i$$

and this is zero for

$$V_i = \frac{\Pi_i}{\rho}$$

If the centre of mass frames for all components are equal, then we can use the ***Single Fluid Approximation*** in which all components have the same centre of mass velocity V_i. In this case the Kinetic Tensor

$$\Pi_{ij} = \sum_a \Pi^a_{ij} = \sum_a \rho^a V_i V_j + \sum_a P^a_{ij}$$
$$= V_i V_j \sum_a \rho^a + \sum_a P^a_{ij}$$
$$= \rho V_i V_j + \sum_a P^a_{ij}$$

and when the distribution is isotropic,

$$\Pi_{ij} = \rho V_i V_j + P\delta_{ij}$$

where

$$\text{Total Pressure} = P = \sum_a P^a$$

i.e the total pressure is the sum of the partial pressures.

In many cases a single fluid approximation is appropriate e.g. stellar winds, hydrostatic atmospheres in elliptical galaxies, supernova blast waves, geophysical fluid dynamics, aerodynamics.

An example of where a single fluid approximation is inadequate is in the study of partially ionised gases where the ions and neutrals may move differently. This leads to ***ambipolar diffusion*** which is discussed later.

5 Conservation of particle number, current and mass

5.1 Conservation of particle number

Integrate the Boltzmann equation wrt momentum

$$\int \frac{\partial f^a}{\partial t} d^3p + \int v_i^a \frac{\partial f^a}{\partial x_i} d^3p + \int \frac{\partial (F_i^a f^a)}{\partial p_i} d^3p = \int \frac{\delta f^a}{\delta t} d^3p$$

The integral of the momentum space divergence can be transformed over an arbitrary surface enclosing all of the momenta. This can be made arbitrarily large so that the distribution function on the surface is zero. Hence this term disappears.

The partial derivative wrt x_i can be taken outside the integral since the integral is over momentum and $v_i^a = (m^a)^{-1} p_i$ is not a function of x_i.

The right hand side represents the rate of scattering of particles out of all momentum space. Although particles are scattered from one value of momentum to another there is a balance, i.e. the total number of particles is not destroyed by the collisions which redistribute momenta. Hence the right hand side equals zero and

$$\frac{\partial}{\partial t} \int f^a d^3p + \frac{\partial}{\partial x_i} \int v_i^a f^a d^3p = 0$$

Using the moments defined earlier:

$$\frac{\partial n^a}{\partial t} + \frac{\partial (n^a V_i^a)}{\partial x_i} = \frac{\partial n^a}{\partial t} + \frac{\partial (\chi_i^a)}{\partial x_i} = 0$$

This represents the conservation of particles of component a.

5.2 Electric current

Multiply above equation by eZ^a

$$\Rightarrow \frac{\partial \rho_e^a}{\partial t} + \frac{\partial J_i^a}{\partial x_i} = 0$$

i.e. the conservation law for electric current associated with component *a*.

5.3 Mass

Multiply number conservation equation by m^a.

$$\Rightarrow \frac{\partial \rho^a}{\partial t} + \frac{\partial(\rho^a V_i^a)}{\partial x_i} = 0$$

i.e. conservation of mass of component *a*.

6 Conservation of Momentum

6.1 Derivation of momentum equation from the Boltzmann equation

Now multiply the Boltzmann equation by momentum and integrate.

$$\int p_i \frac{\partial f^a}{\partial t} d^3p + \int p_i v_j^a \frac{\partial f^a}{\partial x_j} d^3p + \int p_i \frac{\partial (F_j^a f^a)}{\partial p_j} d^3p$$
$$= \int p_i \left(\frac{\delta f^a}{\delta t}\right)_{\text{coll}} f^a d^3p$$

Again, with the second integral, the partial derivative can be taken outside. For the third integral, note that

$$p_i \frac{\partial (F_j^a f^a)}{\partial p_j} = \frac{\partial}{\partial p_j}(p_i F_j^a f^a) - \delta_{ij} F_j^a f^a = \frac{\partial}{\partial p_j}(p_i F_j^a f^a) - F_i^a f^a$$

The divergence integrates to zero as before leaving the integral of $-F_i^a f^a$. This gives:

$$\frac{\partial}{\partial t}\int p_i \frac{\partial f^a}{\partial t} d^3p + \frac{\partial}{\partial x_j}\int p_i v_j f^a d^3p - \int F_i^a f^a d^3p$$
$$= \int p_i \left(\frac{\delta f^a}{\delta t}\right)_{\mathrm{coll}} f^a d^3p$$

Now

$$\begin{aligned}\int F_i^a f^a d^3p &= \int eZ^a E_i f^a d^3p + \int eZ^a \varepsilon_{ijk} v_j^a B_k f^a d^3p + \int m^a g_i f^a d^3p \\ &= eZ^a E_i \int f^a d^3p + eZ^a \varepsilon_{ijk} B_k \int v_j^a f^a d^3p + m^a g_i \int f^a d^3p \\ &= eZ^a n^a E_i + eZ^a \varepsilon_{ijk} \chi_i^a B_k + \rho^a g_i \\ &= \rho_e^a E_i + \varepsilon_{ijk} J_j^a B_k + \rho^a g_i\end{aligned}$$

i.e. integrating the force term in the Boltzmann equation over momentum space gives the Lorentz force on the component.

Rearranging terms and recognizing the kinetic tensor in the above:

$$\frac{\partial \Pi_i^a}{\partial t} + \frac{\partial \Pi_{ij}^a}{\partial x_j} = \rho_e^a E_i + \varepsilon_{ijk} J_j^a B_k + \rho^a g_i + \int p_i \left(\frac{\delta f^a}{\delta t}\right)_{\mathrm{coll}} f^a d^3p$$

The first and second terms on the right represent the Lorentz force on component *a*.

The third term represents the gravitational force.

The last term represents the rate of gain of momentum of component *a* as a result of collisions. Now, contrary to the case of particle conservation, we cannot set the collision term on the right hand side to zero. However, the nett momentum gain to the entire fluid as a result of collisions is zero.

Using the expressions for the momentum density and the kinetic tensor derived above:

$$\frac{\partial}{\partial t}(\rho^a V_i^a) + \frac{\partial}{\partial x_j}(\rho^a V_i^a V_j^a + P_{ij}^a) = \rho_e^a E_i + \varepsilon_{ijk} J_j^a B_k + \rho^a g_i + \left(\frac{\delta \Pi_i^a}{\delta t}\right)$$

where the last term represents the nett rate of gain of momentum to component a from collisions.

6.2 Advective term and pressure forces

The terms

$$\frac{\partial}{\partial t}(\rho^a V_i^a) + \frac{\partial}{\partial x_j}(\rho^a V_i^a V_j^a) = V_i^a\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho^a V_j^a)}{\partial x_j}\right) + \rho^a\left(\frac{\partial V_i^a}{\partial t} + V_j^a \frac{\partial V_j^a}{\partial x_j}\right)$$

and the first group of terms on the RHS is zero as a result of the continuity equation derived earlier.

Hence the a-component momentum equation becomes:

$$\rho^a\left(\frac{\partial V_i^a}{\partial t} + V_j^a \frac{\partial V_j^a}{\partial x_j}\right) = -\frac{\partial P_{ij}^a}{\partial x_j} + \rho_e^a E_i + \varepsilon_{ijk} J_j^a B_k + \rho^a g_i + \left(\frac{\delta \Pi_i^a}{\delta t}\right)$$

or

$$\rho^a \frac{DV_i^a}{Dt} = -\frac{\partial P_{ij}^a}{\partial x_j} + \rho_e^a E_i + \varepsilon_{ijk} J_j^a B_k + \rho^a g_i + \left(\frac{\delta \Pi_i^a}{\delta t}\right)$$

where the operator $\frac{D}{Dt}$ represents differentiation following the motion defined by the streamlines with velocity V_i^a. In the two above equations, the negative gradient of the pressure tensor plays the role of a force.

6.3 Single fluid approximation

Summing the integrated momentum equation over components gives:

$$\frac{\partial \Pi_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = \rho_e E_i + \varepsilon_{ijk} J_j B_k + \rho g_i + \sum_a \int p_i \left(\frac{\delta f^a}{\delta t} \right)_{\text{coll}} f^a d^3 p$$

In this case, the momentum collision term can be set to zero since there is no nett gain to the fluid as a whole as a result of collisions. Thus,

$$\frac{\partial \Pi_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} = \frac{\partial (\rho V_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho V_i V_j + P_{ij}) = \rho_e E_i + \varepsilon_{ijk} J_j B_k + \rho g_i$$

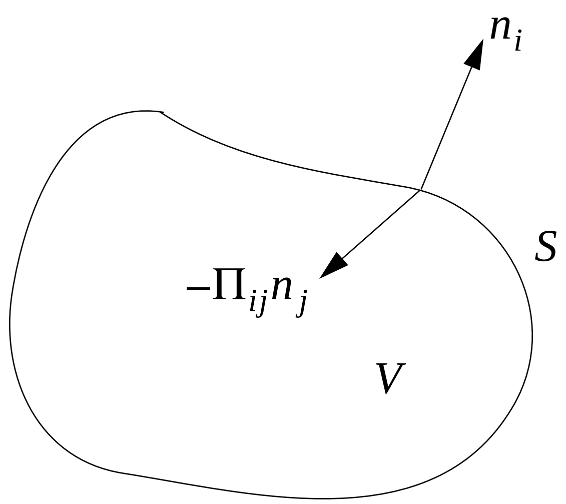
or

$$\rho \left(\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} \right) = -\frac{\partial P_{ij}}{\partial x_j} + \rho_e E_i + \varepsilon_{ijk} J_j B_k + \rho g_i$$

This follows from the continuity equation, as before.

This equation is one of the most widely used forms of the momentum equation in all areas of fluid dynamics.

6.4 Continuum interpretation



Using Green's theorem we can write the momentum equation as

$$\frac{\partial}{\partial t}\int_V \Pi_i d^3v = -\int_S \Pi_{ij} n_j dS + \int_V (\rho_e + \varepsilon_{ijk} J_j B_k + \rho g_i) d^3v$$

i.e. the rate of change of momentum within V is equal to minus the flux of momentum across S as well as the integrated body forces.

Expressing the kimetic tensor in terms of bulk momentum and pressure components, this equation can be written:

$$\frac{\partial}{\partial t}\int_V \rho V_i d^3x = -\int_S \rho V_i V_j n_j dS - \int_S P_{ij} n_j dS + \int_V (\rho_e + \varepsilon_{ijk} J_j B_k + \rho g_i) d^3x$$

The left hand side of this equation represents the time rate of change of the momentum within the volume V; the first term on the right represents the flow of momentum associated with the bulk velocity; the second term represents the flow of momentum associated with the pressure and the last term represents the body forces per unit volue integrated throughout *V*.

This provides another way of thinking of the pressure tensor is that the term $P_{ij}n_j$ represents the force per unit area on the surface *S* due to the flux of momentum or in other words $P_{ij}n_j dS$ is the force exerted on the fluid inside *S* by the fluid outside *S*. When the pressure tensor is isotropic

$$P_{ij}n_j = P\delta_{ij}n_j = Pn_i$$

and the pressure force is normal to the surface. This coincides with the way in which we think of pressure in a continuum approximation.

6.5 Stress tensor of the gravitational field

Consider the tensor

$$\Gamma_{ij} = -\frac{1}{4\pi G}\left(g_i g_j - \frac{1}{2}g^2\delta_{ij}\right)$$

where $G = 6.67\times10^{-8}\,\mathrm{N\,kg^{-2}m^{2}}$ is the constant of gravitation. The divergence of this tensor

$$\Gamma_{ij,\,j} = -\frac{1}{4\pi G}(g_{i,\,j}g_j + g_i g_{j,\,j} - g_j g_{j,\,i})$$

Now the gravitational field is expressed as the gradient of a scalar potential ϕ

$$g_i = -\frac{\partial\phi}{\partial x_i} \qquad \text{with} \quad \nabla^2\phi = 4\pi G\rho$$

Therefore,

$$\Gamma_{ij,\,j} = -\frac{1}{4\pi G}(g_j(\phi_{\,,ij} - \phi_{\,,ji}) + g_i \times -4\pi G\rho) = \rho g_i$$

and

$$\frac{\partial}{\partial t}\int_V \Pi_i d^3x = -\int_S (\Pi_{ij} - \Gamma_{ij}) n_j dS + \int_V (\rho_e + \varepsilon_{ijk} J_j B_k) d^3x$$

For the case of a zero electromagnetic field:

$$\frac{\partial}{\partial t}\int_V \Pi_i d^3x = -\int_S (\Pi_{ij} - \Gamma_{ij}) n_j dS$$

This shows that one can formally identify a flux of momentum per unit area, $-\Gamma_{ij} n_j$, associated with the gravitational field. However, this formal result is rarely exploited.

7 The energy equation

The energy equation follows from multiplying the Boltzmann equation by $K^a = \frac{1}{2} m^a v^2$ and integrating over momentum.

Before carrying out the integration we define a few more terms.

7.1 Bulk kinetic energy and internal energy

Since $v_i^a = V_i^a + v_i^{a\prime}$ then the total energy

$$
\begin{aligned}
E &= \int \frac{1}{2} m^a v^2 f^a d^3 p \\
&= \int \left(\frac{1}{2} m^a V^{a2} + m^a V_i^a v_i^{a\prime} + \frac{1}{2} m^a v^{a\prime 2} \right) f^a d^3 p \\
&= \frac{1}{2} \rho^a V^{a2} + \int \frac{1}{2} m^a v^{a\prime 2} f^a d^3 p \\
&= \frac{1}{2} \rho^a V^{a2} + \varepsilon^a
\end{aligned}
$$

This breaks up the total energy into the kinetic energy associated with the centre of mass velocity and the internal energy density ε^a.

The internal energy density is the kinetic energy of the plasma in the centre of mass frame.

7.2 Heat flux

This is defined for a single component by:

$$q_i^a = \int \frac{1}{2} m^a v'^2 v_i^{a\prime} f^a d^3 p$$

The heat flux is the flux of kinetic energy in the centre of mass frame.

The heat flux for the entire fluid:

$$q_i = \sum_a q_i^a$$

7.3 Derivation of the energy equation

Energy times Boltzmann equation =>

$$\int K^a \frac{\partial f^a}{\partial t} d^3 p + \int K^a v_i^a \frac{\partial f^a}{\partial x_i} d^3 p + \int K^a \frac{\partial (F_i f^a)}{\partial p_i} d^3 p = \int K^a \frac{\delta f^a}{\delta t} d^3 p$$

K does not depend on either x_i or t so that the space and time derivatives can be taken outside the integral. We rearrange the terms within the third integral to give a divergence.This gives us:

$$\frac{\partial}{\partial t}\int K^a f^a d^3p + \frac{\partial}{\partial x_i}\int K^a v_i^a f^a d^3p + \int\left[\frac{\partial}{\partial p_i}(K^a F_i f^a) - \left(\frac{\partial K^a}{\partial p_i}\right)F_i f^a\right]d^3p$$
$$= \int K^a \frac{\delta f^a}{\delta t} d^3p$$

The integral over the momentum space divergence vanishes for the same reasons as before. The term

$$\frac{\partial K^a}{\partial p_i} = \frac{1}{2m^a}\frac{\partial}{\partial p_i}(p_j p_j) = \frac{1}{m^a}p_i = v_i^a$$

Hence,

$$\frac{\partial}{\partial t}\int \frac{1}{2}m_a v^{a2} f^a d^3p + \frac{\partial}{\partial x_i}\int \frac{1}{2}m^a v^{a2} v_i^a f^a d^3p - \int v_i^a F_i f^a d^3p$$
$$= \int K^a \frac{\delta f^a}{\delta t} d^3p$$

Let us take each of these terms separately. As we had above, the first term is:

$$\int \frac{1}{2}m_a v^{a2} f^a d^3p = \frac{1}{2}\rho^a V^2 + \varepsilon^a$$

For the second term (leaving out the a superscript):

$$\text{-}v^2 v_i = \left(\frac{1}{2}V^2 + V_j v_j' + \frac{1}{2}v_j' v_j'\right)(V_i + v_i')$$
$$= \frac{1}{2}V^2 V_i + V_j V_i v_j' + \frac{1}{2}v_j' v_j' V_i + \frac{1}{2}V^2 v_i' + V_j v_i' v_j' + \frac{1}{2}v_j' v_j' v_i'$$

When multiplied by f^a and integrated over momentum space these terms give:

$$\int \frac{1}{2} m^a v^{a2} v_i^a f^a d^3 p = \frac{1}{2}\rho^a V^{a2} V_i + 0 + \varepsilon^a V_i^a + 0 + P_{ij}^a V_j^a + q_i^a$$

We also have:

$$\begin{aligned} \int v_i^a F_i d^3 p &= eZ^a \int v_i E_i f^a d^3 p + \int p_j g_j f^a d^3 p \\ &= J_i^a E_i + \rho^a V_i g_i \end{aligned}$$

Putting all of this together:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho^a V^{a2} + \varepsilon^a\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho^a V^{a2} V_i^a + \varepsilon^a V_i + P_{ij}^a V_j + q_i\right) \\ = J_i^a E_i + \rho^a V_i^a g_i + \int K^a \left(\frac{\delta f^a}{\delta t}\right) d^3 p \end{aligned}$$

As with other expressions we have derived, the collision term is not zero for an individual component. However, if all components are moving with the same mean motion, then we can sum over components to obtain:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \varepsilon\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + q_i\right) \\ = J_i E_i + \rho V_i g_i \end{aligned}$$

where the collision term sums to zero when we assume that energy is conserved in collisions.

The gravitational term $\rho V_i g_i$ can be manipulated in the following way:

$$\rho V_i g_i = -\rho V_i \frac{\partial\phi}{\partial x_i} = -\frac{\partial}{\partial x_i}(\rho\phi V_i) + \frac{\partial}{\partial x_i}(\rho V_i)\phi$$

and using the continuity equation for mass:

$$\rho V_i g_i = -\frac{\partial}{\partial x_i}(\rho\phi V_i) - \phi\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho\phi V_i) - \frac{\partial}{\partial t}(\rho\phi) + \rho\frac{\partial\phi}{\partial t}$$

Hence, the energy equation becomes:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \varepsilon + \rho\phi\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij}V_j + \rho\phi V_i + q_i\right)$$

$$= \rho\frac{\partial\phi}{\partial t} + J_i E_i$$

If $\frac{\partial\phi}{\partial t} = 0$ then the only contribution to the right hand side is $J_i E_i$.

7.4 Continuum interpretation

Integrating over a finite volume:

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho V^2 + \varepsilon + \rho\phi\right) d^3x =$$

$$\int_S \left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij}V_j + \rho\phi V_i + q_i\right) n_i dS + \int_V J_i E_i d^3x$$

The second integral on the right represents the work done by the electromagnetic field on the gas.

Restricting ourselves to the zero electromagnetic field case and also taking the gravitational field to be time independent (a good approximation for many circumstances). Integrating throughout a finite volume:

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho V^2 + \varepsilon + \rho\phi\right) d^3x =$$

$$\int_S \left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + \rho\phi V_i + q_i\right) n_i dS$$

The group of terms

$\frac{1}{2}\rho V^2 + \varepsilon + \rho\phi$ integrated over volume, represent the total energy (kinetic plus internal plus gravitational potential energy) within *V*. The integral over the bounding surface *S* represents the flux of energy through *S*. This gives the general form for the energy flux:

$$_{E,i} = \frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + \rho\phi V_i + q_i$$

The various terms in this expression have the following meaning:

$$\frac{1}{2}\rho V^2 V_i n_i = \text{Flux of Kinetic Energy per unit area}$$

$$\varepsilon V_i n_i = \text{Flux of internal energy per unit area}$$

$$P_{ij} n_i V_j = \text{Rate of work per unit area by pressure forces on fluid}$$

$$\rho\phi V_i n_i = \text{Flux of gravitational potential energy}$$

$$q_i n_i = \text{Heat flux}$$

When the distribution function is isotropic, the pressure tensor is isotropic and the heat flux is zero. Then

$$_{ij} V_j = P V_i$$

$$q_i = 0$$

Enthalpy:

The case of an isotropic distribution function (perfect fluid) introduces the enthalpy density defined by

$$H = \varepsilon + P$$

and the specific enthalpy (enthalpy per unit mass)

$$h = \frac{(\varepsilon + P)}{\rho}$$

For a monatomic gas

$$\varepsilon = \frac{3}{2}nkT \qquad p = nkT \qquad \text{and } \rho = \mu n m_p$$

where μ is the mean molecular weight ($\mu \approx 0.62$ for a completely ionized gas) and m_p is the mass of a proton. Hence:

$$h = \frac{5}{2}\frac{kT}{\mu m_p}.$$

Using the specific enthalpy, the energy flux

$$F_E = \rho\left(\frac{1}{2}V^2 + h + \phi\right)V_i$$

It is, at first sight, surprising to see the specific enthalpy in this equation rather than the specific internal energy $\frac{\varepsilon}{\rho}$. However, the derivation above shows why this is the case. The source of the difference is the work done by the pressure in increasing the energy of a given volume of fluid and this contributes an extra component to the energy flux.

8 Inclusion of electromagnetic fields in the energy and momentum fluxes

In order to incorporate electromagnetic fields into the energy and momentum fluxes we need to consider Maxwell's equations and the electromagnetic energy and momentum densities and associated stress tensor.

8.1 Maxwell's equations

The starting point is Maxwell's equations

$$
\begin{aligned}
E_{j,j} &= \frac{\rho_e}{\varepsilon_0} \\
\varepsilon_{ijk}B_{k,j} &= \mu_0 J_i + \mu_0\varepsilon_0\frac{\partial E_i}{\partial t} \\
B_{j,j} &= 0 \\
\varepsilon_{ijk}E_{k,j} &= -\frac{\partial B_i}{\partial t}
\end{aligned}
$$

where

$$
\mu_0 = 4\pi \times 10^{-7} \text{ Henry/m}
$$

$$
\varepsilon_0 = \frac{1}{\mu_0 c^2} = 8.84\times 10^{-16} \text{ Farad/m}
$$

The following quantities come from electromagnetic theory:

$$\text{Electromagnetic energy density} = \varepsilon_{EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu}$$

$$\begin{array}{c}\text{Poynting Flux}\\ \text{(Flux of electromagnetic energy)}\end{array} = S_i = \frac{1}{\mu_0}\varepsilon_{ijk}E_j B_k$$

$$\text{Momentum density} = \frac{S_i}{c^2}$$

$$\text{Maxwell stress tensor} = M_{ij} = M_{E,\,ij} + M_{B,\,ij}$$

$$M_{E,\,ij} = \varepsilon_0\left(E_i E_j - \frac{1}{2}E^2\delta_{ij}\right)$$

$$M_{B,\,ij} = \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right)$$

It can be shown using Maxwell's equations that the following relationships hold (exercise):

$$\frac{\partial\varepsilon_{EM}}{\partial t} + \frac{\partial S_i}{\partial x_i} = -J_i E_i$$

$$\frac{1}{c^2}\frac{\partial S_i}{\partial t} - M_{ij,\,j} = -(\rho_e E_i + \varepsilon_{ijk}J_j B_k)$$

8.2 Momentum

Substituting for the Lorentz force in the momentum equation

$$\frac{\partial}{\partial t}\left(\Pi_i + \frac{S_i}{c^2}\right) + \frac{\partial}{\partial x_j}(\Pi_{ij} - M_{ij} - \Gamma_{ij}) = 0$$

This equation expresses the total conservation of momentum: mechanical, electromagnetic and gravitational.

8.3 Energy

Substituting for $J_i E_i$ in the energy equation gives

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \varepsilon + \varepsilon_{EM} + \rho\phi\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + S_i + \rho\phi V_i + q_i\right)$$

$$= \rho\frac{\partial\phi}{\partial t}$$

expressing the conservation of bulk kinetic, internal, electromagnetic and gravitational potential energy. The RHS is zero if the gravitational field is time independent.

Thus the total energy flux is

$$ {}_{E,\,i} = \frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + S_i + \rho\phi V_i + q_i$$

and when the distribution function is isotropic

$$F_{E,\,i} = \rho\left(\frac{1}{2}V^2 + h + \phi\right)V_i + S_i$$

the important additional term being the Poynting flux.

9 The internal energy equation

The above equation which is useful for indentifying the components of the energy flux applies to the total energy. A separate equation for the internal energy is also useful. This is derived by combining the total energy equation with the momentum equation.

We have for the total energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2 + \varepsilon\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i + \varepsilon V_i + P_{ij} V_j + q_i\right) = J_i E_i + \rho V_i g_i$$

The point of the following is to eliminate the terms involving ρV^2. To this end we look at

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\rho V_j V_j\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V_j V_j V_i\right)$$

$$= \frac{1}{2} V_j V_j \frac{\partial \rho}{\partial t} + \rho V_j \frac{\partial V_j}{\partial t}$$

$$+ \frac{1}{2} V_j V_j \frac{\partial}{\partial x_i}(\rho V_i) + \rho V_i V_j \frac{\partial V_j}{\partial x_i}$$

The first plus third terms on the RHS vanish by virtue of mass conservation.

Therefore,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho V^2\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{2}\rho V^2 V_i\right) = V_i\left(\rho \frac{\partial V_i}{\partial t} + \rho V_j \frac{\partial V_i}{\partial x_j}\right)$$

$$= -V_i \frac{\partial P_{ij}}{\partial x} + \rho_e V_i E_i + \varepsilon_{ijk} V_i J_j B_k + \rho V_i g_i$$

Substituting this into the total energy equation:

$$\frac{\partial \varepsilon}{\partial t} + \frac{\partial}{\partial x_i}(\varepsilon V_i + P_{ij} V_j + q_i) - V_i \frac{\partial P_{ij}}{\partial x_j} + \rho_e V_i E_i + \varepsilon_{ijk} V_i J_j B_k + \rho V_i g_i$$

$$= J_i E_i + \rho V_i g_i$$

A number of terms cancel and

$$\frac{\partial \varepsilon}{\partial t} + V_i \frac{\partial \varepsilon}{\partial x_i} + \varepsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} = E_i(J_i - \rho_e V_i) - \varepsilon_{ijk} V_i J_j B_k$$

Before proceeding further it is useful to distinguish between the conduction and advection currents. The a^{th} partial current is defined in terms of the distribution function by:

$$
\begin{aligned}
{}_i^a &= eZ^a \int v_i^a f^a d^3p = eZ^a \int (V_i^a + v_i^{a\prime}) f^a d^3p \\
&= eZ^a n^a V_i^a + J_{c,i}^a = \rho_e^a V_i^a + J_{c,i}^a
\end{aligned}
$$

where $J_{c,i}$ is the ***conduction current*** and the term $\rho_e V_i^a$ is the current corresponding to the advection of bulk charge by the motion of the gas. Summing over components,

$$
J_i = \rho_e V_i + J_{c,i}
$$

and the energy equation becomes:

$$
\begin{aligned}
\frac{d\varepsilon}{dt} + \varepsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} &= E_i J_{c,i} - \varepsilon_{ijk} V_i J_{c,j} B_k \\
&= J_{c,i}(E_i + \varepsilon_{ijk} V_j B_k)
\end{aligned}
$$

The term $E_i + \varepsilon_{ijk} V_j B_k$ is the electric field E_i' in the comoving frame of the gas, so that

$$
\frac{d\varepsilon}{dt} + \varepsilon \frac{\partial V_i}{\partial x_i} + P_{ij} \frac{\partial V_i}{\partial x_j} + \frac{\partial q_i}{\partial x_i} = J_{c,i} E_i'
$$

(using differentiation following the motion). The term $J_{c,i} E_i'$ describes the amount of Joule heating of the gas.

When the distribution function is isotropic:

$$
\frac{d\varepsilon}{dt} + (\varepsilon + P) \frac{\partial V_i}{\partial x_i} = J_{c,i} E_i'
$$

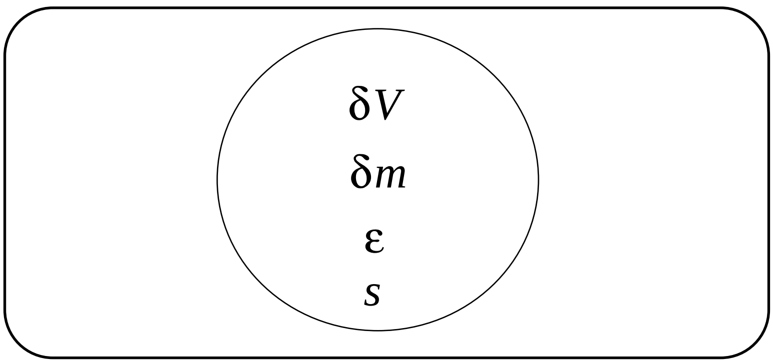
Writing this equation as

$$\frac{d\varepsilon}{dt} = -(\varepsilon + P)\frac{\partial V_i}{\partial x_i} + J_{c,i}E'_i$$

we see that the term $(\varepsilon + P)\frac{\partial V_i}{\partial x_i}$ represents the effect of expansion (contraction) in cooling (heating) the gas.

10 Relationship to thermodynamics

10.1 Entropy



Consider a comoving element of fluid with mass δm, volume $\delta V = \frac{\delta m}{\rho}$.

Let entropy per unit mass be s, then the entropy of the element is $s\delta m$, the internal energy is $\frac{\varepsilon\delta m}{\rho}$, then the relationship between entropy, internal energy and volume is:

$$kTd(s\delta m) = d\left(\frac{\varepsilon\delta m}{\rho}\right) + Pd\left(\frac{\delta m}{\rho}\right)$$

Since this is a comoving element, then $\delta m = \text{constant}$ and

$$kTds = d\left(\frac{\varepsilon}{\rho}\right) + Pd\left(\frac{1}{\rho}\right)$$

Expanding the differentials and multiply by ρ:

$$\rho kTds = d\varepsilon - \frac{(\varepsilon + P)}{\rho}d\rho$$

Express in terms of derivatives along the trajectory of the element:

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + P)}{\rho}\frac{d\rho}{dt}$$

The equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + V_i\frac{\partial\rho}{\partial x_i} + \rho\frac{\partial V_i}{\partial x_i} = 0$$

can be expressed in the form

$$\frac{\partial V_i}{\partial x_i} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

so that

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} + (\varepsilon + P)\frac{\partial V_i}{\partial x_i}$$

and comparing this with the expression above:

$$\rho kT\frac{ds}{dt} = J_{c,i}E'_i$$

When there is no dissipation of energy by electromagnetic effects:

$$\rho kT\frac{ds}{dt} = 0$$

i.e. the flow is adiabatic.

10.2 Other forms of the entropy equation

Other forms of the relationship between entropy and other thermodynamic variables are also useful, e.g. in terms of the specific enthalpy, the above equation for entropy can be written:

$$\rho kT\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{dP}{dt}$$

10.3 Equation of state

In general for an ideal gas with internal degrees of freedom the pressure is still given by

$$p = nkT = \frac{\rho kT}{\mu m_p}$$

but the internal energy may be partitioned amongst extra degrees of freedom. e.g. rotational and vibrational. For the case of constant specific heats we have

$$p = (\gamma - 1)\varepsilon \Rightarrow \varepsilon = \frac{1}{\gamma - 1}p \text{ and } h = \frac{\varepsilon + P}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

where $\gamma = \dfrac{c_p}{c_v}$ the specific heat ratio. This gives rise to the well known relation between pressure and density $p \propto \rho^{\gamma}$ which is worthwhile rederiving here in a slightly different form than is given in the usual Statistical Mechanics texts. Since

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon + P)}{\rho}\frac{d\rho}{dt}$$

then

$$\mu m_p p\frac{ds}{dt} = \frac{1}{\gamma - 1}\left(\frac{dp}{dt} - \gamma\frac{p}{\rho}\frac{d\rho}{dt}\right)$$

$$\Rightarrow \mu m_p\frac{ds}{dt} = \frac{1}{\gamma - 1}\left(\frac{1}{p}\frac{dp}{dt} - \frac{\gamma}{\rho}\frac{d\rho}{dt}\right)$$

Integrating:

$$\mu m_p s = \frac{1}{\gamma - 1}(\ln p - \gamma \ln \rho)$$

$$p = \exp[(\gamma - 1)(\mu m_p s)]\rho^{\gamma}$$

This is often written

$$p = K(s)\rho^{\gamma}$$

where $K(s) = \exp[(\gamma - 1)(\mu m_p s)]$ is known as the ***pseudo-entropy***.

Some terms

Adiabatic Flow: s = constant along a streamline.

Isentropic Flow: s = constant everywhere (space and time).

11 Summary of single fluid equations for an isotropic distribution function

When the distribution function of a component is isotropic, i.e.

$$f(x_i, p_i) = f(x_i, p)$$

then the presure tensor is isotropic and the heat flux is zero. If we further restrict ourselves to the case where all of the components of a fluid have the same mean velocity, then the fluid equations become:

Continuity:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0$$

Momentum:

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j) = -\frac{\partial P}{\partial x_i} - \rho\frac{\partial\phi}{\partial x_i} + \rho_e E_i + \varepsilon_{ijk} J_j B_k$$

where the gravitational field

$$g_i = -\frac{\partial\phi}{\partial x_i}$$

The advective terms in the momentum equation can also be expressed as:

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}(\rho V_i V_j) = \rho\frac{\partial V_i}{\partial t} + \rho V_j \frac{\partial V_i}{\partial x_j}$$

Total energy:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho V^2 + \varepsilon + \varepsilon_{EM} + \rho\phi\right] + \frac{\partial}{\partial x_i}\left[\rho\left(\frac{1}{2}V^2 + h + \rho\phi\right)V_i + S_i\right]$$

$$= \rho\frac{\partial\phi}{\partial t}$$

where the electromagnetic terms are:

$$\text{Poynting flux} = S_i = \frac{1}{\mu_0}\varepsilon_{ijk}E_jB_k$$

$$\text{Energy density} = \varepsilon_{EM} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu}$$

Internal energy equation

$$\frac{d\varepsilon}{dt} + (\varepsilon + P)\frac{\partial V_i}{\partial x_i} = J_{c,i}E'_i$$

where the term on the right represents the Joule dissipation.

Equation of state:

For constant specific heats and $\gamma = \frac{c_p}{c_v}$ the equation of state is

$$p = K(s)\rho^\gamma$$

where the function $K(s)$ of the specific entrop s is called the pseu-do-entropy.

Relationship between pressure and temperature:

$$p = \frac{\rho k T}{\mu m_p}$$

where $\mu \approx 0.62$ for a completely ionised gas and $\mu \approx 1.4$ for a neutral gas.

These equations do not take into account the energy lost by radiation. Also, the electromagnetic terms can be simplified under certain approximations. These features will be considered later in the course.